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Super curves

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Contents

1 Abstract

Understanding super curves is one part of a highly fascinating and active field of research on super spaces, with applications to super string theory and the search for a quantum field of gravity. In this report, we convert simple geometric questions into algebraic ones via elementary algebraic geometry, and we explore both geometric and analytic solutions. We then define a complex supermanifold, and pose a related question regarding super curves.

2 Prelude

2.1 Introduction

Consider a complex polynomial $p(x, y) \in \mathbb{C}[x, y]$ in two variables. The roots of this polynomial form a plane algebraic curve C in $\mathbb{C}^2$. Studying maps (e.g. functions $F : C \to \mathbb{C}$) on curves such as these uncovers a plethora of rich questions in algebraic geometry whose solutions borrow from a diverse range of mathematical fields. By extending the ring in which roots to p can lie, we delve into challenging questions about super curves, and scratch the surface of a burgeoning field of research on super spaces that galvanises the vanguard of mathematicians and mathematical physicists worldwide. This is because it can lead to applications in super string theory, and the search for a quantum theory of gravity (see Dewitt [2], Keßler [3]).

In 3, we provide both an algebraic and a non-algebraic definition of the degree of a map such as F, and prove that these definitions coincide. In 3.1, we prove this definition is well-defined using a geometric argument, and in 3.2 we prove an analogous definition for maps on compact Riemann surfaces is well-defined using an analysis-based approach. We conclude 3 with the result that the existence of a degree one map on a compact Riemann surface implies this surface must have genus 0.

In 4, we introduce Grassmann algebras and the complex supermanifold, and pose a question about super curves that is related to our earlier explorations. We conclude with a discussion about possible approaches to answering this question, and the big-picture relevance of the preceding arguments.

2.2 Statement of Authorship

The bulk of the material presented in this report is standard graduate-level coursework (in some parts we lean heavily on DeWitt [2], and Cavalieri and Miles [1]). However, all questions, conjectures and proofs presented are original, with guidance from my supervisor Paul Norbury, unless otherwise cited.

3 Degree of Algebraic Maps Between Plane Curves

3.1 Defining Degree

In this subsection, we construct the definition of the degree of a map between curves using algebra.

Proposition. *Let $p \in k[x, y]$ be a non-constant polynomial where k is an algebraically closed field, and let C be the algebraic curve $(p(x, y) = 0) \subset \mathbb{A}^2_k$. Then*

(i) any polynomial map $f : C \to \mathbb{A}^1$ induces a ring homomorphism $\phi : k[t] \to k[x, y]/p(x, y)$ via composition;

(ii) assuming f is a non-constant polynomial, for most (i.e. all but finitely many) maximal ideals $M \subset k[t]$, the ideal generated by $\phi(M)$ (i.e. $\langle\phi(M)\rangle$), is contained in d maximal ideals. Here, d is defined to be the degree of the map f.

Proof. **3.1.1 Proving (i)**

Observe that $k[t] = k[\mathbb{A}^1]$ and $k[x, y]/p(x, y) = k[C]$, where $\mathbb{A}^1$ and C are algebraic sets. So f is indeed a map between algebraic varieties. Define $\phi : k[t] \to k[x, y]/p(x, y)$ by composition, i.e. $\phi(g) = g \circ f$. Then clearly we have $\phi(1) = 1$, $\phi(g_1g_2) = (g_1g_2) \circ f = (g_1 \circ f)(g_2 \circ f)$, and $\phi(g_1 + g_2) = (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$ for all $g_1, g_2 \in k[t]$. This concludes the proof of (i).

3.1.2 Proving (ii)

For (ii), the solution is trickier. Our approach will be to convert the definition of the degree to a non-algebraic equivalent using the Nullstellensatz, and then to utilise Bezout's theorem to show that the degree is always well-defined. Hence, consider the following theorems and lemmas:

Hilbert's Nullstellensatz. *Let k be an algebraically closed field. Every maximal ideal of the polynomial ring $k[X_1, \dots, X_n]$ is of the form $(X_1 - a_1, \dots, X_n - a_n)$ for some point $Q = (a_1, \dots, a_n) \in \mathbb{A}^n_k$; that is, it is the ideal $I(Q)$ of all functions vanishing at Q.*

Note that most sources include a corollary about the radical, but we omit this as it is unnecessary for our discussion. The proof is non-technical and widely available (see for example section 3.9 in Reid [4]).

Lemma 1. *For an algebraically closed field k and a polynomial $p(x, y) \in k[x, y]$, the maximal ideals of $k[x, y]/p(x, y)$ are exactly given by $(x - u, y - v)$ for all u, v satisfying $p(u, v) = 0$.*

Proof. We first claim that if $\pi : k[x, y] \to k[x, y]/p(x, y)$ is the natural projection ring homomorphism, then $I \subset k[x, y]/p(x, y)$ is a maximal ideal if and only if $\pi^{-1}(I) \subset k[x, y]$ is a maximal ideal containing $p(x, y)$.

To show this is true, let $I \subset k[x, y]/p(x, y)$ be maximal, and suppose there exists an ideal $J \subset k[x, y]$ such that $\pi^{-1}(I) \subsetneq J \subsetneq k[x, y]$. First note that it is easy to verify $\pi^{-1}(I) = (\pi^{-1}(I))$ is an ideal, and it contains $p(x, y)$

since $0 \in I$. Now take $j \in J \setminus \pi^{-1}(I)$; if $\pi(J) = I$, then $\pi(j) \in I \implies j \in \pi^{-1}(I)$, a contradiction. Similarly, take $q \in k[x,y] \setminus J$; if $\pi(J) = k[x,y]/p(x,y)$, then $\pi(q) \in \pi(J) \implies q = j + rp$ for some $j \in J$, $r \in k[x,y]$. But $p \in \pi^{-1}(I) \subsetneq J$ so that $q = j + rp \in J$, a contradiction. Hence, we have found that $I \subsetneq \pi(J) \subsetneq k[x,y]/p(x,y)$. Since $\pi(J)$ is trivially an ideal, this contradicts the maximality of I, and hence no such ideal J exists. Thus if $I \subset k[x,y]/p(x,y)$ is maximal, then $\pi^{-1}(I) \subset k[x,y]$ is a maximal ideal containing $p(x,y)$.

For the other direction, let $I \subset k[x,y]/p(x,y)$ be some arbitrary subset, and $\pi^{-1}(I) \subset k[x,y]$ be a maximal ideal containing $p(x,y)$. Note that since π is surjective, we have $I = \pi(\pi^{-1}(I))$ and thus I must be an ideal. Suppose that there exists some ideal $J \subset k[x,y]/p(x,y)$ such that $I \subsetneq J \subsetneq k[x,y]/p(x,y)$. Using again that π is surjective, $\pi(\pi^{-1}(S)) = S$ for any $S \subset k[x,y]/p(x,y)$, and thus we must have $\pi^{-1}(I) \subsetneq \pi^{-1}(J) \subsetneq k[x,y]$. As before, it is easy to verify $\pi^{-1}(J) = (\pi^{-1}(J))$ is an ideal. But then we have contradicted the maximality of $\pi^{-1}(I)$. So no such ideal J exists, and thus I is maximal. This completes the proof of our initial claim.

Having characterised the maximal ideals of $k[x,y]/p(x,y)$ in terms of those of $k[x,y]$, it suffices to find all maximal ideals of $k[x,y]$ containing $p(x,y)$. Observe that since k is algebraically closed, the Nullstellensatz tells us that the maximal ideals of $k[x,y]$ are exactly $(x-u, y-v)$ for all $u, v \in k$. Now clearly we have $p(x,y) \in (x-u, y-v) \implies p(u,v) = 0$, and the usual argument with polynomial division of $p(x,y)$ by $x-u$ in $k[y][x]$ and then polynomial division of the remainder by $y-v$ in $k[y]$ gives $p(u,v) = 0 \implies p \in (x-u, y-v)$. So all the maximal ideals of $k[x,y]/p(x,y)$ are exactly given by $(x-u, y-v)$ for all u, v satisfying $p(u,v) = 0$, and this concludes the proof of lemma 1. □

Bézout's theorem. *Suppose that X and Y are two plane projective curves defined over an algebraically closed field k that do not share a non-constant factor. Then the total number of intersection points of X and Y in $\mathbb{P}_k^2$ counted with their multiplicities, is equal to the product of the degrees of X and Y.*

Note that we have merely stated the theorem for two plane curves, but a version of the theorem exists for n polynomials in n indeterminates. This version as well as the elementary proof of the version concerning plane curves (using the resultant) is available on Wikipedia [5].

Lemma 2. *Let k be an algebraically closed field, $C : (p(x,y) = 0) \subset \mathbb{A}_k^2$ be a nonempty algebraic plane curve, and $f : C \to \mathbb{A}_k^1$ be a non-constant polynomial map. Then $f(x,y) - a$ has a constant d distinct roots for all but finitely many points $a \in \mathbb{A}^1$. We also refer to this d as the degree of the map f.*

Proof. Henceforth, let $d'(a)$ and $d(a)$ respectively denote the number of roots of $f(x,y) - a$ counting multiplicity and not counting multiplicity. Let $P(X,Y,Z) := Z^{D_p} p(X/Z, Y/Z)$ where D_p is the total degree of p. Note that P is a homogeneous polynomial, so $C' : (P(X,Y,Z) = 0) \subset \mathbb{P}_k^2$. Clearly C' meets the affine piece $\mathbb{A}_k^2$ in the affine curve C. Similarly, let $F_a(X,Y,Z) := Z^{D_f}[f(X/Z, Y/Z) - a]$ where D_f is the total degree of $f(x,y) - a$ so that $M_a : (F_a(X,Y,Z) = 0) \subset \mathbb{P}_k^2$ meets the affine piece $\mathbb{A}_k^2$ in the affine curve $(f(x,y) - a = 0)$.

Since D_P is finite, there exist at most finitely many $a \in \mathbb{A}^1$ for which $F_a \mid P$. Let A be the finite set of all such a. Then by Bézout's Theorem, for each $a \in \mathbb{A}^1 \setminus A$, C' and M_a intersect at $D_P \cdot D_{F_a} = D_p \cdot D_f$ points counting multiplicity (f is non-constant, so D_{F_a} is clearly independent of a).

Consider that since $D_f > 0$ by assumption, $F_a(X, Y, 0)$ is independent of a and thus the number of intersections of C' and M_a at infinity (counting multiplicity) remains constant at say R as a varies over $\mathbb{A}^1 \setminus A$. Hence, the number of roots of $f(x, y) - a$ (counting multiplicity) is a constant $d'(a) = D_p \cdot D_f - R$ for $a \in \mathbb{A}^1 \setminus A$.

Finally, we can argue that the number of *distinct* roots of $f - a$ is constant for all but finitely many points $a \in \mathbb{A}^1$ (i.e. the degree is well-defined). First, suppose that p is irreducible. A root $\vec{r}$ of $f(x, y) - a$ of order $j \leq D_p \cdot D_f$ must satisfy

$$\left.\frac{d^i y}{dx^i}_{\text{on } (p=0)}\right|_{\vec{r}} = \left.\frac{d^i y}{dx^i}_{\text{on } (f=0)}\right|_{\vec{r}} \quad \text{or} \quad \left.\frac{d^i x}{dy^i}_{\text{on } (p=0)}\right|_{\vec{r}} = \left.\frac{d^i x}{dy^i}_{\text{on } (f=0)}\right|_{\vec{r}}$$

for all $1 \leq i \leq j - 1$. So if we define the operator $\nabla_i := (\partial_x^i, \partial_y^i)$, then $\vec{r} \in (\nabla_i p \cdot \nabla_i f^{\perp} = 0)$ for all $1 \leq i \leq j - 1$. Let $1 \leq m \leq D_p \cdot D_f$ be the largest index for which p shares a non-constant factor with $\nabla_i p \cdot \nabla_i f^{\perp}$ for all $1 \leq i \leq m$ (set $m = 0$ if no such index exists). Since p is irreducible, we must have $p \mid \nabla_i p \cdot \nabla_i f^{\perp}$ for all $1 \leq i \leq m$. Hence, each point c in C is a root of order $m + 1$ of $f - a$ for $a = f(c) \in \mathbb{A}^1$.

Now p shares no common factor with $\nabla_{m+1} p \cdot \nabla_{m+1} f^{\perp}$. So using Bézout's theorem in a similar manner as earlier, we can deduce that $B' := (\nabla_{m+1} p \cdot \nabla_{m+1} f^{\perp} = 0) \cap C$ is a finite set, and thus $B := f(B')$ is finite. Therefore, $d = D_p \cdot D_f - R - m$ for all $a \in \mathbb{A}^1 \setminus (A \cup B)$. As a visual aid, Figure 1 (Appendix 7.1) depicts the real projection of C and $(f - a) = 0$ for $p(x, y) = x^2/4 + y^2 - 1$, $f(x, y) = x^2 + y^2$, and $a \in B = \{1, 4\}$. In this case, $m = 0$ and $A = \varnothing$.

So if p is irreducible, the degree of the map $f : C \to \mathbb{A}^1$ is indeed well-defined. Now suppose p is reducible, i.e. $p = q_1 \cdots q_n$, where $q_1, \ldots, q_n \in k[x, y]$ are irreducible. Note that for any $h \in k[x, y]$, $(h = 0) = (h^i = 0)$ for all $i \in \mathbb{Z}^+$. Hence, we can assume without the loss of generality that $q_1, \ldots, q_n$ are pairwise coprime. Let $C_i := (q_i = 0)$ for $i = 1, \ldots, n$ so that by the Null factor law, $C = \bigcup_{i=1}^n C_k$. Now we can define the restrictions $f_i := f|_{C_i}$ for $i = 1, \ldots, n$. By the argument in the previous paragraph, each f_i has well-defined degree, say d_i, on all but finitely many points (call the set of these points B_i) in the codomain. Then

$$d = d(a) = \sum_{i=1}^{n} d_i \quad \forall \quad a \in \mathbb{A}^1 \setminus \left[T \cup \bigcup_{i=1}^{n} B_i \right],$$

where $T := \{f(c) \mid c \in C_i \cap C_j,\ i \neq j\}$. Clearly $\bigcup_{i=1}^n B_i$ is finite, and also T is finite using Bézout's theorem in a similar manner as earlier, and the fact that $q_1, \ldots, q_n$ are pairwise coprime. Hence, the degree d of f is indeed well-defined, and this concludes the proof of lemma 2. $\square$

Finally, we have enough to prove (ii). By the Nullstellensatz, the maximal ideals $M \subset k[t]$ correspond to points, i.e. $M = (t - a)$. Then for every element $E \in \langle \phi((t - a)) \rangle$, we have $f(x, y) - a \mid E$ in $k[x, y]/p(x, y)$. Since p non-constant implies C nonempty, we can apply lemma 2 to deduce that $f(x, y) - a$ has exactly d distinct roots $(u_1, v_1), \ldots, (u_d, v_d)$ in C for all but finitely many $a \in \mathbb{A}^1$. Thus by lemma 1, $\langle \phi((t - a)) \rangle$ is contained in exactly the d maximal ideals $(x - u_1, y - v_1), \ldots, (x - u_d, y - v_d)$ of $k[x, y]/p(x, y)$ for all but finitely many $a \in \mathbb{A}^1$. Not only does this conclude the proof of (ii), but this also shows that the definition of the degree given in lemma 2 indeed coincides with the definition of the degree given in (ii) of the proposition. $\square$

Comments. Note that no definition of multiplicity was given, but for the purposes of our discussion in lemma 2 we assumed it meant the usual vanishing higher derivatives. One gains intuition by taking $k = \mathbb{C}$ as the natural field extension of $\mathbb{R}$, but the argument should generalise as in the case of Bézout's theorem. Also note that all prior discussion regarding multiplicity is mostly useless for the next subsection, as we discuss degree one maps.

3.2 Complex Analysis Approach and Degree One Maps

In this subsection, we prove a variation of the proposition in 3.1 using a different approach: complex analysis.

Proposition. *Let $P \in \mathbb{C}[X, Y, Z]$ be a non-constant homogeneous polynomial satisfying*

$$\left\{(X, Y, Z) \in \mathbb{C}^3 \mid \frac{\partial P}{\partial X} = \frac{\partial P}{\partial Y} = \frac{\partial P}{\partial Z} = 0\right\} \subset \{(0, 0, 0)\}. \tag{0}$$

(i) Given a non-constant polynomial map $F : S \to \mathbb{CP}^1$ where $S : (P = 0) \subset \mathbb{CP}^2$ is connected, the number of preimages via F is constant for all but finitely many points in $\mathbb{CP}^1$, and is called the degree d of F;

(ii) if $d = 1$, the genus of S must be 0.

Proof. **3.2.1 Compact Riemann Surface**

First, we show that S is a compact Riemann surface using a proof by Cavalieri and Miles [1]:

Firstly, for any $p(x, y) \in \mathbb{C}[x, y]$, we define the affine algebraic curve $C : (p = 0) \subset \mathbb{C}^2$ to be *smooth* if there is no $(x_0, y_0) \in C$ such that $\frac{\partial p}{\partial x}(x_0, y_0) = \frac{\partial p}{\partial y}(x_0, y_0) = 0$. Similarly, we define a the vanishing set $(P = 0) \subset \mathbb{CP}^2$ of a homogeneous polynomial $P \in \mathbb{C}[X, Y, Z]$ to be *smooth* if it satisfies the condition (0). Hence, S as given in the proposition is smooth.

To show that S is compact, it suffices to show that S is a closed set in $\mathbb{CP}^2$, as $\mathbb{CP}^2$ is a compact topological space. Consider the diagram

$$\begin{array}{ccc} \mathbb{C}^3 \setminus \{(0,0,0)\} & \xrightarrow{P} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{CP}^2 & & \end{array}$$

where π is the natural projection function and P is the (continuous) function defined by the homogeneous polynomial P. By definition, S is a closed subset of $\mathbb{CP}^2$ if $\pi^{-1}(S)$ is closed in $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$. But $\pi^{-1}(S) = P^{-1}(0)$ is the inverse image of the closed set $\{0\} \subset \mathbb{C}$, therefore it is closed.

To prove that S is a Riemann surface, it is sufficient to show that its intersection with any of the coordinate open sets of $\mathbb{CP}^2$ is a Riemann surface. Consider (without loss of generality) the chart

$$U_Z := \{[X : Y : Z] \mid Z \neq 0\} \subset \mathbb{CP}^2$$

with affine coordinates

$$(x, y) = \varphi_Z(X, Y, Z) = (X/Z, Y/Z).$$

The set $\varphi_Z(S \cap U_Z)$ is equal to $V(p)$ (a.k.a. $(p = 0)$), where $p(x, y) := P(x, y, 1)$ is called the *dehomogenisation* of P with respect to Z. For any $(x, y) \in \mathbb{C}^2$,

$$\frac{\partial p}{\partial x}(x, y) = \frac{\partial P}{\partial X}(x, y, 1) \tag{1}$$

$$\frac{\partial p}{\partial y}(x, y) = \frac{\partial P}{\partial Y}(x, y, 1). \tag{2}$$

We claim there can be no $(\tilde{x}, \tilde{y}) \in \mathbb{C}^2$ such that

$$p(\tilde{x}, \tilde{y}) = \frac{\partial p}{\partial x}(\tilde{x}, \tilde{y}) = \frac{\partial p}{\partial y}(\tilde{x}, \tilde{y}) = 0. \tag{3}$$

The claim implies that $V(p)$ is a smooth affine plane curve. But then by the Complex Regular Value Theorem, $V(p) = p^{-1}(0)$ is a Riemann surface. Since the restriction of S with any affine chart is a Riemann surface, so is S.

To prove the claim, assume there is $(\tilde{x}, \tilde{y}) \in \mathbb{C}^2$ satisfying the system of equations in (3). By (1) and (2) and the smoothness of S, it must be that

$$\frac{\partial P}{\partial Z}(\tilde{x}, \tilde{y}, 1) \neq 0.$$

But now Euler's Identity gives us a contradiction, since

$$0 \neq \frac{\partial P}{\partial X}(\tilde{x}, \tilde{y}, 1) + \frac{\partial P}{\partial Y}(\tilde{x}, \tilde{y}, 1) + \frac{\partial P}{\partial Z}(\tilde{x}, \tilde{y}, 1) = 0.$$

Comments. Note that Cavalieri and Miles [1] do note cite the Complex Regular Value Theorem in their proof, although this is the common name given to this result. However, they do present a proof of the result using the Implicit Function Theorem, which we omit for the sake of brevity.

3.2.2 Defining a Holomorphic Map

The following definition is from Cavalieri and Miles [1]:

Let X and Y be Riemann surfaces and $f : X \to Y$ a set function.

1. We say that f is *holomorphic at* $x \in X$ if for every choice of charts $\varphi_x, \varphi_{f(x)}$ the function $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic at x.

2. If $U \subset X$ is open, we say that f is *holomorphic on* U if f is holomorphic at each $x \in U$.

3. If f is holomorphic on $U = X$, we say that f is a *holomorphic map*.

The function $L := \varphi_{f(s)} \circ f \circ \varphi_s^{-1}$ is called a *local form* for f (note that some sources including Miles and Cavalieri call it a *local expression*).

Using this definition, it is now clear that F (given in the proposition) is a holomorphic map, as $\mathbb{CP}^2$ is a Riemann surface, and the coordinate functions $X, Y, Z : S \to \mathbb{CP}^1$ are obviously holomorphic in the usual sense.

3.2.3 Showing Cardinality of the Fibre is Finite

We first show that $|F^{-1}(a)|$ is finite for all $a \in \mathbb{CP}^1$; consider the following famous theorem (adapted from Theorem 4.2.1 by Cavalieri and Miles [1]):

Theorem on Local Form *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. For any $x \in X$ there are charts φ_x, $\varphi_{f(x)}$ such that $\varphi_x(x) = \varphi_{f(x)}(f(x)) = 0$, and the local expression of f using these charts is $z \mapsto z^k$ for some $k \geq 1$.*

Suppose for the sake of contradiction there exists some $a \in \mathbb{CP}^1$ for which $F^{-1}(a)$ is an infinite set. Since S is compact, there must then exist a cluster point of roots of $F - a$ on S. Now by the continuity of F and the fact that $\{a\} \subset \mathbb{CP}^1$ is closed, the fibre $F^{-1}(a)$ of a is closed. Let I be the interior of $F^{-1}(a)$; if $I \neq S$, the boundary of I must be nonempty (since S is connected). Hence, let s be a point on the boundary of I. Then F is non-constant near s, so the Theorem on Local Form applies. But there must be a sequence of points from $F^{-1}(a)$ converging to s on which $F = a$, contradicting the theorem. So we must have $S = I = F^{-1}(a)$, in contradiction to F being non-constant. So the cardinality of the fibre $F^{-1}(a)$ is indeed finite.

Comments. This argument is analogous to the proof that a holomorphic function with a cluster point of roots in a domain Ω is vanishing on Ω.

3.2.4 Proving (i)

To prove (i), we could use Bézout's theorem as in 3.1.2. But as mentioned earlier, here we present a different proof making use of the following theorems from Cavalieri and Miles [1]:

Theorem 1 (c.f Lemma 4.2.5) *If X is a compact Riemann surface and $f : X \to Y$ is a non-constant holomorphic map, then the branch locus is a finite set.*

Theorem 2 (c.f Theorem 4.3.3) *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. If $y_1, y_2 \in Y$ are not in the branch locus of f, then $|f^{-1}(y_1)| = |f^{-1}(y_2)|$.*

By the fact that F is a holomorphic map, Theorem 2 implies that the number of preimages via F is constant for all points in $\mathbb{CP}^1$ excluding the branch locus. Since S is a compact Riemann surface, Theorem 2 implies the branch locus is finite. This (in combination with our proof that this constant number of preimages is finite) shows that the degree of F in (i) is indeed well-defined. Note that $d = |F^{-1}(a)|$ for any $a \in \mathbb{CP}^1$ not in the branch locus. This coincides with the definition of the degree given by Cavalieri and Miles.

Comments. The proof of Theorem 1 relies on the compactness of S in a similar manner to our proof in 3.2.3.

Refer to [1] for a definition of the branch locus; for the purposes of our proof above, it suffices to know only that this set is finite.

3.2.5 Proving (ii)

Consider the following theorem from Cavalieri and Miles [1]:

Theorem 3 (c.f Exercise 4.3.2) *Let $f : X \to Y$ be a holomorphic map of compact Riemann surfaces of degree $d > 0$. For any $y \in Y$ and $f^{-1}(y) = \{x_1, \ldots, x_n\}$, we have $d = \sum_{i=1}^{n} k_{x_i}$ where $k_{x_i} \geq 1$ is the multiplicity of the root x_i in $f(x) - y$ (also called the ramification index).*

First note that Theorem 3 implicitly gives that F is surjective (one can also deduce this via Liouville's theorem and a 'clopen' argument, see Theorem 4.3.1 in Cavalieri and Miles [1]).

Now if $d = 1$, Theorem 3 more explicitly implies F is injective, since $d \geq |F^{-1}(a)| \cdot 1 \implies |F^{-1}(a)| = 1$. Then by the Inverse Function Theorem for holomorphic functions, F is invertible with locally holomorphic inverse. So F induces a bicontinuous bijection, i.e. a homeomorphism, from S to $\mathbb{CP}^1$. It is an elementary result that $\mathbb{CP}^1$ is the one-point compactification of $\mathbb{C}$ and thus homeomorphic to the 2-sphere S^2. Therefore, $S \cong S^2$ must have genus 0. $\square$

Comments. There is some natural (although perhaps imprecise) intuition to understanding (ii). Intuitively, genus for plane algebraic curves is similar to topological genus. So if S has genus greater than 0, then it has a handle. It seems like this would have more than one intersection with $F - a$ for most values of a, a contradiction. For some additional terminology, we say F is a $d : 1$ branched cover of $\mathbb{CP}^1$ if F has degree d.

Finally, if P is homogeneous P and $F - a$ have no common roots at infinity, then we can deduce an analogous result to (ii) for the plane algebraic curve $C : (p = 0)$, where p is the dehomogenisation of P.

4 The Super Case

4.1 Grassmann Algebras

All of the definitions in this subsection are due to DeWitt [2]. Let ζ^a, $a = 1, \ldots, N$, be a set of generators for an algebra, which anticommute:

$$\zeta^a \zeta^b = -\zeta^b \zeta^a, \quad (\zeta^a)^2 = 0 \quad \text{for all } a, b.$$

The algebra is called a *Grassmann algebra* and will be denoted by Λ_N. We deal with the formal limit $N \to \infty$. The corresponding algebra will be denoted by Λ_∞. Note that the elements $1, \zeta^{a_1}, \zeta^{a_1}\zeta^{a_2}, \ldots$, where the exponents within each product range over all finite sequences of strictly increasing integers, form an infinite basis for Λ_∞.

The elements of Λ_∞ will be called *supernumbers*. Every supernumber can be expressed in the form

$$z_B + z_S,$$

where the *body* z_B is an ordinary complex number, and the *soul*

$$z_S = \sum_{n=1}^{\infty} \sum_{a_1,\dots a_n} c_{a_1\dots a_n} \zeta^{a_1} \cdots \zeta^{a_n}.$$

Here, the coefficients c are complex numbers anti-symmetric in their indices. Henceforth, we use DeWitt's convention

$$z_S := \sum_{n=1}^{\infty} \frac{1}{n!} c_{a_1\dots a_n} \zeta^{a_1} \cdots \zeta^{a_n},$$

where we adjust the coefficients, and the summation over repeated indices is to be understood unless otherwise stated.

Note: this sum must terminate for each supernumber in Λ_∞, in order for the formal limit to be defined. A useful analogy is that the ring of real polynomials is an infinite-dimensional vector space, but does not contain any infinite power series.

A supernumber z may be split into its *even* and *odd* parts u and v respectively:

$$\begin{aligned} z &= u + v, \\ u &= z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1\dots a_{2n}} \zeta^{a_{2n}} \cdots \zeta^{a_1}, \\ v &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1\dots a_{2n+1}} \zeta^{a_{2n+1}} \cdots \zeta^{a_1}. \end{aligned}$$

The set of all even supernumbers is a commutative subalgebra of Λ_∞, which will be denoted $\mathbb{C}_c$. The set of all odd supernumbers will be denoted $\mathbb{C}_a$; it is *not* a subalgebra.

4.2 Complex Supermanifolds

4.2.1 The Coarse Topology

Before defining the complex supermanifold, we have to first define the *coarse* topology on $\mathbb{C}^{m|n} := \mathbb{C}_c^m \times \mathbb{C}_a^n$. Consider the subspace $\mathbb{C}^m \subset \mathbb{C}^{m|n}$, defined as the set of points whose coordinates have vanishing souls. Let $\pi : \mathbb{C}^{m|n} \to \mathbb{C}^m$ be the natural projection. A subset of $\mathbb{C}^{m|n}$ is said to be *open* in the coarse topology if and only if it has the form $\pi^{-1}(\mathcal{O})$ where $\mathcal{O}$ is some open subset of $\mathbb{C}^m$ in the usual sense.

Comments. The namesake of the topology derives from the fact that if we replace Λ_∞ with Λ_N, the natural topology given to $\mathbb{C}^{m|n}$ (by virtue of the fact that it is a $2^{N+1}(m+n)$-dimensional vector space) is not as coarse.

4.2.2 The Definition

DeWitt defines a *complex supermanifold of dimension* (m, n) to be a space M together with a collection of ordered pairs $(\mathscr{U}_A, \phi_A)$ where each $\mathscr{U}_A$ is a subset of M, and its associated ϕ_A is a one-to-one mapping of $\mathscr{U}_A$ onto an open set in $\mathbb{C}^{m|n}$ (in the coarse sense). The collection of ordered pairs is required to have the following properties:

(1) $\bigcup_A \mathscr{U}_A = M$

(2) $\phi_A \circ \phi_B^{-1}$ is differentiable for all nonempty intersections $\mathscr{U}_A \cap \mathscr{U}_B$.

On the other hand, Keßler defines a *complex supermanifold* $\mathbb{C}^{m|n}$ to be the topological space $\mathbb{C}^m$ together with the sheaf $\mathscr{O}_{\mathbb{C}^{m|n}} = \mathcal{H}_{\mathbb{C}^m} \otimes_{\mathbb{C}} \bigwedge_n^{\mathbb{C}}$.
Comments. DeWitt does not state the earlier definition explicitly, but it can be inferred from his definition of the supermanifold. Note also that the equivalence of the definitions of Keßler and DeWitt is nontrivial and uses the functor of points.

4.3 Posing a Problem in the Super Case

With these definitions, we can now pose a problem in the super case (analogous to (ii) of the proposition in 3.2):

Question. *Suppose $p \in \mathbb{C}[x, y]$, but we interpret $C : (P = 0) \subset \mathbb{CP}^{2|1}$ for the corresponding homogeneous polynomial P. If every point in the codomain of a non-constant polynomial map $F : C \to \mathbb{CP}^{1|1}$ has exactly one preimage, is it true that $C \cong \mathbb{CP}^{1|1}$?*

In the question above, one would call C a *super curve*, the namesake of this report. Upon first inspection, the question is well-defined as P is a polynomial in 3 variables, embedded in a space with 1 odd and 2 even components.

5 Discussion and Conclusion

In the context of question 4.3, it would be fair to question the relevance of many prior explorations. For example, what was the point of defining degree algebraically in 3.1? The answer is, reframing geometric questions as algebraic ones may help one study analogues of such questions in the super case. This is because geometric intuition is quite often useless in anti-commutative spaces.

For question 4.3 in particular, our algebraic definition of the degree is not helpful as it relied on the Nullstellensatz, which does not hold for non-fields $\mathbb{C}_c$ and $\mathbb{C}_a$ (the latter does not even commute as a ring). But there is some reason to believe an algebraic approach would be the most feasible in answering question 4.3 as there does not appear to exist an analogue of Bézout's theorem for super curves.

To understand the relevance of studying super curves, proving analogues of geometric problems (albeit, more general and advanced problems than question 4.3) in the super case can lead to applications in super string theory, and the search for a quantum theory of gravity (see Dewitt [2], Keßler [3]). With respect to question 4.3 specifcally, it is worth mentioning that Keßler defines the notion of genus for complex supermanifolds such as $\mathbb{CP}^{1|1}$, so in the context of 3.2, answering this question could enable discussion of the genus of super curves. Despite the obvious difficulties working with super curves, a surprising number of analogues to statements in the non-super world appear to have been found, and it is presently a highly active field of research.

6 Acknowledgements

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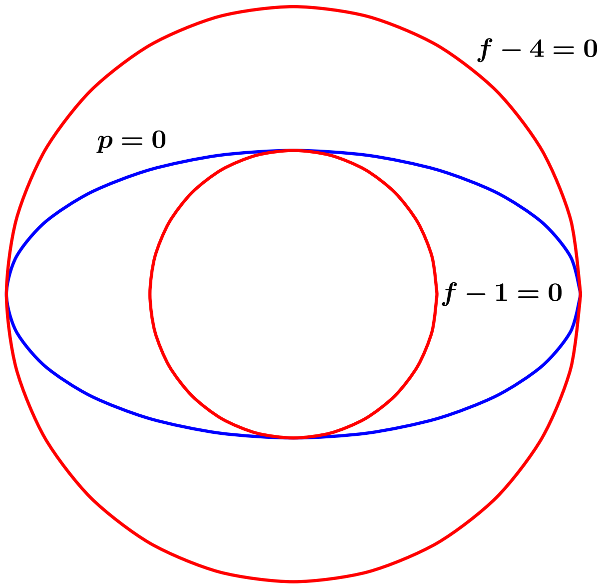
7 Appendices

7.1 Figure 1



7.2 Reid exercises

Below are my solutions to some exercises from Reid's Undergraduate Algebraic Geometry [4], which may help contextualise the notation and algebraic geometry utilised in 3.

0.2 Let $f \in \mathbb{R}[X, Y]$ and let $C : (f = 0) \subset \mathbb{R}^2$; say that $P \in C$ is *isolated* if there is an ε such that $C \cap B(P, \varepsilon) = P$. Show by example that C can have isolated points. Prove that if $P \in C$ is an isolated point then $f : \mathbb{R}^2 \to \mathbb{R}$ must have a max or min at P, and deduce that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at P. This proves that an isolated point of a real curve is singular.

Proof. If $f(x, y) = xy$, then C consists of the single isolated $P = (0, 0)$. If P is isolated, there exists some ε such that $f(v) \neq 0$ for all $v \in D := B(P, \varepsilon) \setminus \{P\}$. Restricting f to the open domain D so that $(f\mid_D)^{-1}(0) = \{P\}$, we see that if $\nabla f \neq \vec{0}$, the regular value theorem would imply $\{P\}$ is a smooth 1-manifold, a contradiction. So $\nabla f(P) = \vec{0}$, and thus P is a critical point. Applying the intermediate value theorem on the path connected domain D shows that f cannot change sign on D, and thus P is a local minimum or local maximum. $\square$

3.5 Let $J = (XY, XZ, YZ) \subset k[X, Y, Z]$; find $V(J) \subset \mathbb{A}^3$; is it irreducible? Is it true that $J = I(V(J))$? Prove that J cannot be generated by 2 elements. Now let $J' = (XY, (X-Y)Z)$; find $V(J')$, and calculate rad J'.

Proof. Observe that J is not prime since $X(Y - Z) \in J$ but $X, Y - Z \notin J$. Therefore, $V(J) = V(J, X) \cup V(J, Y - Z)$ is not irreducible (by 3.7 (a) and the Corollary of NSS, see Reid [4]). Here, $V(J)$ is the union of the X, Y and Z-axes. To see this, note that clearly all members of J vanish when two of X, Y, Z are zero, and for any point where two of X, Y, Z are nonzero it is easy to construct an element of J not vanishing on this point.

Note that $f \in k[X, Y, Z]$ is in J if and only if every monomial is mixed (i.e., every term is the product of powers of at least two of X, Y, Z). So clearly $f \in J \iff f^n \in J$, and thus $J = \text{rad } J$ is radical. So by (c) of NSS (see Reid [4]), $J = I(V(J))$. To see why J cannot be generated by two elements, observe that as a k-vector space, $\dim_k J/(x, y, z)J = 3$. This is because the residue classes of the three mixed quadratic monomials are linearly independent over k.

Now $V(J') = V(J)$ is the union of the three axes by the same reasoning as above. Using part (c) of NSS, we conclude that rad $J' = I(V(J')) = I(V(J)) = J \supsetneq J'$. □

3.9 Let $f = X^2 - Y^2$ and $g = X^3 + XY^2 - Y^3 - X^2Y - X + Y$; find the irreducible components of $V(f, g) \subset \mathbb{A}^2_{\mathbb{C}}$.

Proof. Observe that $f = (X - Y)(X + Y) \in (f, g)$, but $X - Y, X + Y \notin (f, g)$. This is because if $X - Y \in (f, g)$, we would have $X - Y = mf + ng$ where $n \in k$, but

$$(n + 1 - m(X + Y)) \nmid n(X - Y)^2.$$

So $V(f, g) = V(f, g, X - Y) \cup V(f, g, X + Y)$. Observe that $g = (X - Y)(X - Y - 1)(X - Y + 1)$, so $V(f, g, X - Y) = V(X - Y)$, which is clearly irreducible.

Finally,

$$\begin{aligned} V(f, g, X + Y) &= V(4X^3 - 2X, X + Y) \\ &= V(X, X + Y) \cup V(X - 1, X + Y) \cup V(X + 1, X + Y), \end{aligned}$$

assuming char $k > 4$. These components represent single points, and are thus irreducible. So the unique (see 3.7 (b) of Reid [4]) decomposition of $V(f, g)$ into irreducibles is given by

$$V(f, g) = V(X - Y) \cup V(X, X + Y) \cup V(X - 1, X + Y) \cup V(X + 1, X + Y).$$

□

4.5 Let $C : (Y^2 = X^3) \subset \mathbb{A}^2$; prove that

(a) the parametrisation $f : \mathbb{A}^1 \to C$ given by (T^2, T^3) is a polynomial map;

(b) f has a rational inverse $g : C \dashrightarrow \mathbb{A}^1$ defined by $(X, Y) \mapsto Y/X$;

(c) dom $g = C \setminus \{(0, 0)\}$;

(d) f and g give inverse isomorphisms $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0, 0)\}$.

Proof. First note $\mathbb{A}^1$ is the locus of the ideal (0), so is indeed an algebraic set. Now observe that for $F_1 := X^2, F_2 := X^3 \in k[X]$, we have $f(T) = (F_1(T), F_2(T)) \in \mathbb{A}^2_k$ for all $T \in \mathbb{A}^1$. So by definition, the parametrisation f is a polynomial map. This proves (a).

Now by definition, g is a rational map (in fact, a function). Observe that for $g = \alpha/\beta$ with $\alpha, \beta \in k[C]$, we must have $X \mid \beta$ so that β must vanish on the input $(0,0)$. For all $P \in C \setminus \{(0,0)\}$, we have $g = Y/X$ with X not vanishing, so dom $g = C \setminus \{(0,0)\}$. This proves (c).

To prove g is the inverse, we merely need to show $g \circ f = \text{id}_{\mathbb{A}^1}$ on $f^{-1}(\text{dom } g)$ and $f \circ g = \text{id}_C$ on dom g. Clearly $f^{-1}(\text{dom } g) = \mathbb{A}^1 \setminus \{0\}$, and indeed for $T \neq 0$ we have $(g \circ f)(T) = T^3/T^2 = T$. Similarly, for $(X, Y) \neq (0,0)$ we have $(f \circ g)(X, Y) = (Y^2/X^2, Y^3/X^3) = (X, Y)$. This proves (b).

Finally, by definition g restricted to dom g is a morphism. Similarly, f restricted to $\mathbb{A}^1 \setminus \{0\}$ is also a morphism, and as shown above, is a two-sided inverse morphism to g. Thus f and g give inverse isomorphisms $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0,0)\}$. This proves (d). $\square$

4.7 Let $C : (Y^2 = X^3 + X^2) \subset \mathbb{A}^2$; the familiar parametrisation $\varphi : \mathbb{A}^1 \to C$ given by $(T^2 - 1, T^3 - T)$ is a polynomial map, but why is it not an isomorphism? Find out whether the restriction $\varphi' : \mathbb{A}^1 \setminus \{1\} \to C$ is an isomorphism.

Proof. Observe that $\varphi(-1) = \varphi(1)$, so φ is not an isomporhism as it is not injective.

The suggested restriction φ' is indeed a *bijective* morphism, but is still not an isomorphism. To see this, suppose $(\varphi')^{-1}(X, Y) := \alpha(X, Y)/\beta(X, Y)$ is the two-sided inverse morphism, and let $a \in k$ and $b \in k$ be the constant terms of α and β respectively. Now

$$\begin{aligned}
\left((\varphi')^{-1} \circ \varphi\right)(T) = T \ \ \forall \ \ T \in \mathbb{A}^1 \setminus \{1\} \\
\implies \alpha = \beta T \ \ \forall \ \ T \in \mathbb{A}^1 \setminus \{1\} \\
\implies \alpha = \beta T \ \ \forall \ \ T \in \mathbb{A}^1 \qquad & \text{(by FTA, since } k \text{ is an infinite field)} \\
\implies T^2 - 1 \mid Tb - a \\
\implies a = b = 0.
\end{aligned}$$

But $\left((\varphi')^{-1} \circ \varphi\right)(-1) = -1 \implies a = -b \neq 0$, a contradiction.

Note that the restriction $\varphi'' : \mathbb{A}^1 \setminus \{-1, 1\} \to C \setminus \{(0,0)\}$ is an isomorphism via $(\varphi'')^{-1}(X, Y) = Y/X$. $\square$

5.1 Prove a regular function on $\mathbb{P}^1$ is constant. Deduce that there are no nonconstant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ for any m.

Proof. Suppose that $f \in k(\mathbb{P}^1)$ is regular at every point of $\mathbb{P}^1$. Recall that $\mathbb{P}^1 \cong \mathbb{A}^1 \sqcup \mathbb{A}^1/\sim$, where $\mathbb{A}^1_{(\infty)} \ni x_0 \sim 1/x_0 \in \mathbb{A}^1_{(0)}$ for $x_0 \neq 0$. By applying (4.8, II) (see Reid [4]) to the affine piece $\mathbb{A}^1_{(0)}$, we have that $f = p(x_0) \in k[x_0]$. Hence, on the other affine piece $\mathbb{A}^1_{(\infty)}$ (with coordinate y_1), $f = p(1/y_1) \in k[y_1]$.

If p is non constant, then $p(1/y_1)$ is not a polynomial in y_1, a contradiction of (4.8, II). So p is constant and thus f is constant. Since morphisms are regular, it naturally follows that there are no nonconstant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ for any m. $\square$

5.6 Let $C \subset \mathbb{P}^3$ be an irreducible curve defined by $C = Q_1 \cap Q_2$, where $Q_1 : (TX = q_1)$, $Q_2 : (TY = q_2)$, with q_1, q_2 quadratic forms in X, Y, Z. Show that the projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ defined by $(X, Y, Z, T) \mapsto (X, Y, Z)$ restricts to an isomorphism of C with the plane curve $D \subset \mathbb{P}^2$ given by $Xq_2 = Yq_1$.

Proof. First, observe that the given projection π is a rational map, and a morphism outside the point $P_0 = (0, 0, 0, 1) \in C$ (since C is irreducible, it is an affine variety). Next, observe that the map is well-defined, since $(TX = q_1) \wedge (TY = q_2) \implies Xq_2 = TXY = Yq_1$. Now define the inverse morphism $\varphi : D \to C$ by

$$\varphi(X, Y, Z) = \begin{cases} (X, Y, Z, q_1(X, Y, Z)/X) & \text{if } X \neq 0 \\ (X, Y, Z, q_2(X, Y, Z)/Y) & \text{if } X = 0 \text{ and } Y \neq 0. \end{cases}$$

Note that if $X = Y = 0$ but $Z \neq 0$, then C would be reducible as $(X^2 + Y^2 = 0) \subsetneq C$. $\square$

5.12 Let C be the cubic curve $(Y^2Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2$. Prove that any regular function f on C is constant.

Proof. Recall that $C \cong C_0 \sqcup C_1/\sim$, where $C_0 : (y^2 = x^3 + ax + b) \subset \mathbb{A}^2$ and $C_1 : (z_1 = x_1^3 + ax_1z_1^2 + bz_1^3) \subset \mathbb{A}^2$ are affine curves, and $C_0 \ni (x, y) \sim (x/y, 1/y) \in C_1$. Applying (4.8, II) to the affine piece C_0, we have $f = p(x, y) \in k[x, y]$. Subtracting a suitable multiple of $y^2 - x^3 - ax - b$, we can assume $p(x, y) = q(x) + yr(x)$, with $q, r \in k[x]$. Applying (4.8, II) to the affine piece C_1 then gives

$$f = q(x_1/z_1) + (1/z_1)r(x_1/z_1) \in k[C_1],$$

and hence there exists a polynomial $S(x_1, z_1)$ such that

$$q(x_1/z_1) + (1/z_1)r(x_1/z_1) = S(x_1, z_1).$$

Clear the denominator, and use the fact that $k[C_1] = k[x_1, z_1]/g$, where $g = z_1 - x_1^3 - ax_1z_1^2 - bz_1^3$, to deduce a polynomial identity

$$Q_m(x_1, z_1) + R_{m-1}(x_1, z_1) \equiv S(x_1, z_1)z_1^m + A(x_1, z_1)g$$

in $k[x_1, z_1]$, with Q_m and R_{m-1} homogeneous of the indicated degrees.

Now if we write $S = S^+ + S^-$ and $A = A^+ + A^-$ for the decomposition into terms of even and odd degree, and note that g has only terms of odd degree, this identity splits into

$$Q_m \equiv S^+z_1^m + A^-g \quad \text{and} \quad R_{m-1} \equiv S^-z_1^m + A^+g$$

if m is even, and an analogous expression if m is odd.

Suppose $m \geq 1$. Now Q_m is homogeneous of degree m; since the degree of every monomial in $S^+ z_1^m$ is $\geq m$, every monomial in $A^- g$ must also have degree $\geq m$. Then if $z_1 \nmid Q_m$, $A^- g$ must contain the monomial cx_1^m for some nonzero constant c. But then A^- contains the monomial $-cx_1^{m-3}$, and thus $A^- g$ contains the monomial $-cx_1^{m-3} z_1$, a contradiction. So $z_1 \mid Q_m$, and by a similar argument, $z_1 \mid R_{m-1}$.

Finally, since $Q_m(x_1, z_1) = z^m q(x_1/z_1)$ and $R_{m-1}(x_1, z_1) = z^{m-1} r(x_1/z_1)$, taking m minimal gives a contradiction as we must have either $z_1 \nmid Q_m$ or $z_1 \nmid R_{m-1}$. So we cannot possibly have $m \geq 1$. Thus $\deg q = 0$ and $r = 0$, as required. $\square$