Polynomial method in additive combinatorics - Extending the Cap Set Problem to more than three points

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Abstract

In this report, we give an introduction to Additive Combinatorics before introducing the history and background of the Cap Set problem. A concise proof to this problem is given which incorporates Tao's optimisations to the proof of Ellenberg and Gijswijt. A natural generalisation of capsets, known as almost capsets, is introduced and a theorem of Fish and Roy on the cardinality of almost cap sets is generalised. A paper of this is currently in production and will be submitted to a peer-reviewed journal.

1 Introduction

1.1 Introduction to Additive Combinatorics

Additive Combinatorics is the study of the interplay between the combinatorial and additive structure of sets and, as a field, has been very active over the past century. One of the biggest results in this field is the proof of the Erdös-Turán conjecture in which Erdös and Turán published a paper in 1936, containing the following[5]:

Conjecture 1 (Erdös-Turán Conjecture)**.** *If $A \subset \mathbb{N}$ is a set with positive upper density, meaning that*

$$\limsup_{N\to\infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N} > 0,$$

then A contains infinitely many k-term arithmetic progressions, for every $k \in \mathbb{N}$.

The $k = 3$ case was proven by Roth in 1953[11], with it being proved for the $k = 4$ case in 1969[12] and general k in 1975[13] by Szemerédi. Szemerédi's proof of the Erdös-Turán conjecture motivates the consideration of subsets of $\mathbb{N}$ and $\mathbb{Z}$ with strongly additive structure in order to understand arithmetic progressions within them.

1.2 SET® and the Cap Set Problem

In the early 1990's, the card game SET® became very popular. In a standard deck of SET, there are 81 cards with each card having four attributes with three possible values. The attribute/option pairs are as follows:

- Colour (red,green,purple)
- Shape (diamond, oval, squiggle)
- Number (1,2,3)
- Fill (solid, shaded, no-fill)

A *Set* is three cards such that the values across the cards are either the same or distinct for each attribute. The aim of the game is to find Sets as fast as possible. Initially, 12 cards are laid face up on the table. If there is no Set on the table, three more cards are laid face up. This is repeated until eventually someone finds a Set.
One natural question to ask is *what is the maximum number of cards possible such that no Set exists amongst*

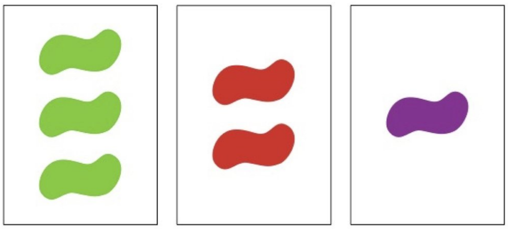

Figure 1: The above is a Set as the cards are (green,squiggle,3,solid), (red,squiggle,2,solid), (purple, squiggle, 1, solid). The colours, shape and fill are the same across the cards whilst the number is distinct.

them?

Each value for each attribute can be mapped to a value in $\mathbb{F}_3$. As there are 4 attributes, we can consider each card as an element of $\mathbb{F}_3^4$.

Theorem 1.1. *Three distinct cards $x, y, z \in \mathbb{F}_3^4$ form a Set if and only if $x + y + z = 0$.*

Proof. Let $\{x_i\}_{i\in I}$, $\{y_i\}_{i\in I}$, $\{z_i\}_{i\in I}$ be the components of x, y, z respectively. We consider the following cases:
Case 1: $x_i = y_i = z_i$

If $x_i = y_i = z_i$, then $x_i + y_i + z_i = 0$.

Case 2: $x_i = y_i$ and $y_i \neq z_i$ up to relabelling

If $x_i = y_i$ and $y_i \neq z_i$, then $x_i + y_i + z_i = 2x_i + z_i = z_i - x_i \neq 0$ as $x_i \neq z_i$.

Case 3: x_i, y_i, z_i distinct

If x_i, y_i, z_i are distinct then $x_i + y_i + z_i = 0$.

Clearly, $x_i + y_i + z_i = 0$ if and only if $x_i = y_i = z_i$ or x_i, y_i, z_i distinct, meaning that $x + y + z = 0$ if and only if $x_i = y_i = z_i$ or x_i, y_i, z_i distinct for each $i \in I$ as $x_i, y_i, z_i \in \mathbb{F}_3$. However, this is only the case if x, y, z form a Set and the theorem is proven. ☐

As such, this leads to the natural mathematical formulation, and generalisation, of our prior question as the following:

Problem (Cap Set Problem)**.** *Let $A \subseteq \mathbb{F}_3^n$ be such that A contains no lines, ie.*

$$x + y + z \neq 0 \quad \forall\, x, y, z \in A \text{ (distinct)}.$$

How does the maximum size of A grow as n grows?

The $n = 4$ case is exactly our question, leading to a set A satisfying the above conditions being called a *capset*.
As such, at most 20 cards can be on the table such that there no Set exists amongst them and thus 21 cards are needed to guarantee the existence of a Set.

n	2	3	4	5	6
$\lvert A\rvert$	4	9	20	45	112

Table 1: Known maximal capset sizes for different n

With regard to the Cap Set Problem for general n, one of the first bounds on $|A|$ was by Meshulam[10] in 1995 who showed that $|A| \leq 2 \cdot \frac{3^n}{n}$. In 2011, Bateman and Katz[1] improved this bound to $\mathcal{O}(3^n/n^{1+\varepsilon})$ for some $\varepsilon > 0$. It was not until 2016 that Croot, Lev and Pach[3] published a paper which used a revolutionary new idea called "the polynomial method" to solve the problem in the case where $A \subseteq \mathbb{Z}_4^n$. This was then extended and generalised by Ellenberg and Gijswijt[4] (independently) in 2017 to finally solve the Cap Set problem and attain an upper bound for $|A|$ of $\mathcal{O}(2.756^n)$.

In this report, a proof of the Cap Set problem using the polynomial method will be shown. Almost capsets will be introduced and ideas from the aforementioned proof will be used to generalise a theorem from Fish and Roy[6] about the cardinality of almost capsets.

2 Statement of Authorship

Theorem 4.1 and its proof is our own original work whilst all other results are from others and their sources have been referenced.

3 Proof of the Cap Set Problem

3.1 Introduction to Tensors

An integral part of the Cap Set problem is the polynomial method which analyses the properties of tensors.

Definition 1 (k-tensor)**.** *Let A be a finite set and $\mathbb{F}_q$ be the finite field with q elements. A k-tensor is a function $T : A^k \to \mathbb{F}_q$.*

A special type of k-tensor is a *diagonal* k-tensor.

Definition 2 (Diagonal tensor)**.** *A k-tensor $T : A^k \to \mathbb{F}_q$ is diagonal if*

$$T(x_1, x_2, \ldots, x_k) \neq 0 \implies x_1 = x_2 = \cdots = x_k.$$

A fundamental concept in the polynomial method is the notion of the slice-rank of a tensor.

Definition 3 (Slice-rank of a tensor)**.** *Let $F : A^k \to \mathbb{F}_q$ be a polynomial function in kn variables from $\mathbb{F}_q$ where $\dim(A) = n$. F is called a slice and has slice-rank 1 if $\exists g : A \to \mathbb{F}_q$, $h : A^{k-1} \to \mathbb{F}_q$ with g, h polynomials such* *that there exists* $j \in \{1, 2, \ldots, k\}$ *such that*

$$F(x_1, x_2, \ldots, x_k) = g(x_j)h(\tilde{x_j})$$

where $\tilde{x_j}$ *denotes the set* $\{x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k\}$*. For a general* F*, the slice-rank of a general function* F *is defined by*

$$\text{slice-rank}(F) = \min\left\{ s \middle| F = \sum_{i=1}^{s} F_i \textit{ where } \text{slice-rank}(F_i) = 1 \right\}.$$

Theorem 3.1. *Let* $T : A^k \to \mathbb{F}_q$ *be a* k*-tensor. Then* $\text{slice-rank}(T) \leq |A|$.

Proof. Every k-tensor can be written as

$$T(x_1, x_2, \ldots, x_k) = \sum_{x \in A} \delta_{x_1}(x) T(x, x_2, \ldots, x_k)$$

where $\delta_{x_1}(x) = 1$ if $x = x_1$ and 0 otherwise. However, each term on the RHS is a slice so $\text{slice-rank}(T) \leq |A|$. □

A more precise version of the above theorem is possible in the case that T is diagonal due to Tao[14].

Theorem 3.2. *If* $T : A^k \to \mathbb{F}_q$ *is a diagonal* k*-tensor with non-zero diagonal entries, then* $\text{slice-rank}(T) = |A|$

Theorem 3.2 is the crux of the polynomial method and allows us to solve general problems in additive and extremal combinatorics involving restrictions on the k-tuples of sets. In general, one aims to construct a tensor T such that when restricted to a set A^k, where A is the set with the largest cardinality satisfying the initial restriction, then T is diagonal. Computing an upper bound on $\text{slice-rank}(T)$ results in an upper bound on $|A|$ by Theorem 3.2[9]. We shall now apply the above method to solve the Cap Set problem.

3.2 Proof of the Cap Set Problem

Suppose that $A \subset \mathbb{F}_3^n$ is a capset. Define the function $F : \mathbb{F}_3^n \to \mathbb{F}_3$ as

$$\begin{aligned} F(x, y, z) &= \delta_{\mathbf{0}^n}(x + y + z) \\ &= \begin{cases} 1 & x + y + z = 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

However, if F is resricted to $A \times A \times A$, then by definition of a capset

$$F|_{A\times A\times A}(x, y, z) = \begin{cases} 1 & x = y = z \\ 0 & \text{otherwise} \end{cases}.$$

Delta functions are clumsy, however, and it'll much more prudent to convert F to a polynomial.

Let x, y, z have components $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n), (z_1, z_2, \dots, z_n)$ respectively. Now, $x + y + z = 0 \implies x_i + y_i + z_i = 0 \ \forall i \in \{1, 2, \dots, n\}$. Now as $x_i, y_i, z_i \in \mathbb{F}_3$, $x_i + y_i + z_i = 0, 1, 2$ so if $x = y = z$ then $x_i + y_i + z_i \neq 1, 2 \ \forall i \in \{1, 2, \dots, n\}$. Thus, $F(x, y, z)$ can be expressed as

$$\begin{aligned} F(x,y,z) &= \prod_{i=1}^{n} (x_i + y_i + z_i - 1)(2 - x_i - y_i - z_i) \\ &= \prod_{i=1}^{n} 1 - (x_i + y_i + z_i)^2 \end{aligned}$$

Note that $F(x, x, x) = 1 \neq 0 \ \forall n \in \mathbb{N}$ and so $F(x, y, z)$ is a diagonal 3-tensor for $x, y, z \in A$ and as such slice-rank$(F) = |A|$. We now prove the following critical lemma due to Croot, Lev, and Pach[**?**] (which was generalised by Ellenberg and Gisjwist[4]).

Lemma 3.3. *The slice-rank of the function defined by $F : A^3 \to \mathbb{F}_3$ as above is at most $3N$ where*

$$N := \sum_{\substack{a,b,c \in \mathbb{N}_0 \\ a+b+c=n \\ b+2c \leq 2n/3}} \frac{n!}{a!b!c!}$$

.

Proof. Clearly $F(x, y, z)$ is a polynomial of degree $2n$ in the $3n$ variables $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_n$. In fact, every monomial m in F (ignoring constant coefficients) is of the form

$$m = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} y_1^{\beta_1} \dots y_n^{\beta_n} z_1^{\gamma_1} \dots z_n^{\gamma_n}$$

where

$$\sum_{i=1}^{n} \alpha_i + \beta_i + \gamma_i \leq 2n$$

and

$$\alpha_i, \beta_i, \gamma_i \in \{0, 1, 2\} \ \ \forall i \in \{1, 2, \dots, n\}\,.$$

Let $d_x(m), d_y(m)$ and $d_z(m)$ be the degrees of the terms in the monomial m containing components of x,y and z respectively. That is to say that

$$d_x(m) = \sum_{i=1}^{n} \alpha_i, d_y(m) = \sum_{i=1}^{n} \beta_i, d_z(m) = \sum_{i=1}^{n} \gamma_i$$

so that

$$\deg(m) = d_x(m) + d_y(m) + d_z(m) \leq 2n.$$

However, by the Pigenhole Principle, this means that at least one of $d_x(m), d_y(m), d_z(m) \leq 2n/3$. Let M_x be the set of monomials in F with $d_x(m) \leq 2n/3$, M_y be the set of monomials in F for which $d_y(m) \leq 2n/3$, and let M_z be the set of monomials in F with $d_z(m) \leq 2n/3$. For simplicity, we force M_x, M_y, M_z to be disjoint by removing anything in $M_y \cap M_x$ from M_x and then removing anything in $M_z \cap M_y$ or $M_z \cap M_x$ from M_z. As a result, we have that

$$F(x,y,z) = \sum_{m \in M_x} m + \sum_{m \in M_y} m + \sum_{m \in M_z} m.$$

As such, one can express $F(x,y,z)$ as

$$F(x,y,z) = \sum_{\alpha,\beta,\gamma \in \mathbb{F}_3^n} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^n x_i^{\alpha_i}\right)\left(\prod_{i=1}^n y_i^{\beta_i}\right)\left(\prod_{i=1}^n z_i^{\gamma_i}\right)$$

where $c_{\alpha,\beta,\gamma}$ is some constant depending on α, β, γ. But,

$$\sum_{m \in M_x} m = \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \alpha_i \leq 2n/3}} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^n x_i^{\alpha_i}\right)\left(\prod_{i=1}^n y_i^{\beta_i}\right)\left(\prod_{i=1}^n z_i^{\gamma_i}\right)$$

and similarly for the summations over M_y and M_z. Thus, we have that

$$\begin{aligned} F(x,y,z) = &\sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \alpha_i \leq 2n/3}} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^n x_i^{\alpha_i}\right)\left(\prod_{i=1}^n y_i^{\beta_i}\right)\left(\prod_{i=1}^n z_i^{\gamma_i}\right) \\ &+ \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \beta_i \leq 2n/3}} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^n x_i^{\alpha_i}\right)\left(\prod_{i=1}^n y_i^{\beta_i}\right)\left(\prod_{i=1}^n z_i^{\gamma_i}\right) \\ &+ \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \gamma_i \leq 2n/3}} c_{\alpha,\beta,\gamma} \left(\prod_{i=1}^n x_i^{\alpha_i}\right)\left(\prod_{i=1}^n y_i^{\beta_i}\right)\left(\prod_{i=1}^n z_i^{\gamma_i}\right). \end{aligned}$$

But this is the same as saying

$$\begin{aligned} F(x,y,z) = &\sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \alpha_i \leq 2n/3}} \left(\prod_{i=1}^n x_i^{\alpha_i}\right) f_\alpha(y,z) \\ &+ \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \beta_i \leq 2n/3}} \left(\prod_{i=1}^n y_i^{\beta_i}\right) g_\beta(x,z) \\ &+ \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{F}_3^n \\ \sum_{i=1}^n \gamma_i \leq 2n/3}} \left(\prod_{i=1}^n z_i^{\gamma_i}\right) h_\gamma(x,y) \end{aligned}$$

for 2-tensors $f_\alpha(y,z), g_\beta(x,z), h_\gamma(x,y)$. Now, all terms on the RHS are slices. As such, we see that

$$\text{slice-rank}(F) \leq 3\left|\left\{(\alpha_1,\dots,\alpha_n)\in\{0,1,2\}^n \,|\sum_{i=1}^n \alpha_i \leq 2n/3\right\}\right|.$$

We now show that

$$\left|\left\{(\alpha_1,\dots,\alpha_n)\in\{0,1,2\}^n \,|\sum_{i=1}^n \alpha_i \leq 2n/3\right\}\right| = \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n\\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!} = N$$

from which the lemma is proven. Let the number of α_i which equal $0,1,2$ be a,b,c respectively. Since every α_i takes exactly one of these values we have that

$$a+b+c=n.$$

However,

$$\sum_{i=1}^n \alpha_i \leq 2n/3 \Leftrightarrow 0a+1b+2c\leq 2n/3 \Leftrightarrow b+2c\leq 2n/3$$

As such, we have that

$$\left|\left\{(\alpha_1,\dots,\alpha_n)\in\{0,1,2\}^n \,|\sum_{i=1}^n \alpha_i \leq 2n/3\right\}\right| = \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n\\ b+2c\leq 2n/3}} \left|\left\{(\alpha_1,\dots,\alpha_n)\in\{0,1,2\}^n \,\middle|\, \text{with } a \text{ 0's, } b \text{ 1's and } c \text{ 2's}\right\}\right|.$$

We wish to find the number of vectors $(\alpha_1,\dots,\alpha_n)$ which all have components in $\mathbb{F}_3$ which have a 0's, b 1's and c 2's for given values of a,b,c. Now, all possible vectors are permutations of the vector $(\underbrace{0,0,\dots,0}_{a\text{ 0's}},\underbrace{1,1,\dots,1}_{b\text{ 1's}},\underbrace{2,2,\dots,2}_{c\text{ 2's}})$ where $a+b+c=n$. As such, the number of distinct vectors (that is no two vectors having all identical components) is $\frac{n!}{a!b!c!}$ and so the above is equal to

$$= \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n\\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!}$$
$$= N$$

and the lemma is proven. □

All that is left is to evaluate N. The standard continuation from here is to use Stirling's approximation as well as probabilistic arguments (Cramer's theorem for large deviations and Shannon entropy) to show that

$$N \sim \theta^{n(1+o(1))} \qquad \text{where} \qquad \theta = \max_{\substack{a,b,c\geq 0\\ a+b+c=1\\ b+2c\leq 2/3}} e^{h(a,b,c)}$$

where

$$h(a,b,c) = a\ln\left(\frac{1}{a}\right) + b\ln\left(\frac{1}{b}\right) + c\ln\left(\frac{1}{c}\right)$$

is the Shannon entropy. A routine Lagrange Multiplier calculation reveals that the values for a, b, c corresponding to θ are

$$a = \frac{32}{3(15+\sqrt{33})}, b = \frac{4(\sqrt{33}-1)}{3(15+\sqrt{33})}, c = \frac{\left(\sqrt{33}-1\right)^2}{6\left(15+\sqrt{33}\right)}$$

so that

$$\theta = 1.01345 \implies N \approx 2.755^n e^{o(1)n} \implies |A| \leq \mathcal{O}(2.756)^n.$$

This proof is, however, quite convoluted and cumbersome and we provide a more elementary way to evaluate N below.

Lemma 3.4. *N as defined in Lemma 3.3 is at most c_3^n where $c_3 = \frac{3}{8}\sqrt[3]{207+33\sqrt{33}} \approx 2.756$ is a constant.*

Proof.

$$(1+x+x^2)^n = \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n}} \frac{n!}{a!b!c!}x^{b+2c} \text{ by the Multinomial Theorem}$$

which upon dividing both sides by $x^{2n/3}$ implies

$$\frac{(1+x+x^2)^n}{x^{2n/3}} = \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n}} \frac{n!}{a!b!c!}x^{b+2c-2n/3}.$$

Let $f(x) = \dfrac{(1+x+x^2)^n}{x^{2n/3}} = (x^{-2/3}+x^{1/3}+x^{4/3})^n$. Then,

$$\begin{aligned} f(x) &> \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n\\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!}x^{b+2c-2n/3} \text{ if } x>0 \\ &> \sum_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c=n\\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!} \text{ if } 0<x<1 \text{ as } b+2c-2n/3\leq 0. \end{aligned}$$

Therefore, finding the minimum of $f(x)$ for $0 < x < 1$ will allow us to find an upper bound for N and thus an upper bound on $|A|$. A routine calculation shows that $f(x)$ is minimised when $x = \frac{\sqrt{33}-1}{8}$. Thus,

$$f\left(\frac{\sqrt{33}-1}{8}\right) = \frac{3}{8}\sqrt[3]{207+33\sqrt{33}}$$
$$\implies N \geq \frac{3}{8}\sqrt[3]{207+33\sqrt{33}}$$

□

As such, we can combine Theorem 3.2, Lemma 3.3 and Lemma 3.4 to show that

$$|A| \leq 3(2.756)^n$$
$$\leq \mathcal{O}(2.756)^n$$

and the Cap Set problem is solved! In fact the above argument can be generalised to find the largest subset of $\mathbb{F}_q^n$ that has no non-trivial solutions to the equation $ax + by + cz = 0$, where $a, b, c \in (\mathbb{F}_q \setminus \{0\})^3$ such that $a + b + c = 0$ (as noted in [4]).To do this, the function $\delta_{\mathbf{0}^n}(x + y + z)$ is replaced by $\delta_{\mathbf{0}^n}(ax + by + cz)$.

4 Almost Capsets

4.1 Introduction to Almost Capsets

A natural multivariate generalisation of capsets are subsets of $\mathbb{F}_q^n$ that have no non-trivial solutions to the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$, where $a_1, a_2, \ldots, a_k \in (\mathbb{F}_q \setminus \{0\})^k$ such that $a_1 + a_2 + \cdots + a_k = 0$. One can also consider *almost capsets* where the condition "non non-trivial solutions" is replaced with "not too many non-trivial solutions". These were first introduced and studied by Fish and Roy[6] in the three variable case. In this section we obtain analogous upper bounds on the cardinality of almost cap sets in the multivariate case, which satisfy a much weaker structural rigidity than cap sets in $\mathbb{F}_q^n$ and build upon techniques developed by Tao[14] to analyse them.

Definition 4 (Almost Capset)**.** *An (ϵ, δ)-cap set, sometimes referred to as an almost capset, is a set $A \subset \mathbb{F}_q^n$ for $\epsilon, \delta > 0$ corresponding to a k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in (\mathbb{F}_q \setminus \{0\})^k$ with $a_1 + a_2 + \cdots + a_k = 0$, if there exists $A' \subset A$ with $|A'| \geq \delta A$ such that for every $x_1 \in A'$, the number of $(k-1)$-tuples $(x_2, x_3, \ldots, x_k) \in A^{k-1}$ satisfying $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$ is less than $|A|^\epsilon$.*

In a similar fashion, for an arbitrary $\epsilon > 0$, k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in (\mathbb{F}_q \setminus \{0\})^k$ with $a_1 + a_2 + \cdots + a_k = 0$, and set $A \subset \mathbb{F}_q^n$, we denote $A_{\mathbf{a}}^\epsilon$ by the following

$$A_{\mathrm{a}}^\epsilon = \{x_1 \in A \mid \exists \text{ at least } |A|^\epsilon \ (k-1)\text{-tuples } (x_2, x_3, \ldots, x_k) \in A^{k-1} \text{ with } a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0\}.$$

Theorem 4.1. *Suppose that $k, q \in \mathbb{N}_{>2}$. There exist $\epsilon > 0$ and $c_q < q$ such that for any $\delta > 0$, $A \subset \mathbb{F}_q^n$ with $|A| > c_q^n$ and $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in (\mathbb{F}_q \setminus \{0\})^k$, then for sufficiently large n*

$$|A_{\mathbf{a}}^{\epsilon}| \geq (1-\delta)|A|.$$

The above theorem was proven for the case $k = 3$ in Fish and Roy[6].

4.2 Proof of Theorem 4.1

Let $\delta, \epsilon > 0$ and $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in (\mathbb{F}_q \setminus \{0\})^k$ with $a_1 + a_2 + \cdots + a_k = 0$. Furthermore, assume that $A \subset \mathbb{F}_q^n$ is an (ϵ, δ)-cap set. Now, we introduce the function $F : A^k \to \mathbb{F}_q$ to be

$$\begin{aligned} F(x_1, x_2, \ldots, x_k) &= \delta_{\mathbf{0}^n}(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k) \\ &= \sum_{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in A^k} c_\alpha \prod_{i=1}^{k} \delta_{\alpha_i}(x_i) \end{aligned} \tag{1}$$

where

$$c_\alpha = \begin{cases} 1 & \text{if } a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

4.2.1 Proof of the lower bound on slice-rank(F)

It was assumed that $\exists A' \subset A$ with $|A'| \geq \delta|A|$ such that for every $x_1 \in A'$, the number of $(k-1)$-tuples $(x_2, x_3, \ldots, x_k) \in A^{k-1}$ with $a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0$ is smaller than $|A|^\epsilon$. From this we deduce that

$$\begin{aligned} \left|\left\{c_\alpha \neq 0 \mid \alpha \subset (A')^k\right\}\right| &\leq |A'|\,|A|^\epsilon \\ &\leq |A'|(\delta^{-1}|A'|)^\epsilon \text{ as } |A'| \geq \delta|A| \\ &\leq \delta^{-1}|A'|^{1+\epsilon} \end{aligned} \tag{2}$$

We introduce an important definition.

Definition 5 (Independent Set)**.** *A set $\mathcal{I} \subset \{1, \ldots, N\}$ is an independent set for a tensor T if for any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathcal{I}^d$ such that $c_\alpha \neq 0$ we have that $\alpha_1 = \ldots = \alpha_d$.*

As F only takes values 0 and 1, we can consider F to be the adjacency matrix for a k-uniform hypergraph with $|A'|$ vertices. We invoke the generalised Caro-Wei lower bound for the maximal cardinality of an independent set for this hypergraph.

Theorem 4.2 (Caro-Tuza [2])**.** *Let H be a k-uniform hypergraph for $k \geq 3$ with edges $E(H)$ and vertices $V(H)$. A set $I \subset V(H)$ is an independent set of H if for every $e \in E(H)$, we have that $e \not\subseteq I$. Then there exists $d_k > 0$*

such that the maximum size of an independent set in H, denoted by $\alpha(H)$, is given by

$$\alpha(H) \geq d_k \sum_{v \in V(H)} \frac{1}{(d(v)+1)^{\frac{1}{k-1}}}$$

where $d(v)$ is the degree of the vertex v.

By the above theorem, there exists $\mathcal{I} \subset A'$ an independent set satisfying

$$|\mathcal{I}| \geq C_1 \sum_{x \in A'} \frac{1}{(d_x+1)^{\frac{1}{k-1}}},$$

where $d_x = \left|\left\{c_\alpha \neq 0 \mid \alpha_1 = x, \alpha \subset (A')^k\right\}\right|$ and $C_1 > 0$ is a constant.

Lemma 4.3. *Assuming $\sum_{x \in A'} d_x = d_{sum}$ is constant, $\sum_{x \in A'} \frac{1}{(d_x+1)^{\frac{1}{k-1}}}$ is minimised whenever all the summations are equal - that is when all vertices have equal degree.*

Proof. Let $a_1, a_2, \ldots, a_{|A'|}$ be the elements of A' and $d_1, d_2, \ldots, d_{|A'|}$ be the degree of these vertices of the hypergraph respectively. Thus,

$$\begin{aligned}
\sum_{x \in A'} \frac{1}{(d(x)+1)^{1/k-1}} &= \sum_{i=1}^{|A'|} \frac{1}{(d_i+1)^{1/k-1}} \\
&\geq \frac{|A'|^2}{\sum_{i=1}^{|A'|}(d_i+1)^{1/k-1}} \text{ by the AM-HM inequality} \\
&\geq \frac{|A'|^2}{|A'|^{1-\frac{1}{k-1}}\left(\sum_{i=1}^{|A'|} d_i + 1\right)^{1/k-1}} \text{ by the Power Mean Inequality} \\
&= \frac{|A'|^{1+\frac{1}{k-1}}}{(d_{\text{sum}} + |A'|)^{1/k-1}}
\end{aligned}$$

with equality if and only if $(d_1+1) = (d_2+1) = \cdots = (d_{|A'|}+1) \implies d_1 = d_2 = \cdots = d_{|A'|} = d_{\text{sum}}/|A'| = d_{\text{avg}}$. ∎

Now, it follows from (2) that $\sum_{x \in A'} d_x \leq \delta^{-1} |A'|^{1+\varepsilon}$. Therefore, there exists a constant $C_2 = C(q,k,\delta) > 0$ such that

$$|\mathcal{I}| \geq C_2 |A'|^{1-\frac{\epsilon}{k-1}}$$

We invoke a theorem by Lovett.

Theorem 4.4 (Lovett[8], Theorem 1.7)**.** *There exists a positive constant $c = C(d,q)$ such that for any d-tensor T we have*

$$\text{slice-rank}(T) \geq c|\mathcal{I}|$$

for any independent set $\mathcal{I} \subset \{1, \ldots, N\}$.

As such, there exists a constant $C_3 = C(q,k,\delta) > 0$ such that

$$\text{slice-rank}(F) \geq C_3|A|^{1-\frac{\epsilon}{k-1}}. \tag{3}$$

4.2.2 Proof of the upper bound on slice-rank(F)

It should be seen that the nature of the proof of this part is inspired by that of Ellenberg and Gijswijt[4] with some additional techniques being used to prove the generalisation.

Lemma 4.5. *The slice-rank of the function defined by $F : A^k \to \mathbb{F}_q$ as in (1) is at most kN where*

$$N := \sum_{\substack{N_0,N_1,\ldots,N_{q-1}\in\mathbb{N}_0\\ \sum_{i=0}^{q-1} N_i = n\\ \sum_{i=1}^{q-1} iN_i \leq (q-1)n/k}} \frac{n!}{\prod_{i=0}^{q-1} N_i!}.$$

Proof. One can consider $F(x_1, x_2, \ldots, x_k)$ as a polynomial in the components of $x_1, x_2, \ldots, x_k$. Let the components of $x_1, x_2, \ldots, x_k$ be $(x_{11}, x_{12}, \ldots, x_{1n}), (x_{21}, x_{22}, \ldots, x_{2n}), \ldots, (x_{k1}, x_{k2}, \ldots, x_{kn})$ respectively where

$$x_{ij} \in \mathbb{F}_q \ \forall i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, n\}.$$

Clearly, $F(x_1, x_2, \ldots, x_k)$ can be expressed as

$$\begin{aligned} F(x_1, x_2, \ldots, x_k) &= \delta_{\mathbf{0}^n}(a_1x_1 + a_2x_2 + \cdots + kx_k) \\ &= \prod_{j=1}^{n} 1 - (a_1x_{1j} + a_2x_{2j} + \cdots + a_kx_{kj})^{q-1}. \end{aligned}$$

F is a polynomial of degree $n(q-1)$ in the kn variables $x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{k1}, x_{k2}, \ldots, x_{kn}$. In fact, every monomial m in F (ignoring constant coefficients) is of the form

$$x_{11}^{\beta_{11}} x_{12}^{\beta_{12}} \ldots x_{1n}^{\beta_{1n}} x_{21}^{\beta_{21}} x_{22}^{\beta_{22}} \ldots x_{2n}^{\beta_{2n}} \ldots x_{k1}^{\beta_{k1}} x_{k2}^{\beta_{k2}} \ldots x_{kn}^{\beta_{kn}}$$

where $\sum_{i=1}^{k}\sum_{j=1}^{n} \beta_{ij} \leq n(q-1)$.
Let $d_{x_i}(m)$ be the degrees of the terms in the monomial m containing components of x_i for $i \in \{1, 2, \ldots, k\}$. That is to say that

$$d_{x_i}(m) = \sum_{j=1}^{n} \beta_{ij}.$$

However, by the Pigeonhole Principle, this means that at least one of $d_{x_1}(m), d_{x_2}(m), \ldots, d_{x_k}(m) \leq n(q-1)/k$. Let M_i be the set of monomials in F with $d_{x_i}(m) \leq n(q-1)/k$. For simplicity, we force $M_1, M_2, \ldots, M_k$ to be disjoint by removing anything in $M_2 \cap M_1$ from M_1, and then removing anything in $M_3 \cap M_1$ or $M_3 \cap M_2$ from M_3 and so on.

It should be clear that $F(x_1, x_2, \ldots, x_k)$ can be expressed as

$$F(x_1, x_2, \ldots, x_k) = \sum_{\beta_1, \beta_2, \ldots, \beta_k \in [0, q-1]^n} c_{\beta_1, \beta_2, \ldots, \beta_k} \prod_{i=1}^{k} \prod_{j=1}^{n} x_{ij}^{\beta_{ij}}$$

where $c_{\beta_1, \beta_2, \ldots, \beta_k}$ is some constant depending on $\beta_1, \beta_2, \ldots, \beta_k$. But,

$$\sum_{m \in M_i} m = \sum_{\substack{\beta_1, \beta_2, \ldots, \beta_k \in [0, q-1]^n \\ \sum_{j=1}^{n} \beta_{ij} \leq n(q-1)/k}} c_{\beta_1, \beta_2, \ldots, \beta_k} \prod_{l=1}^{k} \prod_{j=1}^{n} x_{lj}^{\beta_{lj}}.$$

As such, we have that

$$\begin{aligned} F(x_1, x_2, \ldots, x_k) &= \sum_{i=1}^{k} \sum_{m \in M_i} m \\ &= \sum_{i=1}^{k} \sum_{\substack{\beta_1, \beta_2, \ldots, \beta_k \in [0, q-1]^n \\ \sum_{j=1}^{n} \beta_{ij} \leq n(q-1)/k}} c_{\beta_1, \beta_2, \ldots, \beta_k} \prod_{l=1}^{k} \prod_{j=1}^{n} x_{lj}^{\beta_{lj}} \\ &= \sum_{i=1}^{k} \sum_{\substack{\beta_i \in [0, q-1]^n \\ \sum_{j=1}^{n} \beta_{ij} \leq n(q-1)/k}} \left(\prod_{l=1}^{n} x_{il}^{\beta_{il}} \right) f_{\beta_i}(\overline{x_i}) \end{aligned}$$

where f_{β_i} are $(k-1)$-tensors and $\overline{x_i}$ denotes the set $\{x_1, x_2, \ldots, x_k\} \setminus \{x_i\}$.

All terms on the RHS have slice rank 1. As such, we see that

$$\text{slice-rank}(F) \leq k \left| \left\{ (\gamma_1, \gamma_2, \ldots, \gamma_n) \in [0, q-1] | \sum_{i=1}^{n} \gamma_i \leq n(q-1)/k \right\} \right|$$

where $\beta_{11}, \beta_{12}, \ldots, \beta_{1n}$ have been relabeled as $\gamma_1, \gamma_2, \ldots, \gamma_n$ for simplicity. Let the number of γ_i which equal to $0, 1, \ldots, q-1$ be $N_0, N_1, \ldots, N_{q-1}$ respectively. Since γ_i takes exactly one of these values, we have that $N_0 + N_1 + \cdots + N_{q-1} = n$. However,

$$\sum_{i=1}^{n} \gamma_i \leq (q-1)n/k \Leftrightarrow \sum_{i=0}^{q-1} i N_i \leq (q-1)n/k \Leftrightarrow \sum_{i=1}^{q-1} i N_i \leq (q-1)n/k.$$

As such, we have that

$$\begin{aligned}&\left|\left\{(\gamma_1,\dots,\gamma_n)\in[0,q-1]|\sum_{i=1}^{n}\gamma_i\leq(q-1)n/k\right\}\right|\\ =&\sum_{\substack{N_0,\dots,N_{q-1}\in\mathbb{N}_0\\ \sum_{i=0}^{q-1}N_i=n\\ \sum_{i=1}^{q-1}iN_i\leq(q-1)n/k}}\left|\left\{(\gamma_1,\dots,\gamma_n)\in[0,q-1]\right|\text{ with }N_i\ i\text{'s for }i\in\mathbb{F}_q\right\}\right|.\end{aligned}$$

We wish to find the number of vectors $(\gamma_1,\dots,\gamma_n)$ which all have components in $\mathbb{F}_q$ which have N_0 0's, N_1 1's, ..., N_{q-1} $(q-1)$'s for given values of $N_0,N_1,\dots,N_{q-1}$. Now, all possible vectors are permutations of the vector $(\underbrace{0,0,\dots,0}_{N_0\text{ 0's}},\underbrace{1,1,\dots,1}_{N_1\text{ 1's}},\dots,\underbrace{q-1,q-1,\dots,q-1}_{N_{q-1}\ (q-1)\text{'s}})$ where $N_0+N_1+\cdots+N_{q-1}=n$. As such, the number of distinct vectors (that is no two vectors having all identical components) is $\frac{n!}{\prod_{i=0}^{q-1}N_i!}$ and so the above is equal to

$$=\sum_{\substack{N_0,N_1,\dots,N_{q-1}\in\mathbb{N}_0\\ \sum_{i=0}^{q-1}N_i=n\\ \sum_{i=1}^{q-1}iN_i\leq(q-1)n/k}}\frac{n!}{\prod_{i=0}^{q-1}N_i!}:=N$$

so that

$$\text{slice-rank}(F)\leq kN.$$

and the proof of the lemma is complete. □

Lemma 4.6. *For $k,q\in\mathbb{N}_{>2}$ and sufficiently large n, $kN<b_q^n$ where $b_q<q$.*

Proof. As seen in Appendix 1. □

It should be noted that the $k=3$ case of Lemma 4.6 was proven by Ellenberg and Gijswijt[4].

4.2.3 Tying it all together

Combining (3) and Lemma 4.5 and Lemma 4.6, one has that

$$C_3|A|^{1-\frac{\epsilon}{k-1}}\leq\text{slice-rank}(F)<b_q^n \tag{4}$$

for some $b_q<q$. Finally, we choose $\epsilon>0$ to satisfy

$$\begin{aligned}&b_q^{\frac{1}{1-\frac{\epsilon}{k-1}}}<q\\ &\implies\epsilon<(k-1)\left(1-\frac{\ln(b_q)}{\ln(q)}\right)\end{aligned}$$

and c_q such that

$$b_q^{\frac{1}{1-\frac{\epsilon}{k-1}}} < c_q < q.$$

Thus, Theorem 4.1 holds true for these choices of ϵ and c_q.

5 Discussion and Conclusion

In this report, we gave a background for the Cap Set problems and provided an elegant proof of such (based on that of Ellenberg and Gijswijt[4] and Tao[14]). Almost capsets were introduced and a theorem from Fish and Roy[6] regarding their cardinality was generalised and improved in the case of more than three variables.

One possible future direction is the incorporation of a generalisation of Green's regularity type lemma[7] into the proof of Theorem 4.1. A corollary of the generalisation for the case of 3 variables is

Theorem 5.1. *If $A \subseteq \mathbb{F}_q^n$ where $q = p^r$ for some integer r, the number of $(x, y, z) \in A^3$ satisfying $ax+by+cz = 0$ is at least $\left(\frac{|A|}{3q^n}\right)^{1+\frac{1}{c_p}} q^{2n}$ where c_p is defined as*

$$c_p = 1 - \frac{1}{p} \ln \left(\min_{0<x<1} x^{-(p-1)/3} \left(x^0 + x^1 + \cdots + x^{p-1}\right) \right).$$

Generalising Theorem 5.1 for multiple variables would allow for the tightening of ϵ and c_q in Theorem 4.1 and the upper and lower bounds for slice-rank(F) where F is defined in (1).

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7 Appendix 1: Proof of Lemma 4.3

By the Multinomial Theorem,

$$\begin{aligned}
(x^0 + x^1 + \cdots + x^{q-1})^n &= \sum_{\substack{N_0, N_1, \ldots, N_{q-1} \in \mathbb{N}_0 \\ \sum_{i=0}^{q-1} N_i = n}} \frac{n!}{\prod_{i=0}^{q-1} N_i!} x^{\sum_{i=1}^{q-1} iN_i} \\
\implies \frac{(x^0 + x^1 + \ldots + x^{q-1})^n}{x^{(q-1)n/k}} &= \sum_{\substack{N_0, N_1, \ldots, N_{q-1} \in \mathbb{N}_0 \\ \sum_{i=0}^{q-1} N_i = n}} \frac{n!}{\prod_{i=0}^{q-1} N_i!} x^{\left(\sum_{i=1}^{q-1} iN_i\right) - q(n-1)/k}. \\
\text{Let } f(x) &= \frac{(x^0 + x^1 + \ldots + x^{q-1})^n}{x^{(q-1)n/k}}. \text{ Then,} \\
f(x) &> \sum_{\substack{N_0, N_1, \ldots, N_{q-1} \in \mathbb{N}_0 \\ \sum_{i=0}^{q-1} N_i = n \\ \sum_{i=1}^{q-1} iN_i \le (q-1)n/k}} \frac{n!}{\prod_{i=0}^{q-1} N_i!} x^{\left(\sum_{i=1}^{q-1} iN_i\right) - q(n-1)/k} \quad \text{if } x > 0 \\
&> \sum_{\substack{N_0, N_1, \ldots, N_{q-1} \in \mathbb{N}_0 \\ \sum_{i=0}^{q-1} N_i = n \\ \sum_{i=1}^{q-1} iN_i \le (q-1)n/k}} \frac{n!}{\prod_{i=0}^{q-1} N_i!} \quad \text{if } 0 < x < 1 \text{ as } \left(\sum_{i=1}^{q-1} iN_i\right) - q(n-1)/k \le 0. \\
&= N
\end{aligned}$$

Therefore, finding the minimum of $f(x)$ for $0 < x < 1$ will allow us to find an upper bound for N and thus an upper bound on slice-rank(F). The x-value which minimises $g(x) = f(x)^{1/n} = \sum_{i=0}^{q-1} x^{(ki-(q-1))/k}$ on $0 < x < 1$ will also minimise $f(x)$. But,

$$\begin{aligned}
g(x) &= \sum_{i=0}^{q-1} x^{(ki-(q-1))/k} \\
&= \frac{(x^q - 1)\, x^{(1-q)/k}}{x - 1}
\end{aligned}$$

so that

$$g'(x) = \frac{x^{-(k+q-1)/k}\left(q(x-1)\left((k-1)x^q + 1\right) - \left((k-1)x + 1\right)(x^q - 1)\right)}{k(x-1)^2}.$$

Define $h(x) := g(x) - q$. Then,

$$\lim_{x \to 0^+} h(x) = +\infty \text{ and } \lim_{x \to 1^-} h(x) = 0^-.$$

However, $h(x)$ is clearly continuous on $(0, 1)$ and so there must $\epsilon_2 > 0$ and $\epsilon_3 > 0$ such that $h(x) > 0$ for $0 < x \le \epsilon_2$ and $h(x) < 0$ for $1 > x \ge \epsilon_3$. As such, by the Intermediate Value Theorem there exists a root of $h(x)$ in the interval (ϵ_2, ϵ_3) and by extension, a root of $h(x)$ for $x \in (0, 1)$.

We now prove that this root is unique. Let the x-value of this root be ζ and let's assume that there exists a root ζ_2 of $h(x)$ in the interval $(\zeta, 1)$. By Rolle's theorem there must exist $x \in (\zeta_2, 1)$ where $h'(x) = 0$ (as $h(1) = 0$). It can be shown, through some algebraic manipulation, that $h'(x)$ is strictly increasing for $x > 0$ and as such the equation $h'(x) = 0$ has only 1 solution for $x > 0$. However, by Rolle's theorem one has that $h'(x) = 0$ for some $x \in (\zeta, \zeta_2)$ as well as for some $x \in (\zeta_2, 1)$. This implies that $h'(x) = 0$ has two solutions for $x > 0$ which is a contradiction. Thus, there is a unique root of $h(x) = 0$ for $0 < x < 1$.

Furthermore, by Rolle's theorem, there exists an extrema, x_{ext} of $h(x)$ in the interval $(\zeta, 1)$. However, as $h'(x)$ is strictly increasing for $x > 0$, the only extrema of $h(x)$ occurs at $x = x_{\text{ext}}$. Moreover, as $h'(x)$ is strictly increasing, we have that $h'(x) < 0$ for $0 < x < x_{\text{ext}}$ and $h'(x) > 0$ for $x > x_{\text{ext}}$. As such, the extrema at $x = x_{\text{ext}}$ is a minimum. As $\zeta \neq x_{\text{ext}}$, however, one has that $h(x_{\text{ext}}) < 0$ which implies that $g(x_{\text{ext}}) < q$.

Thus as $f(x) = g(x)^n$, and as $\min_{0<x<1}(g(x)) = g_{\text{ext}} < q$, then $\min_{0<x<1}(f(x)) = g_{\text{ext}}^n < q^n$. However, we have that $N < \min_{0<x<1}(f(x)) = g_{\text{ext}}^n$ and so we have that

$$\begin{aligned}\text{slice-rank}(F) &= kN \\ &< kg_{\text{ext}}^n \\ &= (k^{1/n} g_{\text{ext}})^n \\ &< b_q^n\end{aligned}$$

for some $b_q < q$ if n is big enough as $g_{\text{ext}} < q$.

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