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C^*-Algebras of Discrete Groups

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Abstract

Group C^*-algebras have a variety of applications in many different areas of mathematics, such as representation theory, topology and differential geometry. In this report, we will offer an introduction to the theory of group C^*-algebras by considering the case of discrete groups. In particular, we will be partially focusing on a representation-theoretic approach, as these algebras encode all of the information regarding the unitary representations of the groups. We then give a brief overview of amenability, including some of its various characterisations; for instance, Følner's characterisation, Reiter's characterisation, and some characterisations that follow from these.

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Statement of Authorship

No new results are included in this report, and indeed the theory presented is quite standard in the literature. In particular, Putnam's excellent notes [18] are central to the first few chapters on the initial theory of C^*-algebras, with *Kazhdan's Property (T)* [1] guiding most of the remaining chapters on amenability. Various other books and papers have also been cited to supplement the theory presented by those sources.

Instead of providing any new results, we aim to offer our own distinct, example-driven treatment of the topics from the aforementioned sources. We attempt to give more detailed proofs given where space permits, as well as an explanation of the intuition behind the concepts involved.

1 Introduction

The study of group C^*-algebras originated from the work of Murray and von Neumann in the late 1930s [12][13][16][14]. Soon after these influential series of papers on rings of operators, which laid the groundwork for the study of C^*-algebras, Gelfand and Rykov extended the theory to locally compact groups [6]. We typically continue to define group C^*-algebras on locally compact groups – a general collection of groups that contains discrete groups. Much of the basic theory is, however, very similar between these two cases, with the discrete case simply admitting a more approachable treatment. As a result, understanding the theory of discrete group C^*-algebras is often quite illuminating with respect to the locally compact case, and hence well worth studying.

Very relevant to the study of group C^*-algebras is the topic of amenability. Amenability was originally proposed by von Neumann in 1929 to explain the Banach-Tarski paradox [15]; since then, it has found applications in many areas of mathematics, such as ergodic theory, group theory and operator algebras. What makes the amenability so rich as a concept is how it admits a multitude of equivalent characterisations, that often appear far removed from one another at first glance. One such characterisation is that the reduced and universal group C^*-algebras of amenable groups can, in some sense, be seen as equal – quite a remarkable result indeed!

We will begin the report by introducing the concept of unitary representations, their implications for discrete groups and discrete group algebras, and how C^*-algebras can be defined on these group algebras. We will then move towards a discussion of amenability and a selection of its various characterisations.

2 Group Representations

Definition 2.1 (Topological Group). *A* topological group $(G, \cdot, \tau)$ *is a group* $(G, \cdot)$ *endowed with some topology* τ*, under which the group binary operation* $\cdot : (g, h) \mapsto gh$ *and inversion map* $g \mapsto g^{-1}$ *are both continuous, for all* $g, h \in G$*.*

Perhaps the simplest topology we can think about on a group G is the *discrete topology*. Under this topology, every subset of G is open, and hence it naturally endows G with a topological group structure. A group with the discrete topology is referred to as a *discrete group*. Discrete groups have many nice topological properties. In particular, a discrete group is automatically a locally compact Hausdorff space; furthermore, every function on a discrete group is continuous. A consequence of this is that we can easily study continuous representations.

Definition 2.2 (Unitary Representation). *A* representation *of a discrete group* G *on a Hilbert space* H*, sometimes written* (u, H)*, is a group homomorphism* $u : G \to \mathcal{B}(H)$*. A representation is said to be* unitary *if it maps into* $\mathcal{U}(H) \subset \mathcal{B}(H)$*. We often let* u_g *denote the map* $u(g)$*, the image of* g *under* u*. Two unitary representations* (u, H_u) *and* (v, H_v) *are said to be* unitarily equivalent *if there exists another unitary operator* $w : H_u \to H_v$ *such that* $w \circ u_g \circ w^{-1} = v_g$*, for all* $g \in G$*. We denote this by* $u \sim_w v$*, or simply* $u \sim v$*.*

In the more general locally compact case, we tend to require that our representations be strongly continuous; however, this is of course automatically satisfied if G is discrete. Moreover, if u is a unitary representation, then it follows from the properties of group homomorphisms that $u_{g^{-1}} = u_g^{-1} = u_g^*$, for all $g \in G$.

Due to a theorem of Gelfand and Rykov, a locally compact group is completely determined by its unitary representations [6][19]. As a result, we can restrict our study to strictly unitary representations of the underlying group.

Our notion of unitary representations is still somewhat abstract; as a result, it may be helpful to see some examples. Of course, we can always define the *trivial representation*, $1_G : g \mapsto \mathrm{id}_H$, where id_H is the identity operator in $\mathcal{U}(H)$. A more interesting and useful representation is the left regular representation. We define this on the Hilbert space $\ell^2(G)$, from definition A.1.

Theorem-Definition 2.3 (Left Regular Representation)**.** *Suppose we define an action of a discrete group G on $\ell^2(G)$ by $(g \cdot \xi)(x) := \xi(g^{-1}x)$, for all $\xi \in \ell^2(G)$ and all $g, x \in G$. Then the* left regular representation *of G is the unitary representation $\lambda_G : G \to \mathcal{U}(\ell^2(G)) : g \mapsto \lambda_g$, where $\lambda_g : \ell^2(G) \to \ell^2(G)$ is the the unitary convolution map on $\ell^2(G)$, defined by $\lambda_g : \xi \mapsto g \cdot \xi$.*

Proof. Consider the canonical basis $\{\delta_a\}_{a \in G}$ for $\ell^2(G)$; we note that $\langle \lambda_g(\delta_a), \lambda_g(\delta_b) \rangle = \langle \delta_a, \delta_b \rangle$. This holds for all δ_a, δ_b in our basis, so by linearity and density it must hold for all elements of $\ell^2(G)$, and hence λ_g is an isometry. It is also invertible; in particular, $\lambda_g^{-1} : \xi(g^{-1}x) \mapsto \xi(x)$, whence it follows that $\lambda_g^{-1} = \lambda_{g^{-1}}$. Hence λ_g is unitary, as it is invertible and isometric. Because λ_G is a group homomorphism, the result follows. This completes the proof. ∎

Something we should observe is that we have $\lambda_g \circ \delta_h = \delta_{gh}$, for δ_h given in definition A.1.

Example 2.4. A simple example that we can use to illustrate the concepts we have introduced is that of the cyclic groups, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. The left regular representation sends an integer $p \in \mathbb{Z}_n$ to the unitary map $\lambda_p : \xi(q + n\mathbb{Z}) \mapsto \xi(q - p + n\mathbb{Z})$. In other words, it behaves as a translation!

Definition 2.5 (Irreducible Representation)**.** *Let (u, H) be a unitary representation of a discrete group G. A closed subspace $N \subseteq H$ is said to be an* invariant subspace *of u if $u_g(N) \subseteq N$, for all $g \in G$. We say that the representation is* irreducible *if its only invariant subspaces are the trivially invariant subspaces $\{0_H\}$ and H; otherwise, it is said to be* reducible*. We generally let $\widehat{G}$ denote the set of unitary equivalence classes of irreducible representations.*

Note that we require invariant subspaces be closed, ensuring that they are Hilbert subspaces. This allows us to decompose unitary representations in terms of irreducible representations.

Furthermore, it is worth thinking for a moment about $\widehat{G}$. Its definition is actually quite subtle, as the collection of all unitary representations of G does not necessarily form a set! However, it turns out that the equivalence classes of irreducible representations *do* form a set, so the definition is fine as long as we restrict our attention to these.

Definition 2.6. *Let $\{(u_i, H_i)\}_{i \in I}$ be a collection of unitary representations of a discrete group G. Their* direct sum*, denoted by $(\oplus_{i \in I} u_i, \oplus_{i \in I} H_i)$, is the unitary representation defined by $\left(\bigoplus_{i \in I} u_i\right)_g = \bigoplus_{i \in I} (u_i)_g$, for all $g \in G$.*

Lemma 2.7. *Let G be a discrete group, and $N \subset H$ an invariant subspace of the unitary representation (u, H). Then its orthogonal complement, $N^\perp$, is also invariant. Moreover, u is unitarily equivalent to the orthogonal direct sum ${}_N|u \oplus {}_{N^\perp}|u$, the direct sum of the restrictions of the form ${}_K|u : G \xrightarrow{u} \mathcal{U}(H) \xrightarrow{|_K} \mathcal{U}(K)$ for the Hilbert subspace $K \in \{N, N^\perp\}$.*

Proof. Suppose we let $\xi_1 \in N^\perp$. We would first like to show that $u_g(\xi_1) \in N^\perp$. We will proceed by showing that $u_g(\xi_1)$ is perpendicular to every $\xi_2 \in N$. But to see this, we simply recall that $u_g^* = u_g^{-1}$, whence $\langle u_g(\xi_1), \xi_2\rangle = \langle \xi_1, u_g^*(\xi_2)\rangle = \langle \xi_1, u_g^{-1}(\xi_2)\rangle = 0$ for all $\xi_2 \in N$ and $g \in G$, as the invariance of N ensures that $u_g^{-1}(\xi_2) \in N$. Thus $N^\perp$ is invariant under u. All that remains is to show that u is unitarily equivalent to ${}_N|u \oplus {}_{N^\perp}|u$. That is, we wish to find a unitary representation $w : H \to N \oplus N^\perp$ such that $w \circ u_g \circ w^{-1} = u_g|_N \oplus u_g|_{N^\perp}$, for all $g \in G$. But this is as simple as setting $w(\xi) := \text{proj}_N(\xi) \oplus \text{proj}_{N^\perp}(\xi)$ and $w^{-1}(\xi_1 \oplus \xi_2) := \xi_1 + \xi_2$ where proj_K is the projection onto the subspace K. Note that by a corollary of the Hilbert space projection theorem, the closure of N ensures that $H \cong N \oplus N^\perp$, which is required in order for w to be bijective. Thus the result follows. This completes the proof. ■

The above lemma can be applied in the following simplified restatement of the classical Peter-Weyl theorem, which essentially states that the unitary representations of G are completely characterised by the irreducible representations alone [4].

Theorem 2.8 (Peter-Weyl)**.** *Let (u, H) be a unitary representation of a discrete group G. Then it is unitarily equivalent to an orthogonal direct sum of irreducible representations of G.*

Proof. This proof essentially follows from transfinite induction on the result of lemma 2.7. ■

Proposition-Definition 2.9 (Universal Representation)**.** *Let G be a discrete group, and suppose we choose a representative u for each $[u] \in \widehat{G}$. The direct sum of all such unitarily distinct representations is called the* universal representation *of G, and denoted (U, H_U). For any unitary representation u of G, we have that $\|u(g)\| \leq \|U(g)\|$, for all $g \in G$.*

Proof. By definition of the operator norm, the norm of an element $g \in G$ under U is given by

$$\|U(g)\| = \sup\{\|u(g)\| : [u] \in \widehat{G}\}.$$

That this holds for any arbitrary unitary representation, rather than specifically the irreducible representations, is just an application of the Peter-Weyl theorem. This completes the proof. ■

3 Group Algebras

Definition 3.1 (Complex Group Algebra)**.** *The* (complex) group algebra *of a group G, denoted $\mathbb{C}G$, is the set of all complex linear combinations of finitely many elements of G. That is,*

$$\mathbb{C}G := \left\{ \sum_{g\in G} a_g g : a_g \in \mathbb{C}, \text{ where } a_g = 0 \text{ for all but finitely many } g \in G \right\}. \tag{3.1.1}$$

We can make $\mathbb{C}G$ into a complex $*$-algebra by defining

(3.1.2). scalar multiplication: $c\sum_{g\in G} a_g g := \sum_{g\in G}(ca_g)g$, for $c \in \mathbb{F}$;

(3.1.3). vector addition: $\sum_{g\in G} a_g g + \sum_{g\in G} b_g g := \sum_{g\in G}(a_g + b_g)g$;

(3.1.4). vector multiplication: $\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) := \sum_{g,h\in G}(a_g b_h)gh$;

(3.1.5). $*$-operation: $\left(\sum_{g\in G} a_g g\right)^* := \sum_{g\in G} \overline{a}_g g^{-1}$.

It's easy to see that $\mathbb{C}G$ is Abelian if and only if G is; furthermore, denoting $1g = g$ for all $g \in G$, we may essentially say that $G \subset \mathbb{C}G$. We know from Gelfand and Rykov that a discrete group is completely determined by its unitary representations [6][19]. Happily, there is a bijective correspondence between these unitary representations and the unital $*$-representations of $\mathbb{C}G$.

Theorem 3.2 (Universal Property)**.** *Let $u : G \to \mathcal{U}(H)$ be a unitary representation of a discrete group G on the Hilbert space H. There exists a bijection from u to the extension π_u: a unital $*$-representation of $\mathbb{C}G$ on H given by*

$$\pi_u : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g u_g. \tag{3.2.1}$$

The inverse is given by restricting the unital $*$-representation $\pi : \mathbb{C}G \to \mathcal{B}(H)$ to G. Finally, this bijection preserves irreducibility; that is, u is irreducible if and only if π_u is irreducible.

The main implication of this theorem is that we only really need to care about the unital $*$-representations of $\mathbb{C}G$; indeed, these representations happen to completely characterise the group algebra, just as G is fully characterised by its unitary representations. Finally, it is worth considering what happens to the left regular representation in $\mathbb{C}G$.

Proposition 3.3. *The left regular representation π_{λ_G} of $\mathbb{C}G$ is injective.*

4 Discrete Group C^*-Algebras

We are now ready to define a C^*-algebra for discrete groups. We begin by introducing the reduced group C^*-algebra. Some partial intuition for this definition comes from the Gelfand-Naimark representation theorem (see A.20), which tells us that every C^*-algebra is embedded in $\mathcal{B}(H)$, for some Hilbert space H. Thus, to build a C^*-algebra from $\mathbb{C}G$, it makes sense to try and map it into $\mathcal{B}(H)$. The simplest way of doing this is to use π_{λ_G}, which we already know is injective by proposition 3.3. It turns out that this does give us a C^*-algebra!

Proposition-Definition 4.1 (Reduced Group C^*-Algebra). *Let G be a discrete group. We define the* reduced norm *on $\mathbb{C}G$ by*

$$\|a\|_r = \|\pi_{\lambda_G}(a)\|, \tag{4.1.1}$$

for each $a \in \mathbb{C}G$, where π_{λ_G} denotes the left regular representation. The completion of $\mathbb{C}G$ with respect to this norm is a referred to as the reduced group C^*-algebra *of G, and denoted $C_r^*(G)$.*

Note that we can also define a C^*-algebra $C_\pi^*(G) := \overline{\pi(\mathbb{C}G)}$ for any other injective, unital $*$-representation π, not just π_{λ_G}. Later, however, we will see why the left regular representation specifically is important. Finally, note that if G is finite, we can simply say $C_\pi^*(G) \cong \pi(\mathbb{C}G)$.

We now introduce one more kind of completion of $\mathbb{C}G$, that also turns it into a C^*-algebra.

Proposition-Definition 4.2 (Universal Group C^*-Algebra). *Let G be a discrete group. We define the* universal norm *on $\mathbb{C}G$ by*

$$\|a\|_U := \sup\{\|\pi_u(a)\| : u \text{ is a unitary representation of } G\}, \tag{4.2.1}$$

for each $a \in \mathbb{C}G$, is a well-defined norm on $\mathbb{C}G$. The completion of $\mathbb{C}G$ with respect to this norm is a referred to as the universal group C^*-algebra *of G, and denoted $C^*(G)$. In particular, $C^*(G)$ is a C^*-algebra containing $\mathbb{C}G$ as a dense $*$-subalgebra.*

Note that we have a canonical surjective $*$-homomorphism from $C^*(G)$ into $C_\pi^*(G)$, which just extends the map $\mathbb{C}G \xrightarrow{\pi} C_\pi^*(G)$; however, this will *not necessarily* give an inclusion of $C_\pi^*(G)$ in $C^*(G)$! Before we take (4.2.1) for granted, it is worth pausing for a moment to consider whether or not the supremum exists, or is well-defined. As we noted in our remark to definition 2.5, the set of unitary representations do not form a set. However, because we are taking an operator norm, we can essentially map them down to a manageable subset of $\mathbb{R}$. That the supremum actually exists is maybe a bit more subtle; we will need to introduce one more completion of $\mathbb{C}G$.

Definition 4.3. *Let G be a discrete group. We define the ℓ^1-*norm *on $\mathbb{C}G$ by*

$$\left\|\sum_{g\in G} a_g g\right\|_1 := \sum_{g\in G} |a_g|. \tag{4.3.1}$$

This definition can actually be made consistent with definition A.1 by associating with each element of $\mathbb{C}G$ a functional $a \in \ell^1(G)$ of the form $a(g) = a_g$.

In light of this definition, we actually find that the completion of $\mathbb{C}G$ with respect to this norm is isomorphic to $\ell^1(G)$! Furthermore, we can even extend our bijective correspondence between the unitary representations of G and the unital $*$-representations of $\mathbb{C}G$ [18].

Theorem 4.4. *Let G be a discrete group. There are bijective correspondences between the unitary representations of G, the unital $*$-representations of $\mathbb{C}G$ and the unital $*$-representations of $\ell^1(G)$. Furthermore, these bijective correspondences preserve irreducibility. Finally, every unital $*$-representation of $\ell^1(G)$ is a contraction.*

Essentially, because each unitary representation u of G extends bijectively to a contraction on $\ell^1(G)$, we can say that $\|\pi_u(a)\| \leq \|a\|_1$, for each corresponding unital $*$-representation π_u of $\mathbb{C}G$. Thus $\|\pi_u(a)\|$ is bounded, and hence the universal norm (4.2.1) is, happily, well-defined!

Proposition 4.5. *Let G be a discrete group. Then*

$$\|a\|_U = \sup\{\|\pi_u(a)\| : u \text{ is an irreducible representation of } G\} = \|\pi_U(a)\|, \tag{4.5.1}$$

for each $a \in \mathbb{C}G$, where π_U is the $*$-representation of $\mathbb{C}G$ corresponding to the universal representation of G.

Because the group is described completely by not just its unitary representations, but its *irreducible* unitary representations, it should not be too surprising that we can write the universal norm in this way. Indeed, it follows as a direct corollary of proposition 2.9.

Finally, we have the following extension to theorem 4.4; a wonderful result indeed!

Proposition 4.6. *Let G be a discrete group. Then there is a bijective correspondence between the unital $*$-representations of $\ell^1(G)$ and the unital $*$-representations of $C^*(G)$.*

5 Pontryagin Duality

We have already remarked that finite group C^*-algebras can just be seen as the image of $\mathbb{C}G$ under π_{λ_G}, with no completion argument required. It is also worth exploring what happens in the case of Abelian groups. To this end, we introduce the notion of Pontryagin duals.

Definition 5.1 (Pontryagin Dual). *Let G be a group. The* Pontryagin dual *of G, denoted $\hat{G}$, is the set of all continuous group homomorphisms from G to the circle group $\mathbb{T}$, where $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for all $\chi_1, \chi_2 \in \hat{G}$ and $g \in G$.*

Theorem 5.2. *Let G be a discrete, Abelian group. The corresponding dual group $\hat{G}$ is nothing but the set of irreducible representations of G over the Hilbert space $\mathbb{C}$.*

Proof. Suppose we have some irreducible representation $\pi : G \to \mathcal{U}(\mathbb{C})$. Because we have the correlation $\mathcal{U}(\mathbb{C}) \cong \mathcal{U}(1) \cong \mathbb{T}$, the composition of π with this isomorphism naturally results in a homomorphism $\pi' : G \to \mathbb{T}$. Conversely, suppose we have some group homomorphism $\pi' : G \to \mathbb{T}$. As before, we note that $\mathbb{T} \cong \mathcal{U}(\mathbb{C})$. Furthermore, because $\mathbb{C}$ is one-dimensional as a complex Hilbert space, so too is $\mathcal{U}(\mathbb{C})$. Thus the representation is irreducible, as every subspace of a one-dimensional space is trivial. This completes the proof. ■

The main result of this chapter is the following: the group C^*-algebras of Abelian groups are given by the "topological dual" of their Pontryagin dual [18].

Theorem 5.3. *Let G be a discrete, Abelian group. Then the corresponding group C^*-algebra, $C^*(G)$, is isomorphic to $\mathcal{C}(\hat{G})$, the space of continuous functionals on $\hat{G}$. Furthermore, this isomorphism takes a group element $g \in \mathbb{C}G \subset C^*(G)$ to the function $\hat{g} : \chi \mapsto \chi(g)$.*

Example 5.4. The Pontryagin dual of the cyclic group $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_n$, for all $n > 0$. Hence $C^*(\mathbb{Z}_n)$ is isomorphic to $\mathbb{Z}_n^*$, the topological dual of $\mathbb{Z}_n$! In particular, the evaluation map $\hat{\mathbb{Z}}_n \to \mathbb{Z}_n : \chi \mapsto \chi(1)$ is a group isomorphism, as the image of every possible χ is just the nth roots of unity in $\mathbb{T} \subset \mathbb{C}^2$. Furthermore, this generalizes beautifully to the infinite cyclic group, $\mathbb{Z}$; the same evaluation map gives a group isomorphism $\hat{\mathbb{Z}} \to \mathbb{T}$. In fact, this statement is even stronger: because the Pontryagin dual is compact (actually, it is also Hausdorff), it is not terribly difficult to show that this map is homeomorphic as well.

6 Introduction to Amenability

Definition 6.1 (Mean on a Group). *A* mean *on a group G is a finitely additive measure on the space $\ell^\infty(G)$ of ℓ^∞-bounded functionals on G; that is, a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ satisfying the following properties:*

(6.1.1). $m(\chi_G) = 1$ (normalization);

(6.1.2). $m(f) \geq 0$, for all $f \in \ell^\infty(G)$ with $f(g) \geq 0$ for all $g \in G$ (non-negativity).

*Suppose we now define a group action on $\ell^\infty(G)$ by left translation, similarly to what we did for the left regular representation. A mean is then said to be G-*invariant *if it further satisfies*

(6.1.3). $m(g \cdot f) = m(f)$, for all $g \in G$ and $f \in \ell^\infty(G)$ (left-invariance).

Definition 6.2 (Amenable Group). *A group G is said to be* amenable *if it admits a G-invariant mean.*

While this definition may be slightly abstract, in many cases we really can think of this mean as nothing but an averaging operator. For instance, consider the following example.

Example 6.3. All finite groups are amenable. In fact, given a finite group G, it is easy to verify that the averaging operator defined by

$$f \mapsto \frac{1}{|G|} \sum_{g \in G} f(g),$$

for all $f \in \ell^\infty(G)$, is a G-invariant mean on $\ell^\infty(G)$.

With this example in mind, it makes sense to ask about the amenability of Abelian groups, just as we sought to understand their C^*-algebras earlier.

Theorem-Definition 6.4 (Fixed Point Property). *A group G is said to satisfy the* fixed point property *if any continuous, affine action of G on a non-empty, compact, convex subset X of a locally convex topological vector space has a fixed point. Moreover, a group is amenable if it satisfies this property.*

Theorem 6.5 (Markov-Kakutani Fixed Point Theorem). *Let G be a discrete, Abelian group. Then any continuous, affine action of G on a non-empty, compact, convex subset X of a locally convex topological vector space V has a fixed point. In other words, G satisfies the fixed point property, and is hence amenable.*

The proofs for these two results are actually quite approachable, though slightly long; as a result, they have been deferred to A.2 and A.3 in the appendix.

We have now seen two broad classes of groups that are amenable: finite groups and Abelian groups. At this point, it is natural to wonder what a non-amenable group might look like. A particularly influential example of a non-amenable group is the free group of rank 2. Please refer to definition A.4 in the appendix for an explanation of the free group.

Theorem 6.6. *The free group on two generators, F_2, is not amenable.*

Proof. To show this, we will proceed by contradiction. Suppose that F_2 is amenable, with some invariant mean m. Furthermore, let $\{a, b\}$ be a generating set for F_2, and consider a set $A \subset F_2$ consisting of words that start with a non-trivial power of a. It is straightforward to observe that

$$A \cup (a^{-1}A) = F_2.$$

Suppose we consider $\chi_{F_2} = \chi_{A\cup a^{-1}A}$, where χ denotes the indicator function. Because A and $a^{-1}A$ are not disjoint, there will be some overlap when we sum their corresponding indicator functions χ_A and $\chi_{a^{-1}A}$, whence it follows from linearity that

$$\begin{aligned} 1 = m(\chi_{F_2}) = m(\chi_{A\cup a^{-1}A}) &\leq m(\chi_A + \chi_{a^{-1}A}) \\ &= m(\chi_A) + m(\chi_{a^{-1}A}) \\ &= m(\chi_A) + m(\lambda_a(\chi_A)) \\ &= 2m(\chi_A); \end{aligned}$$

thus $m(\chi_A) \geq 1/2$. However, suppose we instead consider the disjoint sets A, bA and b^2A, which do not cover the entirety of F_2; then

$$\begin{aligned} 1 = m(\chi_{F_2}) \geq m(\chi_{A\cup bA\cup b^2A}) &= m(\chi_A + \chi_{bA} + \chi_{b^2A}) \\ &= m(\chi_A) + m(\chi_{bA}) + m(\chi_{b^2A}) \\ &= m(\chi_A) + m(\lambda_b(\chi_A)) + m(\lambda_{b^2}(\chi_A)) \\ &= 3m(\chi_A). \end{aligned}$$

But this is a contradiction; thus F_2 cannot be amenable. This completes the proof. ∎

To understand why this particular example is so important, we consider the following results.

Proposition 6.7. *Amenable groups satisfy the following inheritance properties.*

(6.7.1). Subgroups of amenable groups are amenable.

(6.7.2). Homomorphic images of amenable groups are amenable.

(6.7.3). Let G, N and Q be groups, with G an extension of Q by N. That is, we have a sequence $\{e\} \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow \{e\}$ of group homomorphisms, where ι is injective and π is surjective, and where the image of each homomorphism is equal to the kernel of the next in the sequence. Then G is amenable if and only if both N and Q are amenable.

(6.7.4). Let $(G_i)_{i\in I}$ be a directed set of amenable subgroups of a group G, such that $G_i \subseteq G_j$ for $i \leq j$ and $G = \bigcup_{i\in I} G_i$. Then G is amenable.

Proof. (6.7.1). Let G be an amenable group with a left-invariant mean m, and let $H \subset G$ be a subgroup. Furthermore, let $R \subset G$ be a set of representatives for $H\backslash G$, the quotient group consisting of right cosets of H in G. That is, choose R such that for every element $Hg \in H\backslash G$, there is exactly one $r \in R$ such that $Hr = Hg$. Finally, let $s : G \to H$ be the map with $g \in s(g)R$, for all $g \in G$. This map certainly exists, as $\{Hr\}_{r\in R} = \{Hg\}_{g\in G}$ covers G. Then the map $\widetilde{m} : \ell^\infty(H) \to \mathbb{C}$, defined by $\widetilde{m}(f) := m(f \circ s)$, is a left-invariant mean on H.

(6.7.2). Let G be an amenable group with a left-invariant mean m, and let $\pi : G \to Q$ be a surjective group homomorphism. Then $f \mapsto m(f \circ \pi)$ is a left-invariant mean on Q.

(6.7.3). Suppose G is amenable; the first direction is implied directly by (6.7.1) as follows. Restricting the codomain of ι such that it becomes surjective, we see that N is isomorphic to a subgroup of G, and similarly by restricting the domain of π we require that Q is also isomorphic to a subgroup of G. Conversely, suppose N and Q are amenable, with left-invariant means m_N and m_Q respectively. Without loss of generality, we may assume $N \subset G$ by our prior reasoning, and $Q = G/N$ by the first isomorphism theorem. Then $f \mapsto m_Q(g \cdot N \mapsto m_N(n \mapsto f(g \cdot n)))$ is a well-defined, left-invariant mean on G.

(6.7.4). For each $i \in I$, let m_i be a left-invariant mean on G_i. Furthermore, suppose we consider the maps $\widetilde{m}_i : \ell^\infty(G) \to \mathbb{C}$ of the form $f \mapsto m_i(f|_{G_i})$. By the Banach-Alaoglu theorem, there is a subnet of $(\widetilde{m}_i)_{i\in I}$ that converges to a functional $m : \ell^\infty(G) \to \mathbb{C}$. In fact, this limit is a left-invariant mean on G as desired. ■

As it turns out, many non-amenable groups seem to have free subgroups. If we are able to find some such free subgroup of a group G, for instance using the ping-pong lemma, then it is a trivial consequence of theorem 6.6 and (6.7.1) that G is not amenable! Note that the converse to this remark – that every non-amenable group has a free subgroup of rank 2 – actually does *not* hold. This converse was known as the *von Neumann problem*; it was disproven by Olshansky, who constructed the Tarski monster groups as a counterexample [17].

We conclude the chapter with a brief example using theorem A.6, the ping-pong lemma. For those unfamiliar with braid groups, a description is given across definitions A.7, A.8 and A.11.

Theorem 6.8. *The braid group on n strands is amenable if and only if $n \in \{1, 2\}$.*

Proof. By proposition A.12, we know that $B_1 \cong \{e\}$ and $B_2 \cong F_1 \cong \mathbb{Z}$, which are both amenable by theorem 6.3 and theorem 6.5, respectively. Observe now that B_n is a proper subgroup of B_{n+1}, for all $n \in \mathbb{Z}_+$; thus if we are able to show that B_3 is not amenable, any higher order braid groups must also not be amenable by the first result of proposition 6.7. Well, let G be the subgroup of B_3 generated by σ_1^2 and σ_2^2, and consider the group homomorphism $\varphi : B_3 \to M_n(\mathbb{R})$ defined by

$$\varphi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \varphi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Note that $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ both, as matrices, satisfy the relation demanded by (A.12.1). We can therefore define a group action of G on $\mathbb{R}^2$ by $f \cdot v := \varphi(f)v$. Suppose we now define two subsets $A, B \subset \mathbb{R}^2$ as follows:

$$A := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| > |y| \right\}, \quad B := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : |x| < |y| \right\}.$$

Given $v = (x, y)^T \in B$, we see that

$$\sigma_1^{2n} \cdot v = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ny \\ y \end{pmatrix}.$$

But it follows from the definition of B that

$$|x + 2ny| \geq |2ny| - |x| > 2|y| - |y| = |y|,$$

hence $\sigma_1^{2n} \cdot B \subset A$. A similar process yields $\sigma_2^{2n} \cdot A \subset B$. Thus $G \cong F_2$ by the ping-pong lemma given in A.6, whence the result follows from proposition 6.7. This completes the proof. ■

7 Følner's Characterisation of Amenability

In this chapter, we give an overview of Følner's condition. It is recommended to first read definitions A.13, A.14 and A.15. A good reference for this material is the book of Löh [10].

Definition 7.1 (Følner Sequence). *Let (X, d) be a* UDBG space*; that is, a metric space that is uniformly discrete and of bounded geometry. If $F \subset X$ and $r \in \mathbb{N}$, we define the* r-boundary *of F in X by*

$$\delta_r^X F := \{x \in X \backslash F : d(x, f) \leq r, \text{ for some } f \in F\}. \tag{7.1.1}$$

A Følner sequence *for X is then a sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of X satisfying*

$$\lim_{n \to \infty} \frac{\left|\delta_r^X F_n\right|}{|F_n|} = 0, \tag{7.1.2}$$

for all $r \in \mathbb{N}$. These subsets are sometimes referred to as Følner sets.

Note that finite subsets of discrete spaces are naturally compact; thus in the case of subsets of discrete (or, in fact, UDBG) spaces, we can use the terms interchangeably.

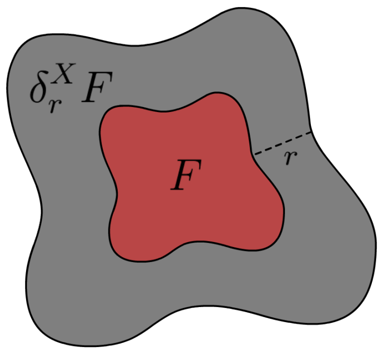

Figure 7.2: Illustration of the r-boundary.

Example 7.3. Let $S := \{e_1, \ldots, e_n\}$ be the standard generating set of the group $\mathbb{Z}^n$, for $n \in \mathbb{N}$. Then the set of all balls in $\mathbb{Z}^n$ centered at the origin, $(\{-k, \ldots, k\}^n)_{k \in \mathbb{N}}$, is a Følner sequence for $(\mathbb{Z}^n, d_S)$.

Proposition 7.4 (Følner's Property). *A UDBG space X admits a Følner sequence if and only if, for every $r \in \mathbb{N}$ and all $\varepsilon > 0$, there exists a finite subset $F \subset X$ satisfying*

$$\frac{\left|\delta_r^X F\right|}{|F|} < \varepsilon. \tag{7.4.1}$$

In this case, it is said to satisfy Følner's property.

Theorem 7.5 (Amenability via Følner's Property). *A finitely-generated group G is amenable if and only if (G, d_S) satisfies Følner's property, where d_S is the word metric given by A.15.1.*

As we can see in figure 7.2, the r-boundary is actually exactly what the name implies, as it gives an "outline" of width r around the space F. The existence of a Følner sequence can thus be thought of as a statement regarding the geometric "efficiency" of the space; that is, the space admits subsets with small boundaries but relatively large volumes.

Theorem 7.6 (Amenability via Growth). *Let X be a UDBG space with* subexponential growth*; that is, $\gamma_X \lesssim (n \mapsto 2^n)$ with $\gamma_X \not\sim (n \mapsto 2^n)$, for growth function γ_X given in definition A.16. Furthermore, choose any $x \in X$, and define $F_n \coloneqq B_X(x, n)$. Then $(F_n)_{n\in\mathbb{N}}$ contains a Følner subsequence for X, and is hence amenable.*

Proof. Let γ_X be a growth function for X with respect to the element $x \in X$, exhibiting subexponential growth. Then there exists, for all $j, K \in \mathbb{N}$ and $\varepsilon > 0$, some $n \geq K$ such that

$$\frac{\gamma_X(n+j)}{\gamma_X(n)} < 1 + \varepsilon.$$

Suppose $K = n_{j-1} + 1$ and $\varepsilon = 1/j$. Then there is a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ with

$$\frac{\gamma_X(n_j+j)}{\gamma_X(n_j)} < 1 + \frac{1}{j}$$

is satisfied for all $j \in \mathbb{N}$. We claim that $(F_{n_j})_{j\in\mathbb{N}}$ is a Følner sequence for X. Let $r \in \mathbb{N}$. Then by definition of the r-boundary, it follows that $\delta_r^X F_n \subset B_X(x, n+r) \backslash B_X(x, n)$ for all $n \in \mathbb{N}$, whence

$$\left|\delta_r^X F_n\right| \leq |B_X(x, n+r) \backslash B_X(x, n)| = \gamma_X(n+r) - \gamma_X(n).$$

Therefore, for all $j \geq r$, we obtain

$$\frac{\left|\delta_r^X F_{n_j}\right|}{\left|F_{n_j}\right|} \leq \frac{\gamma_X(n_j+r) - \gamma_X(n_j)}{\gamma_X(n_j)} = \frac{\gamma_X(n_j+r)}{\gamma_X(n_j)} - 1 \leq \frac{\gamma_X(n_j+j)}{\gamma_X(n_j)} \leq 1 + \frac{1}{j} - 1 = \frac{1}{j}.$$

This clearly tends to 0 as j tends to infinity; hence $(F_{n_j})_{j\in\mathbb{N}}$ is a Følner sequence for X as desired. This completes the proof. ■

Example 7.7. The infinite dihedral group, $D_\infty = \langle x, y \mid x^2 = y^2 = e\rangle$, has a linear growth rate. In particular, $\gamma_G^S(n) = 2n + 1$, for $S \coloneqq \{x, y\}$. This is actually quite easy to see, as any x^2 or y^2 term will reduce to identity; hence all minimum-length words must have terms alternating between x and y. There are only two distinct words of length n that satisfy this condition; one beginning with x, and the other beginning with y.

8 Reiter's Characterisation of Amenability

We now look over the characterisation of amenability that we are primarily interested in; namely, the condition of Hulanicki-Reiter. It is worth becoming familiar with the concept of weak containment; more information on this is provided across definition A.18 and theorem A.19 from the appendix.

Definition 8.1 (Reiter's Property (P_2)). *Let G be a discrete group, and suppose we define $\ell^2(G)_{1,+} := \{s \in \ell^2(G) : s \geq 0, \|s\|_2 = 1\}$. Then G is said to satisfy* Reiter's property (P_2) *if, for every finite subset $Q \subset G$ and all $\varepsilon > 0$, there exists some $s \in \ell^2(G)_{1,+}$ such that*

$$\|[\lambda_G(q)](s) - s\|_2 < \varepsilon, \tag{8.1.1}$$

for all $q \in Q$, where λ_G is the left regular representation on $\ell^2(G)$.

Theorem 8.2 (Amenability via Reiter's Property). *A discrete group G is amenable if and only if it satisfies Reiter's property.*

The following result essentially follows from Reiter's characterisation of amenability. We once again defer the proofs to *Kazhdan's Property (T)* [1], where equivalent statements have been given in theorems G.3.1 and G.3.2, respectively.

Theorem 8.3 (Hulanicki-Reiter). *Let G be a discrete group and λ_G its corresponding left regular representation on $\ell^2(G)$. Then the following conditions are equivalent:*

(8.3.1). G is amenable;

(8.3.2). $1_G \prec \lambda_G$;

(8.3.3). $u \prec \lambda_G$, for every unitary representation u of G.

Corollary 8.4. *Let G be a discrete group. Then G is amenable if and only if $C^*(G) = C^*_r(G)$.*

Proof. This is a direct consequence of Hulanicki-Reiter, theorem 4.4 and theorem A.19. ■

Note that we have written "$C^*(G) = C^*_r(G)$" in the theorem above; when G is amenable, the two group C^*-algebras are not only isomorphic, but refer to the exact same completion of $\mathbb{C}G$. In other words, the identity map on $\mathbb{C}G$ itself extends to an isomorphism from the universal group C^*-algebra to the reduced group C^*-algebra!

9 Further Study

The most natural direction for further study is towards the more general theory of locally compact groups.

Definition 9.1 (Locally Compact Group). *A* locally compact space *is a topological space such that that every element belongs to a compact neighbourhood. A* locally compact group *is a topological group for which the underlying topological space is locally compact.*

Many of the results presented in this paper actually do hold in some form in the locally compact case, albeit with some important differences. One such difference is the introduction of the Lebesgue spaces, $L^p(G)$.

In the discrete case, we have the nice situation where $L^p(G) \cong \ell^p(G)$. However, this does not hold in general for locally compact groups; as a consequence, most of the theory must be modified to use the Lebesgue spaces instead. In particular, the group C^*-algebras must be given as the completion of the full convolution algebra $L^1(G)$, rather than of the group algebra.

This is where the more general case becomes slightly more technical than the discrete case, as it essentially necessitates a treatment of measure theory. Every locally compact group admits a unique Haar measure, which is central to constructing the convolution algebra $L^1(G)$.

Some good resources regarding the group C^*-algebras of locally compact groups are *Kazhdan's Property (T)* [1] and the classical *C^*-Algebras* [5]. Fortunately, as we mentioned before, much of the theory of locally compact groups still ends up being very similar to the theory of discrete groups; thus understanding the discrete case well translates to a good understanding of the locally compact case.

More recently, there has also been some work by Hendrik Grundling on extending the concept of group C^*-algebras beyond the locally compact case [8].

Other areas of interest that follow from our investigation of amenability would be the study of Haagerup property and Kahzdan's property (T); two more group properties that can each respectively be seen as an "extension" to and an "opposite" of amenability.

A Appendix

Definition A.1 (Sequence spaces). *Let $\ell^p(X)$ be the space of ℓ^p-bounded, complex-valued functionals on X, defined by*

$$\ell^p(X) \coloneqq \begin{cases} \left\{\xi : X \to \mathbb{C} : \sum_{x \in X} |\xi(x)|^p < \infty\right\}, & \text{for } 1 \leq p < \infty; \\ \{\xi : X \to \mathbb{C} : \sup\{|\xi(x)| : x \in X\} < \infty\}, & \text{for } p = \infty. \end{cases} \tag{A.1.1}$$

*These spaces form groups under function composition. They also become Banach spaces under pointwise operations when endowed with their respective ℓ^p-*norm*, defined by*

$$\|\xi\|_p \coloneqq \begin{cases} \left(\sum_{x \in X} |\xi(x)|^p\right)^{1/p}, & \text{for } 1 \leq p < \infty; \\ \sup\{|\xi(x)| : x \in X\}, & \text{for } p = \infty, \end{cases} \tag{A.1.2}$$

for all $\xi \in \ell^p(X)$. If A is a subset of X, we let $\chi_A \in \ell^\infty(X)$ be the indicator function

$$\chi_A(x) \coloneqq \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

We let $\delta_a \in \ell^p(X)$, for $a \in X$, be the delta function

$$\delta_a(x) \coloneqq \begin{cases} 1, & \text{if } x = a; \\ 0, & \text{if } x \neq a. \end{cases}$$

The set $\{\delta_x\}_{x \in X}$ forms an orthonormal linear basis for $\ell^p(X)$.

The space $\ell^2(X)$ will be of particular interest to us. It is unique in that it can made into a Hilbert space by endowing it with the inner product $\langle \xi_1, \xi_2 \rangle_2 \coloneqq \sum_{x \in X} \overline{\xi_1(x)}\xi_2(x)$. We will often find it natural to let $\ell^2(G)$ be the Hilbert space for representations of our discrete group G.

Theorem-Definition A.2 (Fixed Point Property). *A group G is said to satisfy the* fixed point property *if any continuous, affine action of G on a non-empty, compact, convex subset X of a locally convex topological vector space has a fixed point. Moreover, a group is amenable if it satisfies this property.*

Proof. Assume that G satisfies the fixed point property, and denote by $\mathcal{L}$ the topological dual space of $\ell^\infty(G)$ with respect to the weak*-topology. Consider the set $\mathcal{M}(G)$ of all means on $\ell^\infty(G)$; this set is certainly non-empty, as the identity evaluation $f \mapsto f(e)$ is a mean on $\ell^\infty(G)$. Furthermore, because it is a weak*-closed subset of $\mathcal{L}$, by the Banach-Alaoglu theorem it is a compact, convex subset of the unit ball in $\mathcal{L}$. Thus by hypothesis, the continuous, affine action $m \mapsto g \cdot m$ of G on $\mathcal{M}(G)$, given by $(g \cdot m)(f) = m(\lambda_g(f))$ for all $g \in G$ and $f \in \ell^\infty(G)$, has a fixed point. This fixed point is precisely our left-invariant mean. This completes the proof. ■

It turns out that the converse to the above theorem is also true; that is, amenable groups automatically satisfy the fixed point property. However, this direction takes a bit more effort to prove, and is slightly more technical. For a full proof of both directions, the book *Kazhdan's Property (T)* is an excellent resource [1].

A wonderful application of this fixed point property is as follows.

Theorem-Definition A.3 (Markov-Kakutani Fixed Point Theorem). *Let G be a discrete, Abelian group. Then any continuous, affine action of G on a non-empty, compact, convex subset X of a locally convex topological vector space V has a fixed point. In other words, G satisfies the fixed point property, and is hence amenable.*

Proof. Suppose we define a map $A_n(g) : X \to X$ by

$$[A_n(g)](x) = \frac{1}{n+1} \sum_{i=0}^{n} g^i \cdot x,$$

for $n \in \mathbb{N}$ and $g \in G$. Then $A_n(g)$ is a continuous, affine transformation of X. Let $\mathcal{A}$ denote the Abelian semigroup of continuous, affine transformations of X generated by the set $\{A_n(g) : n \in \mathbb{N}, g \in G\}$. Because X is compact, $\alpha(X)$ is a closed subset of X, for all $\alpha \in \mathcal{A}$. We would first like to show that the set

$$\bigcap_{\alpha \in \mathcal{A}} \alpha(X)$$

is non-empty. Because X is compact, it is covered by finitely many sets of the form $\alpha_i(X)$. It is therefore sufficient to show that $\alpha_1(X) \cap \cdots \cap \alpha_m(X)$ is non-empty for any $\alpha_1, \ldots, \alpha_m \in \mathcal{A}$. Suppose we let $\alpha = \alpha_1 \circ \cdots \circ \alpha_m \in \mathcal{A}$. Because $\mathcal{A}$ is Abelian, $\alpha(X) \subset \alpha_i(X)$ for each

$1 \leq i \leq m$. Hence $\alpha(X) \subseteq \alpha_1(X) \cap \cdots \cap \alpha_m(X)$ and is this non-empty. Our final claim is that this intersection is a fixed point set for G; that is, for any

$$x_0 \in \bigcap_{\alpha \in \mathcal{A}} \alpha(X),$$

we have that $g \cdot x_0 = x_0$ under any continuous, affine group action. Well, we know that for every $n \geq 0$ and $g \in G$, there exists an $x \in X$ such that $x_0 = [A_n(g)](x)$. Thus for every $\varphi \in V^*$ and $g \in G$, we have the inequality

$$\begin{aligned} |\varphi(x_0 - g \cdot x_0)| &= |\varphi([A_n(g)](x) - g \cdot [A_n(g)](x))| \\ &= \frac{1}{n+1} \big|\varphi(x) - \varphi(g^{n+1} \cdot x)\big| \\ &\leq \frac{2}{n+1} \sup\{|\varphi(y)| : y \in X\}, \end{aligned}$$

where this supremum exists due to the compactness of X. Because this inequality must hold for all $n \in \mathbb{N}$, it follows that $\varphi(x_0) = \varphi(g \cdot x_0)$, for every $\varphi \in V^*$, and so $x_0 = g \cdot x_0$ for all $g \in G$ as required. This completes the proof. ∎

Definition A.4 (Free Group). *The* free group *with* free generating set S, *denoted* F_S, *is the group consisting of all words in* S. *A* word *in* S *is defined to be any finite product of elements and formal inverses of elements in* S, *where the equality of two words follows strictly from the group axioms. The identity element* e *corresponds to the empty word with no symbols. A group* G *is said to be* free *if it is isomorphic to* F_S, *for some* $S \subseteq G$.

Lemma A.5 (Universal Property). *Let* G *be a group and* F_S *the group that is freely generated by* S. *Then for any function* $\varphi : S \to G$, *there exists a unique group homomorphism* $\overline{\varphi} : F_S \to G$ *such that* $\overline{\varphi}|_S = \varphi$.

Proof. Suppose we have some function $\varphi : S \to G$, and any word $w = s_0^{k_0} \cdots s_n^{k_n}$ in F_S, where $s_i \in S$ and $k_i \in \{-1, 1\}$ for all $0 \leq i \leq n$. We define an extended map $\overline{\varphi} : F_S \to G$ by

$$\overline{\varphi}(w) = \overline{\varphi}(s_0^{k_0} \cdots s_n^{k_n}) := \varphi(s_0)^{k_0} \cdots \varphi(s_n)^{k_n}.$$

This map is certainly a well-defined homomorphism by construction, and furthermore it is unique. This completes the proof. ∎

We will mainly be looking at the *free group on two generators*, sometimes also referred to as the *free group of rank 2*. This is simply the free group F_2, with generating set $\{a, b\}$.

Theorem A.6 (Ping-Pong Lemma). *Let G be a group generated by elements α and β, and let $(X, \cdot)$ be a G-set. Furthermore, suppose there exist non-empty, disjoint subsets $A, B \subset X$, with*

$$\alpha^n \cdot B \subset A \qquad \text{and} \qquad \beta^n \cdot A \subset B, \tag{A.6.1}$$

for all $n \in \mathbb{Z}\backslash\{0\}$. Then G is a free group on two generators; that is, $G \cong F_2$.

Proof. Let a and b be the generators of the free group F_2. We wish to find an isomorphism between F_2 and G mapping $\{a, b\}$ to $\{\alpha, \beta\}$. By the universal property of the free group, we know that a map $\varphi : F_2 \to G$ preserving the group generators exists. Furthermore, because G is generated by $\{\alpha, \beta\}$, this map is certainly surjective. We need only show that it is injective, and we will be done.

In order to show that φ is injective, we will proceed by contradiction. Assume to this end that φ is *not* injective; then there must be some non-identity word $w \in F_S\backslash\{e\}$ with $\varphi(w) = \varepsilon$, where ε is the identity element of G. Suppose we write $w = a^{n_0}b^{m_1}a^{n_1} \cdots b^{m_k}a^{n_k}$, where we allow $n_0, n_k \in \mathbb{Z}$ but require $n_1, \ldots, n_{k-1}, m_1, \ldots, m_k \in \mathbb{Z}\backslash\{0\}$. Then letting $r \in \mathbb{Z}\backslash\{0, -n_0, n_k\}$,

$$\begin{aligned}
B = \varepsilon \cdot B = \varphi(a^r)\varepsilon\varphi(a^{-r}) \cdot B &= \varphi(a^r w a^{-r}) \cdot B \\
&= \alpha^{n_0+r}\beta^{m_1}\alpha^{n_1} \cdots \alpha^{n_{k-1}}\beta^{m_k}\alpha^{n_k-r} \cdot B \\
&\subset \alpha^{n_0+r}\beta^{m_1}\alpha^{n_1} \cdots \alpha^{n_{k-1}}\beta^{m_k} \cdot A \\
&\subset \alpha^{n_0+r}\beta^{m_1}\alpha^{n_1} \cdots \alpha^{n_{k-1}} \cdot B \\
&\subset \cdots \\
&\subset \alpha^{n_0+r} \cdot B \\
&\subset A.
\end{aligned}$$

This is a contradiction, as we have $B \not\subset A$ by hypothesis. The reason for choosing r as above is so that α^{n_0+r} and α^{n_k-r} do not reduce to trivial powers of α (that is, identity); the latter ensures that we can make the first jump from B to A, while the former ensures that we will eventually land in A. This is actually where the lemma gets its name, as this process of jumping between A and B mimics a game of table tennis! Thus φ is an isomorphism, and hence $G \cong F_2$. This completes the proof. ∎

Definition A.7 (Homotopy). *Let X and Y be topological spaces, and consider two continuous maps $f, g : X \to Y$. These maps are said to be* homotopic *if there exists a continuous function $H : X \times [0,1] \to Y$ such that $H(t,0) = f(t)$ and $H(t,1) = g(t)$. Such a function is referred to as a* homotopy *from f to g.*

Proposition-Definition A.8 (Geometric Braid on n Strands). *Let $n \in \mathbb{Z}_+$, and fix n distinct points $z_1, \dots, z_n \in \mathbb{R}^2$. We define a* (geometric) braid on n strands *to be an n-tuple $(f_1, \dots, f_n)$ of continuous maps of the form $f_i : [0,1] \to \mathbb{R}^2$, such that $f_i(0) = z_i$ and $f_i(1) = z_{s(i)}$ are satisfied for some permutation $s : \{1, \dots, n\} \to \{1, \dots, n\}$, as well as $f_i(t) \neq f_j(t)$ for all $i \neq j$. Two such braids on n strands $(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ are said to be* equivalent *if, for each $i \in \{1, \dots, n\}$, there exists a homotopy H_i from f_i to g_i such that $\{H_i(t,r)\}_{i=1}^n$ is a braid for all fixed values of $r \in [0,1]$. This, in fact, defines an equivalence relation on braids.*

Before we continue, it's a good idea to pause and think about what this all means. To this end, consider the following figure of some braid on three strands, $f := (f_1, f_2, f_3)$.

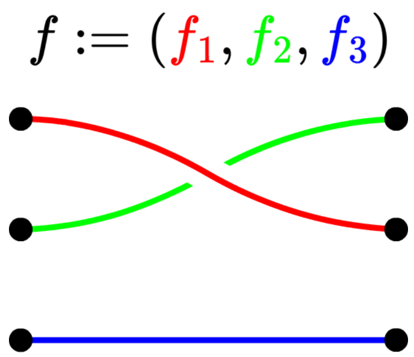

Figure A.9: Geometric interpretation of a braid on 3 strands. This is essentially a slight abstraction of the definition, formed by plotting the three paths $\{(f_i(t), t)\}_{i=1}^3$, for $t \in [0,1]$.

This figure represents an abstraction given by plotting the paths individually as $\{(f_i(t), t)\}_{i=1}^3$. Although our braids are formally defined as n-tuples of functions mapping to $\mathbb{R}^2$, we can think of them as collections of intertwining strands in $\mathbb{R}^3$. To see that these are equivalent, we simply view the former as living in the xy-plane, while the latter extends to the $z = t$ plane.

In a moment, we will show that these braids form a group under concatenation. Note that the layering of the braids *does* actually matter here; for instance, consider now the following braid.

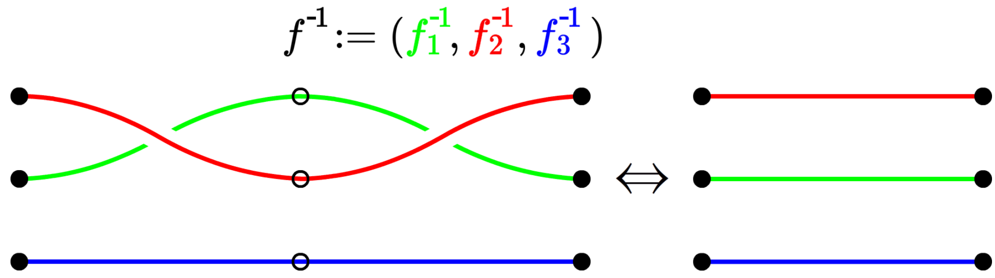

Figure A.10: Concatenation of the braid from figure A.9 with its inverse.

By joining the braids f and f^{-1} as shown in figure A.10 and then pulling the ends taught, we end up with the identity braid, composed of three straight, horizontal strands; this would not occur if we instead continued to twist the first two strands around each other. This act of "pulling tightly on the ends" is essentially what our homotopic equivalence relation encodes.

Proposition-Definition A.11 (Geometric Braid Group on n Strands). *Let $f := (f_1, \ldots, f_n)$ and $g := (g_1, \ldots, g_n)$ be two n-strand braids, where f has the associated permutation s_f. Defining a concatenating product on braids by*

$$(f \cdot g)_i(t) := \begin{cases} f_i(2t), & t \in [0, 1/2]; \\ g_{s_f(i)}(2t-1), & t \in [1/2, 1], \end{cases} \tag{A.11.1}$$

we can make the set of equivalence classes of braids into a group. This group is referred to as the (geometric) braid group on n strands*, and denoted B_n.*

Proof. Fix n distinct points $z_1, \ldots, z_n \in \mathbb{R}^2$. Consider three braids on n strands $f, g, h \in B_n$, with associated permutations s_f, s_g and s_h, respectively. We see that

$$((f \cdot g) \cdot h)_i(t) := \begin{cases} f_i(4t), & t \in [0, 1/4]; \\ g_{s_f(i)}(4t-1), & t \in [1/4, 1/2]; \\ h_{s_g(s_f(i))}(2t-1), & t \in [1/2, 1]; \end{cases}$$

$$(f \cdot (g \cdot h))_i(t) := \begin{cases} f_i(2t), & t \in [0, 1/2]; \\ g_{s_f(i)}(4t-2), & t \in [1/2, 3/4]; \\ h_{s_g(s_f(i))}(4t-3), & t \in [3/4, 1], \end{cases}$$

whence it is clear that $((f \cdot g) \cdot h)$ is equivalent to $(f \cdot (g \cdot h))$, and so associativity is satisfied. The identity braid is simply the braid $e := (e_1, \ldots, e_n)$ where we define $e_i(t) := z_i$, for all $t \in [0, 1]$. Finally, we construct an inverse to each $f \in B_n$ by taking $f_i^{-1}(t) := f_{s_f^{-1}(i)}(1-t)$, for each i. ■

With this topological interpretation of the braid groups now understood, we can move towards a more abstract, algebraic characterisation. This will prove more useful for our purposes.

Proposition A.12. *The braid group on n strands admits the presentation*

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle, \tag{A.12.1}$$

where the first relation must hold for all $|i-j| = 1$, while the second must hold for all $|i-j| > 1$.

The formal proofs for this result are typically rather long. Consequently, we will instead just take this more abstract characterisation as the definition, and defer the proof that it is equivalent to the geometric characterisation [7]. To see intuitively that they are equivalent, we can identify each so-called *Artin generator* $\sigma_i := (\sigma_{i,1}, \dots, \sigma_{i,n}) \in B_n$ with the braid given by $\sigma_{i,i}(t) := z_i + t(z_{i+1} - z_i)$ and $\sigma_{i,i+1}(t) := z_{i+1} + t(z_i - z_{i+1})$, where $\sigma_{i,j}(t) := e_j$ for all remaining strands. As an example, figure A.9 actually illustrates the first Artin generator σ_1 of B_3. By drawing the remaining generators as typical braid diagrams, it is easy to observe that they form a generating set for B_n, and furthermore satisfy the properties of (A.12.1).

Now that we have reviewed some of the groups cited in the report, we will go over the basics of a group's *growth rate*; a concept with wide applications in geometric group theory.

Definition A.13 (Uniform Discreteness). *A metric space (X, d) is said to be* uniformly discrete *if* $\inf\{d(x, x') : x, x' \in X, x \neq x'\} > 0$.

Definition A.14 (Bounded Geometry). *A metric space (X, d) is said to be of* bounded geometry *if, for all $r \in \mathbb{R}_+$, there exists some $K_r \in \mathbb{N}$ such that $|B(x, r)| \leq K_r$, for all $x \in X$. That is, every ball in (X, d) of radius r contains no more than K_r points.*

Definition A.15 (Word Metric). *Let $(G, \circ)$ be a group with finite generating set S, where we take S to be closed under inverses. We define the* word metric *on G with respect to S by*

$$d_S(g, h) := \min\{n \in \mathbb{N} : \exists x_1, \dots, x_n \in G \text{ for which } g \circ x_1 \circ \cdots \circ x_n = h\}, \tag{A.15.1}$$

for all $g, h \in G$. That is, $d_S(g, h)$ is the smallest number of elements needed to multiply by g to get h. We refer to $d_S(e, g)$ as the word length *of $g \in G$ with respect to the finite generating set S.*

Definition A.16 (Growth Rate). *Let X be a UDBG space, and let $x \in X$. Consider the closed ball of radius $n \in \mathbb{N}$ centered at x, $F_n := B_X(x, n)$. We can then define a* growth function *of X by $\gamma_X : n \mapsto |B_X(x, n)|$. Moreover, because UDBG spaces have bounded geometry, the growth functions corresponding to any two elements of X will be asymptotically equivalent; hence it makes sense to define the* growth rate *of X to be the asymptotic behaviour of any such growth function γ_X.*

For a group G with a finite generating set S, the ball $B_G^S(e, n)$ of radius $n \in \mathbb{N}$ is just the set of all elements in G that can be expressed as a product of no more than n generators. We can therefore naturally define a growth function γ_G^S, that describes precisely how the group grows with respect to S. However, we should note that the word length of a particular element will not necessarily be the same with respect to every generating set. As a result, it is sensible to wonder if this growth rate is a good description of a group in general. This leads to the following proposition.

Proposition A.17. *Let G be a group with finite generating sets S and S'. Then the word metrics d_S and $d_{S'}$ are* bi-Lipschitz equivalent*; that is, there exists some constant $C > 0$ such that*

$$\frac{1}{C} d_S(g, h) \leq d_{S'}(g, h) \leq C d_S(g, h), \tag{A.17.1}$$

for any $g, h \in G$. In particular, γ_G^S and $\gamma_G^{S'}$ are asymptotically equivalent, and hence the growth rate of G is independent of the choice of generating set.

Proof. Let $S = \{s_1, \dots, s_m\}$ and $S' = \{s'_1, \dots, s'_n\}$ be finite generating sets for G. Suppose we define, for instance,

$$C_1 := \max\{d_{S'}(e, s_i) : 1 \leq i \leq m\},$$
$$C_2 := \max\{d_S(e, s'_i) : 1 \leq i \leq n\}.$$

In other words, every generator in S has a representation in S' composed of no more than C_1 elements. Therefore, by simply substituting generators in S with their shortest representations in S', we obtain an upper bound for $d_{S'}(g, h)$; in particular, $d_{S'}(g, h) \leq C_1 d_S(g, h)$. By symmetry, we also obtain $\frac{1}{C_2} d_S(g, h) \leq d_{S'}(g, h)$. Setting $C := C_1 + C_2$, the result follows. This completes the proof. ■

Finally, we include some brief comments on weak containment. First, a definition.

Definition A.18 (Weak Containment). *Let G be a group with two unitary representations $u : G \to \mathcal{U}(H)$ and $v : G \to \mathcal{U}(K)$. The representation u is said to be* weakly contained *in v, denoted $u \prec v$, if for every $\xi \in H$, every compact $Q \subset G$ and all $\varepsilon > 0$, there exists $n \in \mathbb{Z}_+$ and $\eta_1, \ldots, \eta_n \in K$ such that*

$$\left| \langle \xi, u_g(\xi) \rangle - \sum_{i=1}^{n} \langle \eta_i, v_g(\eta_i) \rangle \right| < \varepsilon, \tag{A.18.1}$$

for all $g \in Q$. This definition extends to the unital $*$-representations of the group C^*-algebra in the natural way.

Weak containment has some rather nice implications in the context of our group C^*-algebras. In particular, we have the following restriction on the norms of the images. As the proof for this result is rather involved, we will defer it to theorem 7 of *On simplicity of reduced C^*-algebras of groups* [9].

Theorem A.19. *Let G be a discrete group with unitary representations u and v, corresponding to the non-degenerate $*$-representations π_u and π_v of $C^*(G)$. Then the following conditions are equivalent:*

(A.19.1). $u \prec v$;

(A.19.2). $\ker(\pi_u) \subset \ker(\pi_v)$;

(A.19.3). $\|\pi_u(a)\| \leq \|\pi_v(a)\|$, for all $a \in C^*(G)$.

A deep understanding of C^*-algebras past the most basic, abstract definition is not essential for this report. For the curious, some good resources on the general theory of C^*-algebras are *C^*-Algebras and Operator Theory* [11], *C^*-Algebras by Example* [3] and *C^*-Algebras and Finite-Dimensional Approximations* [2]. We will, however, include one major result here, as it is arguably important for much of the intuition behind forming group C^*-algebras.

Theorem A.20 (Gelfand-Naimark Representation Theorem). *For every C^*-algebra A, there exists a Hilbert space H and a norm-closed $*$-subalgebra $B \subseteq \mathcal{B}(H)$ such that A is isometrically $*$-isomorphic to B.*

This concludes the appendix. Thank you for reading!

R References

[1] Bekka, B, de la Harpe, P & Valette, A 2008, *Kazhdan's Property (T)*, Cambridge University Press, New York.

[2] Brown, NP & Ozawa, N 2008, *C^*-Algebras and Finite-Dimensional Approximations*, American Mathematical Society, Providence.

[3] Davidson, KR 1996, *C^*-Algebras by Example*, American Mathematical Society, Providence.

[4] Deitmar, A & Echterhoff, S 2014, *Principles of Harmonic Analysis*, 2nd edn, Springer, Cham.

[5] Dixmier, J 1977, *C^*-Algebras*, North-Holland Publishing Company, Amsterdam.

[6] Gelfand, IM & Rykov, DA 1943, 'Irreducible unitary representations of locally bicompact groups', *Rec. Math. [Mat. Sbornik] N.S.*, vol. 13, no. 2, pp. 301-316.

[7] Gonzalez-Meneses, J 2010, 'Basic results on braid groups'. ArXiv. Available from: https://arxiv.org/abs/math/1010.0321v1.

[8] Grundling, HB 2004, 'Group Algebras for Groups which are not Locally Compact'. ArXiv. Available from: https://arxiv.org/abs/math/0404020v3.

[9] de la Harpe, P 2005, 'On simplicity of reduced C^*-algebras of groups'. ArXiv. Available from: https://arxiv.org/abs/math/0509450v1.

[10] Löh, C 2017, *Geometric Group Theory: An Introduction*, Springer, Cham.

[11] Murphy, GJ 1990, *C^*-Algebras and Operator Theory*, Academic Press, San Diego.

[12] Murray, FJ & von Neumann, J 1936, 'On Rings of Operators', *Annals of Mathematics*, vol. 37, no. 1, pp. 116-229.

[13] Murray, FJ & von Neumann, J 1937, 'On Rings of Operators II', *Trans. Amer. Math. Soc.*, vol. 41, no. 2, pp. 208-248.

[14] Murray, FJ & von Neumann, J 1943, 'On Rings of Operators IV', *Annals of Mathematics*, vol. 44, no. 4, pp. 716-808.

[15] von Neumann, J 1929, 'Zur allgemeinen Theorie des Masses', *Fund. Math.*, vol. 13, pp. 73–116.

[16] von Neumann, J 1940, 'On Rings of Operators III', *Annals of Mathematics*, vol. 41, no. 1, pp. 94-161.

[17] Olshansky, A 1980, 'On the question of the existence of an invariant mean on a group', *Uspekhi Mat. Nauk*, vol. 35, no. 4, pp. 199-200.

[18] Putnam, IF 2019, 'Lecture Notes on C^*-algebras', lecture notes, University of Victoria, 3rd January.

[19] Yoshizawa, H 1949, 'Unitary representations of locally compact groups', *Osaka Mathematical Journal*, vol. 1, no. 1, pp. 81-89.