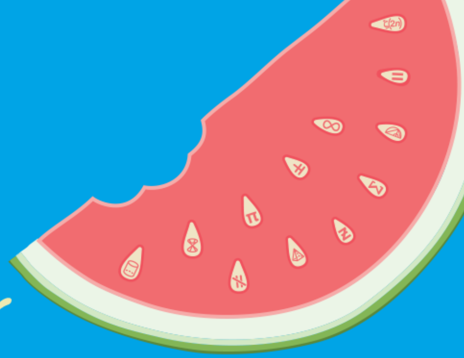

Uniqueness of cohomogeneity one Einstein metrics on $\mathbb{S}^n$ with group action $SO(2) \times SO(n-1)$

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Abstract: We consider the problem of finding and classifying Einstein metrics on n spheres that are invariant under certain cohomogeneity one group action. Using the Böhm-Wilking rounding theorem, we show that a class of these invariant metrics, called doubly warped, must be round to be Einstein.

1 Introduction:

Einstein manifolds constitute solutions of the Einstein field equations for a gravitational field in the special case that there is no matter. They also have applications in many other areas of mathematical physics [1]. Moreover, these objects are also interesting in a geometric sense. Constructing, classifying and studying Einstein manifolds is thereby a crucial part of Riemannian geometry.

Many examples of Einstein metrics have been constructed on $\mathbb{S}^n$. Firstly, there are the well known round metrics induced from the pullback of the Euclidean metric on $\mathbb{R}^{n+1}$ and scalar multiples of it. In 1973, Jensen was able to find more Einstein metrics on $\mathbb{S}^{4m+3}$ for $m > 1$ [11]. The most notable development since, is Böhms construction of infinite sequences of non-isometric Einstein metrics over $\mathbb{S}^5$, $\mathbb{S}^6$, $\mathbb{S}^7$, $\mathbb{S}^8$ and $\mathbb{S}^9$ in 1998 [3]. Boyer, Calicki and Kollár also found many non-isometric families of Einstein metrics on odd dimentional spheres [4] and there have been many other results on this topic. However, uniqueness results are relatively uncommon and far from comprehensive. Most have been predominantly obtained under assumptions of homogeneous symmetry or curvature positivity.

In this document we aim to use the Böhm-Wilking rounding theorem to show that extensions of particular doubly warped product metrics on n spheres must be isometric to a round metric.

Statement of Authorship

My academic supervisor, Dr Timothy Buttsworth, conceived the outline, main aims and ideas of the project. Under his supervision, I analysed eigenvalues of the curvature operator and drafted the report.

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2 Riemannian geometry

In this section we cover most definitions and standard results required for the main research. Firstly we introduce some Riemannian geometry as in [13]. An introduction to necessary information of smooth manifolds and some notation is presented in Appendix 1.

As smooth manifolds alone only give a smooth structure of manifolds, we introduce an inner product over the tangent space to get notions of distances and angles and thereby geometry. The inner product chosen is called a *metric*.

Definition 2.1. A metric g on M defines an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$ on the tangent space for any $p \in M$. A metric g is smooth if for any smooth vector fields $X, Y \in \mathfrak{X}(M)$, the map $p \to g_p(X_p, Y_p)$ is smooth on M. A Riemannian manifold is then defined as a pair (M, g), where M is a smooth manifold and g is a smooth metric on M.

Remark. As the tangent space is a vector space we can consider the metric as a symmetric matrix. Given coordinate maps $(x^1, ..., x^n)$ we can assign components, $g_{ij}\big|_p := g_p\left(\partial_{x^i}\big|_p, \partial_{x^j}\big|_p\right)$. Moreover using the tensor product one can write the metric in terms of this basis as

$$g = g_{ij}\ dx^i \otimes dx^j,$$

where dx^i is the corresponding element of the dual basis in the cotangent bundle. There is a brief introduction to tensors and tensor products in Appendix 2. For use below we give slightly different notation as we will denote $(dx^i)^2 := dx^i \otimes dx^i$. Note that above and for the rest of the document we will also use the Einstein summation convention, see Appendix 1 for more details.

Example 2.1. $\mathbb{R}^n$ with its natural structure is a simple example of a Riemannian manifold where the metric is simply the dot product. That is, over the canonical basis for the tangent plane, $(e^1, ..., e^n)$,

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{or equivalently} \quad g = (de^i)^2.$$

Example 2.2. As $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the tangent plane on the n sphere is contained in the corresponding tangent plane in $n+1$ euclidean space. Then we can use the metric in Example 2.1 on $\mathbb{R}^{n+1}$ and restrict it to $\mathbb{S}^n$. Formally, this is the metric on $\mathbb{S}^n$ defined via pullback of the Euclidean metric on $\mathbb{R}^{n+1}$. We call this a round metric. We also call positive scalar multiples of such a metric, round.

A standard approach to measure the geometry of a Riemannian manifold is to measure its curvature. Before we can define notions of curvature we first need to define some preliminary objects, first of which is the *Lie bracket*, an operator that measures the non commutativity of composition of vector fields.

Definition 2.2. The Lie bracket, $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined as

$$[X, Y](f) := X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$.

Remark. We note that the Lie bracket does indeed define a vector field as it is a linear operator and satisfies the Leibniz rule. For a calculation see Appendix 3.

We now define the *covariant derivative* which is able to measure the change of one vector field along another. It is a very important concept used here for introducing curvature.

Definition 2.3. Fix a Riemannian manifold (M, g). The Levi-Civita connection, also known as the covariant derivative, is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoted by $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;
2. $\nabla_{(fX+gY)} Z = f\nabla_X Z + g\nabla_Y Z$;
3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$;
4. $[X, Y] = \nabla_X Y - \nabla_Y X$; and
5. $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Remark. The fundamental theorem of Riemannian geometry states that such a connection exists and is unique. In fact it shows that the Levi-Civita connection is uniquely determined by the Koszul formula as below:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X).$$

Finally we can introduce some curvature forms, firstly the *Riemannian curvature tensor* from which other forms are built.

Definition 2.4. Fix a Riemannian manifold (M, g). The (1,3) Riemannian curvature tensor is the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by,

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Similarly, the (0,4) Riemannian curvature tensor is defined by lowering the indices of the (1,3) Riemannian curvature tensor as below:

$$\mathrm{R}(X, Y, Z, W) = g(\mathrm{R}(X, Y)Z, W).$$

Remark. We note that indeed the $(1, 3)$ and consequently the $(0, 4)$ Riemannian curvature are tensorial, that is for any $f \in C^\infty(M)$, $R(fX, Y)Z = R(X, fY)Z = R(X, Y)fZ = fR(X, Y)Z$. See Appendix 3 for calculations.

We also introduce important symmetries on the Riemannian curvature tensors as in [13] as they will be used throughout this document.

Proposition 2.1. *The Riemannian curvature tensor $R(X, Y, Z, W)$ satisfies the following properties:*

1. $R(X, Y, Z, W) = -R(Y, X, Z, W) = R(X, Y, W, Z)$; *and*
2. $R(X, Y, Z, W) = R(Z, W, X, Y)$.

As the tensor is antisymmetric within its first and last two components, we can consider it acting over bivectors instead. In fact, we are able to introduce the *curvature operator*, an operator acting over bivectors that describes the (0,4) Riemannian curvature tensor. Note that there is a definition of bivectors in Appendix 2.

Definition 2.5. Fix a Riemannian manifold (M, g). The curvature operator $\mathfrak{R} : \Lambda^2 M \to \Lambda^2 M$ is the linear operator defined to satisfy the following equation:

$$g(\mathfrak{R}(X \wedge Y), U \wedge V) := \mathrm{R}(X \wedge Y, U \wedge V) := \mathrm{R}(X, Y, V, U).$$

Remark. We note that as the $(0, 4)$ Riemanian curvature tensor is indeed tensorial in all components and skew symmetric in its first two and last two components, the curvature operator is indeed a well defined linear operator over $\Lambda^2 M$. Moreover by condition 2 from Proposition 2.1, $\mathfrak{R}$ is symmetric.

Lastly we define *Ricci curvature* by contracting the (1,3) Riemannian curvature tensor.

Definition 2.6. Fix a Riemannian manifold (M, g). The Ricci curvature tensor, $\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$, is the (0,2) tensor defined by

$$\mathrm{Ric}(X, Y) := \mathrm{tr}\{Z \mapsto R(Z, X)Y\}.$$

Remark. Note that locally we can find an orthonormal basis $(e_i)_{i=1}^n$ of vector fields about any chart. That is, there exists functions $f_i \in C^\infty(M)$ such that locally every vector field can be written as $X = f_i e_i$. Then we can compute the Ricci curvature tensor by

$$\mathrm{Ric}(X, Y) = g(R(e_i, X)Y, e_i) = R(e_i, X, Y, e_i).$$

We note Proposition 2.1 shows that the the Ricci curvature tensor is symmetric. As both the Ricci curvature and metric are symmetric (0,2) tensors, it makes sense to compare them and we can give the following definition.

Definition 2.7. A Riemannian manifold (M, g) is called an Einstein manifold if

$$\mathrm{Ric}(X, Y) = \lambda g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$ and a constant $\lambda \in \mathbb{R}$.

2.1 Ricci flow

We now introduce *Ricci flow*, a powerful tool used to yield many results in geometry and topology. It is a way to evolve the metric of a manifold such that the Ricci curvature becomes more uniform. In 2002, Perelman was able to show how Ricci flow could be used to prove the Poincaré Conjecture (one of the renowned millennium problems).

Definition 2.8. Given a smooth manifold M with initial metric g_0 and open interval $(0, l)$, the Ricci flow assigns each $t \in (0, l)$ a metric g_t on M such that for all $X, Y \in \mathfrak{X}(M)$,

$$\partial_t \, g_t(X, Y) = -2\mathrm{Ric}^{g_t}(X, Y).$$

Example 2.3. Consider an Einstein manifold (M, g_0) with Einstein constant λ. Then $g_t = (1 - 2\lambda t)g_0$ is a Ricci flow on M over the interval $(0, \frac{\lambda}{2})$. Indeed,

$$\partial_t \; g_t = \partial_t \; (1 - 2\lambda t)g_0 = -2\lambda g_0 = -2\mathrm{Ric}^{g_0} = -2\mathrm{Ric}^{g_t}.$$

For an explanation of the final step see Appendix 3.

We now introduce *normalised Ricci flow*, which gives a way to evolve the metric according to Ricci flow but rescales to maintain constant volume.

Definition 2.9. Given a smooth compact manifold M with initial metric g_0 and open interval $(0, l)$, the normalised Ricci flow assigns each $t \in (0, l)$ a metric g_t on M such that for all $X, Y \in \mathfrak{X}(M)$

$$\partial_t \; g_t(X, Y) = -2\mathrm{Ric}^{g_t}(X, Y) + \frac{2}{n} \frac{\int_M S \; dV_{g(t)}}{\int_M dV_{g(t)}} g_t(X, Y).$$

Here, we further contract the curvature to get the scalar curvature, $S := \mathrm{tr}_g(\mathrm{Ric})$, the trace of the Ricci curvature with respect to the metric.

Remark. From Example 2.3 we expect that the metrics of Einstein manifolds are themselves solutions of normalised Ricci flow because the Ricci flow only rescales their metrics. Indeed we find that Einstein manifolds are fixed point solutions of the normalised Ricci flow as the scalar curvature simplifies to $n\lambda$.

3 Invariance of doubly warped metrics

3.1 Lie groups

In order to later investigate Einstein manifolds that are invariant under a group action, we first introduce Lie groups and more specifically $SO(n)$.

Definition 3.1. A Lie group $(M, \cdot)$ is a group such that M is a smooth manifold and the map $(x, y) \mapsto x^{-1} \cdot y$ is smooth.

Example 3.1. The well know Lie group, $SO(n + 1)$ is the group of matrix multiplication over the set of $n + 1$ dimensional square orthogonal matrices with determinant 1. The carrier space of $SO(n + 1)$ is $\mathbb{R}^{n+1}$, that is, the group elements act over $\mathbb{R}^{n+1}$. Moreover, as these matrices are orthogonal, they preserve norm. Hence, for all $A \in SO(n + 1)$, $A(\mathbb{S}^n) = \mathbb{S}^n$. Then, as their determinant is 1, $SO(n + 1)$ is simply the set of rotations of $\mathbb{S}^n$.

Example 3.2. We can construct a subgroup $SO(2) \times SO(n - 1)$ of $SO(n + 1)$ as the group of matrix multiplication over the set of $n + 1$ dimensional square matrices of the form:

$$\begin{pmatrix} S_2 & 0_{2\;(n-1)} \\ 0_{(n-1)\;2} & S_{n-1} \end{pmatrix}$$

where $S_2 \in SO(2)$, $S_{n-1} \in SO(n - 1)$ and $0_{i\;j}$ is a $i \times j$ matrix of 0s. Some simple calculations show that $SO(2) \times SO(n - 1)$ is indeed a subgroup and a sub-manifold of $SO(n + 1)$ and hence a Lie group itself.

We note that we can have a manifold with the action of a Lie group. For example $\mathbb{S}^n$ can be acted on via $SO(2) \times SO(n-1)$. This now begs the question of investigating manifolds with actions of a Lie group similar to the manifold itself.

Definition 3.2. A cohomogeneity one manifold M is a manifold with actions of a compact Lie group $(N, \cdot)$ whose quotient M/N is one dimensional.

We say that a Riemannian manifold is invariant under a Lie group if all actions in the group preserve the manifold. For example we see that $\mathbb{S}^n$ with a round metric is invariant under $SO(n+1)$. In fact a metric on $\mathbb{S}^n$ must be round to be invariant under $SO(n+1)$.

3.2 Construction of a doubly warped metric

We aim to find all Einstein metrics on $\mathbb{S}^n$, for $n \geq 4$, that are invariant under the group action $SO(2) \times SO(n-1)$. We note that this action is well defined on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ as $SO(2) \times SO(n-1) \subset SO(n+1)$ by Example 3.1 and 3.2. We also note that the principal orbits of $SO(2) \times SO(n-1)$ are diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-2}$ and hence the action is of cohomogeneity one.

We will only investigate a specific class of invariant metrics, although any invariant metric is isometric to one we give below. To easily define this class we first construct a product manifold, $M := I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ where $I = (0, T) \subset \mathbb{R}$ an open interval. Then $SO(2) \times SO(n-1)$ is also a group action on M, where $SO(2)$ acts on $\mathbb{S}^1$ and $SO(n-1)$ acts on $\mathbb{S}^{n-2}$ by rotations as in Example 3.1. Moreover, the group action is cohomogeneity one again as the principal orbits are diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-2}$. We define the metric on $I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ as doubly warped,

$$g = dt^2 + f_1(t)^2 d\theta^2 + f_2(t)^2 ds^2, \tag{1}$$

where ds^2 is the round metric on $\mathbb{S}^{n-2}$ of Ricci curvature $n-3$, $d\theta^2$ is the canonical metric on $\mathbb{S}^1$ which gives it length 2π and $f_1(t), f_2(t) > 0$ for all $t \in (0, T)$. We note that indeed (M, g) is invariant under $SO(2) \times SO(n-1)$ as for fixed t, $(\mathbb{S}^1, f_1(t)^2 d\theta^2)$ and $(\mathbb{S}^{n-1}, f_2(t)^2 ds^2)$ are invariant under $SO(2)$ and $SO(n-1)$ respectively.

Lastly, we note that M is diffeomorphic to $\mathbb{S}^n \setminus P$, where $P = (\{0\}^2 \times \mathbb{S}^{n-2}) \cup (\mathbb{S}^1 \times \{0\}^{n-1})$ are singular orbits of dimension $n-2$ and 1. The diffeomorphism is given by

$$\Phi(t, \theta, s) = \left(\cos\left(\frac{\pi t}{2T}\right)\theta, \sin\left(\frac{\pi t}{2T}\right)s\right) \tag{2}$$

for $t \in I$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$ and $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$. We verify Φ is a diffeomorphism in Appendix 4. To intuitively see why we get singular orbits we use notation as in Example 3.2. We note that as $t \to 0$, S_{n-1} vanishes and we get a one dimensional orbit corresponding to elements of the form $\begin{pmatrix} S_2 & 0_{2\ (n-1)} \\ 0_{(n-1)\ 2} & 0_{(n-1)\ (n-1)} \end{pmatrix}$. Similarly as $t \to T$, S_2 vanishes and we get a $n-2$ dimensional orbit corresponding to elements of the form $\begin{pmatrix} 0_{2\ 2} & 0_{2\ (n-1)} \\ 0_{(n-1)\ 2} & S_{n-1} \end{pmatrix}$.

If we define a metric $\tilde{g}$ on $\mathbb{S}^{n-1}$ such that

$$\tilde{g}(X, Y) = g(d\Phi(X), d\Phi(Y)), \tag{3}$$

then Φ is an isometric embedding. Thus we get a natural identification of $I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ in $\mathbb{S}^n$. Moreover as $SO(2)$ acts on the first 2 coordinates, θ, and $SO(n-1)$ acts on the last $n-1$ coordinates, s, we see that $(\mathbb{S}^n, \tilde{g})$ will also be invariant under $SO(2) \times SO(n-1)$. Informally we are able to think of M as a subset covering most of $\mathbb{S}^n$, and we use this to get an invariant metric on $\mathbb{S}^n$ as the group action is preserved on the diffeomorphism.

4 Boundary conditions and curvature operator

In this section we give a lemma detailing sufficient conditions of the metric in equation (3), on the singular orbits $\{0\}^2 \times \mathbb{S}^{n-2}$ and $\mathbb{S}^1 \times \{0\}^{n-1}$, such that it is a smooth metric on $\mathbb{S}^n$. We then calculate the curvature operator of the metric.

Note that if we take $t = 0$ and $t = T$ in the diffeomorphism in equation (2) we exactly get the missing singular orbits. However, in these cases, then $\mathbb{S}^{n-2}$ and $\mathbb{S}^1$ will vanish respectively. Moreover, without constraints on f_1 and f_2, the geometry does not make sense on these boundaries. Thus in order to preserve the structure of $\tilde{g}$ on $\mathbb{S}^n$ as $t \to 0$ and $t \to T$, f_2 and f_1 respectively must vanish. Now, by imposing these conditions we are able to get important higher order conditions on f_1 and f_2 on the boundaries. We will soon give Lemma 3.1 which details this, but first we show two useful propositions from [13].

Proposition 4.1. *For a doubly warped product $g = dt^2 + \phi(t)^2 ds_p^2 + \psi(t)^2 ds_q^2$ on $(0, b) \times \mathbb{S}^p \times \mathbb{S}^q$, if $\phi : (0, b) \to (0, \infty)$ is smooth and $\phi(0) = 0$, then we get a smooth metric at $t = 0$ with local topology $\mathbb{R}^{p+1} \times \mathbb{S}^q$, if and only if*

$$\phi \textit{ is odd about 0 }, \phi'(0) = 1$$

and

$$\psi \textit{ is even about 0 }, \psi(0) > 0.$$

Proposition 4.2. *For a doubly warped product $g = dt^2 + \phi(t)^2 ds_p^2 + \psi(t)^2 ds_q^2$ on $(0, b) \times \mathbb{S}^p \times \mathbb{S}^q$, if $\phi : (0, b) \to (0, \infty)$ is smooth and $\phi(b) = 0$, then we get a smooth metric at $t = b$ with local topology $\mathbb{R}^{p+1} \times \mathbb{S}^q$, if and only if*

$$\phi \textit{ is odd about } b\ , \phi'(b) = -1$$

and

$$\psi \textit{ is even about } b\ , \psi(b) > 0.$$

We can now introduce the lemma we will use.

Lemma 4.1. *A Riemannian metric, g, as in equation (1) on $I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ can be extended to a smooth Riemannian metric on $\mathbb{S}^n$ by equation (3) if f_1 and f_2 are smoothly extendable to functions on* $[0, T]$

such that:

$$f_1 \textit{ is even about 0 and } f_1(0) > 0;$$
$$f_1 \textit{ is odd about } T \textit{ and } f_1'(T) = -1;$$
$$f_2 \textit{ is odd about 0 and } f_2'(0) = 1; \textit{ and}$$
$$f_2 \textit{ is even about } T \textit{ and } f_2(T) > 0.$$

Remark. It is possible to give a more general forms such that we get an if and only if assertion. In fact such conditions would be sufficient for Section 6. However, we do not give these forms as the proof is too complicated to be included.

Proof. We cannot immediately extend g to $\mathbb{S}^n$ via equation (3) as g is only defined on M, an embedding which does not cover $\mathbb{S}^n$. Moreover, we then only have smoothness on the image of Φ. Thus we need to ensure smoothness at $t = 0$ and $t = T$ as this covers the missing singular orbits. We noted earlier that as $t \to 0$, f_2 vanishes and thus by Proposition 4.1, we get the conditions at 0. Similarly, as $t \to T$, f_1 vanishes and using Proposition 4.2 we get the conditions at T. Hence we have smoothness at 0 and T, so the extension $\tilde{g}$ will also be smooth. □

Lemma 3.1 immediately gives some second order terms for the functions f_1 and f_2, in fact, for almost all of the results below only second or lower order terms are needed. All second or lower order terms of f_1 and f_2 from Lemma 3.1 are listed below:

$$f_1(0) > 0, f_1(T) = 0, f_1'(0) = 0, f_1'(T) = -1, f_1''(T) = 0,$$
$$f_2(0) = 0, f_2(T) > 0, f_2'(0) = 1, f_2'(T) = 0 \text{ and } f_2''(0) = 0.$$

Let $(e_i)_{i=1}^{n-1}$ be vector fields that form an orthogonal basis for the tangent planes of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ on some chart under the product metric $d\theta^2 + ds^2$. We can construct it such that e_1 is tangent to $\mathbb{S}^1$ and e_j is tangent to $\mathbb{S}^{n-2}$ for $j > 1$. It is easy to then ensure $(e_i)_{i=1}^{n-1}$ will form an orthonormal basis locally in the tangent spaces of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$ under the metric $f_1(t)^2 d\theta^2 + f_2(t)^2 ds^2$ for fixed $t \in (0, T)$. Note that formally we should define a collection of these to cover each $t_0 \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$, then M and later $\mathbb{S}^n$. However, we will only need to use them locally so we do not need to be cautious about using them globally. We note that then e_i extends smoothly over a neighbourhood of M as t varies and so $(\partial_t, e_1, e_2, ...)$ forms a local orthonormal basis for (M, g). Thus $(\partial_t \wedge e_1, ..., \partial_t \wedge e_{n-1}, e_1 \wedge e_2, e_1 \wedge e_3, ..., e_{n-2} \wedge e_{n-1})$ forms an orthonormal basis in $\Lambda^2 M$ which we claim diagonalises the curvature operator. Recall that we defined the metric on bivectors in Appendix 2.

We will now compute the curvature operator. In order to do so we first need to do many calculations and use several propositions from [13]. This lengthy calculation is all in Appendix 5 and we find:

$$\mathfrak{R}(\partial_t \wedge e_1) = -\frac{f_1''(t)}{f_1(t)} \partial_t \wedge e_1; \tag{4}$$

$$\mathfrak{R}(\partial_t \wedge e_i) = -\frac{f_2''(t)}{f_2(t)} \partial_t \wedge e_i; \tag{5}$$

$$\mathfrak{R}(e_1 \wedge e_i) = -\frac{f_1'(t) f_2'(t)}{f_1(t) f_2(t)} e_1 \wedge e_i; \text{ and} \tag{6}$$

$$\mathfrak{R}(e_j \wedge e_k) = \frac{1 - f_2'(t)^2}{f_2(t)^2} e_j \wedge e_k. \tag{7}$$

Therefore equations (4), (5), (6) and (7) show the curvature operator is indeed diagonalised and its eigenvalues are:

1. $-\frac{f_1''(t)}{f_1(t)}$ with multiplicity 1;
2. $-\frac{f_2''(t)}{f_2(t)}$ with multiplicity $n-2$;
3. $-\frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)}$ with multiplicity $n-2$; and
4. $\frac{1-f_2'(t)^2}{f_2(t)^2}$ with multiplicity $\binom{n-2}{2}$.

5 The Einstein equations

We will now construct a system of ODEs which are necassary and sufficient conditions for (M, g) to be an Einstein manifold. We note that as $(\partial_t, e_1, e_2, ..., e_{n-1})$ is an orthonormal basis at any point and the Ricci curvature is a tensor it is sufficient to have,

$$\begin{gathered} \mathrm{Ric}(\partial_t, \partial_t) = \lambda, \\ \mathrm{Ric}(e_1, e_1) = \lambda, \text{ and} \\ \mathrm{Ric}(e_i, e_i) = \lambda \text{ for } i \geq 2, \end{gathered}$$

as the off diagonal terms are indeed 0 by Appendix 5. Using antisymmetry of the Riemannian curvature tensor we find $R(X, X, X, X) = 0$ for all $X \in \mathfrak{X}(M)$ and thus,

$$\begin{aligned} \lambda = \mathrm{Ric}(\partial_t, \partial_t) &= R(\partial_t, \partial_t, \partial_t, \partial_t) + R(\partial_t, e_1, e_1, \partial_t) + R(\partial_t, e_j, e_j, \partial_t) \\ &= g(\mathfrak{R}(\partial_t \wedge e_1), \partial_t \wedge e_1) + g(\mathfrak{R}(\partial_t \wedge e_j), \partial_t \wedge e_j) \\ &= -\frac{f_1''(t)}{f_1(t)} - (n-2)\frac{f_2''(t)}{f_2(t)}. \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda = \mathrm{Ric}(e_1, e_1) &= g(\mathfrak{R}(\partial_t \wedge e_1), \partial_t \wedge e_1) + g(\mathfrak{R}(e_1 \wedge e_j), e_1 \wedge e_j) \\ &= -\frac{f_1''(t)}{f_1(t)} - (n-2)\frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)}. \end{aligned}$$

Lastly, for $i \geq 2$,

$$\begin{aligned} \lambda = \mathrm{Ric}(e_i, e_i) &= g(\mathfrak{R}(\partial_t \wedge e_i)\partial_t \wedge e_i) + g(\mathfrak{R}(e_1 \wedge e_i), e_1 \wedge e_i) + g(\mathfrak{R}(e_j \wedge e_i), e_j \wedge e_i) \\ &= -\frac{f_2''(t)}{f_2(t)} - \frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)} + (n-3)\frac{1-f_2'(t)^2}{f_2(t)^2}. \end{aligned}$$

Therefore the manifold $M = I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ with metric as in equation (1) is Einstein if and only if the following system of ODEs is satisfied:

$$-\frac{f_1''(t)}{f_1(t)} - (n-2)\frac{f_2''(t)}{f_2(t)} = \lambda; \tag{8}$$

$$-\frac{f_1''(t)}{f_1(t)} - (n-2)\frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)} = \lambda; \text{ and} \tag{9}$$

$$-\frac{f_2''(t)}{f_2(t)} - \frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)} + (n-3)\frac{1-f_2'(t)^2}{f_2(t)^2} = \lambda. \tag{10}$$

6 Curvature operator is positive

In order to later use the Böhm-Wilking rounding theorem, we need to show that at any point the sum of the two smallest eigenvalues is positive. In fact we are able to show they are all positive on $\overline{M}$ which we recall we can identify with $\mathbb{S}^n$. That is, we claim the eigenvalues are positive for all $t \in (0, T)$, $t = 0$ and $t = T$.

Firstly, we make use of Theorem 1.84 in [1] which is a result of the Bochner theorem. We use it to ensure positivity of λ.

Theorem 6.1. *Let (M, g) be a compact Riemannian manifold and its Ricci curvature is nonpositive, then any Killing vector fields are parallel and the connected component of the isometry group is a torus.*

As $\mathbb{S}^n$ is compact, suppose it has nonpositive Ricci curvature. As $\mathbb{S}^n$ is invariant under our group actions of $SO(2) \times SO(n-1)$, the isometry group contains $SO(2) \times SO(n-1)$. Moreover, as $SO(2) \times SO(n-1)$ has a sub manifold diffeomorphic to the connected manifold $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and $n \geq 4$, the connected component of the isometry group is not a torus. Thus (M, g) and hence $(\mathbb{S}^n, \tilde{g})$ must have positive Ricci curvature and thus if they are Einstein then $\lambda > 0$.

As $\lambda > 0$ we can begin to analyse equations (8), (9) and (10) to show positivity of the curvature operator, but first we construct a lemma for use below.

Lemma 6.1. *Choose $L > 0$, and let $f : [0, L] \to \mathbb{R}$, $h : [0, L) \to \mathbb{R}$ be smooth functions with $f(0) > 0$. If*

$$\frac{df}{dt} = h(t)f(t) \text{ for all } t \in [0, L)$$

then f is positive on $[0, L)$. Similarly if h is smooth on $[0, L]$ and the ODE is satisfied on $[0, L]$ then f is positive on all $[0, L]$.

Proof. Suppose for sake of contradiction there exists a point where f is non positive, then by intermediate value theorem there exists a $c \in (0, L)$ such that $f(c) = 0$. Using separability the well known solution is

$$f(t) = Ae^{\int_0^t h(\tau)\, d\tau}$$

where $A \in \mathbb{R}$ a fixed constant. By setting $t = 0$, we find $A = f(0) > 0$. Note that as h is smooth on $[0, c]$ it is integrable and bounded here. Thus, set $\beta = \inf_{t \in [0,c]} h(t)$, then

$$0 = f(c) \geq Ae^{c\beta} > 0$$

a contradiction. The same argument holds for $c = L$ if h is smooth on $[0, L]$. □

To show the eigenvalue of multiplicity 1 is positive on $[0, L]$, we first give a variable transformation. Set $L_i = \frac{f_i'}{f_i}$, $R_i = \frac{1}{f_i}$ and $\xi = L_1 + (n-2)L_2$, then equations (8), (9) and (10) become,

$$\xi' = -L_1^2 - (n-2)L_2^2 - \lambda; \tag{11}$$

$$L_1' = -\xi L_1 - \lambda; \tag{12}$$

$$L_2' = -\xi L_2 + (n-3)R_2^2 - \lambda; \text{ and} \tag{13}$$

$$R_2' = -L_2 R_2. \tag{14}$$

See Appendix 6 for details. We now have a proposition, which resembles Proposition 1 from [6], regarding the eigenvalue of multiplicity 1.

Proposition 6.1. $-\dfrac{f_1''}{f_1}$ *is positive on* $[0,T]$.

Proof. Define $K := -\frac{f_1''}{f_1}$. We investigate the smoothness of K, L_1 and L_2 for use later. We note that as f_i is smooth and positive on $(0,T)$, all these transformed variables are smooth on $(0,T)$. Moreover, as f_1 can be smoothly extended to $[0,T]$, we only need to show existence and compatibility of limits of transformed variables at 0 and T to show smoothness.

K is smooth at 0 as $f_1(0) > 0$. At T, we use L'Hôpitals rule to show $\lim_{t\to T} K(t) = -f_1'''(T)$, which must exist as f_1 is smooth and hence K is smooth at T.

L_1 is also smooth at 0 as $f_1(0) > 0$, moreover $L_1(0) = 0$ as $f'(0) = 0$. Also as $f_1(T) = 0$, $f_1'(T) = -1$ and $f_1 > 0$ on $(0,T)$, $L_1(t) \to -\infty$ as $t \to T$. Thus L_1 smooth on $[0,T)$ and similarly, we find L_2 is smooth on $(0,T]$, where instead $L_2(t) \to \infty$ as $t \to 0$.

We now use boundary conditions to find the value of $K(0)$ and we can then show $K > 0$ for all $t \in (0,T]$. In the following calculations, we denote $O_j(t)$ as series expansions with only terms of degree 2 and above. We will calculate $L_1'(0)$ in order to find $K(0)$, and we require the behaviour of ξ near 0. For small t we find

$$\xi(t) = (n-2)\frac{f_2'(t)}{f_2(t)} + L_1(t) = \frac{n-2+O_3(t)}{t+O_2(t)} + L_1'(t)t + O_1(t),$$

as $L_1(0) = 0$, $f_2(0) = 0$, $f_2'(0) = 1$ and $f_2''(0) = 0$. Thus, by equation (12),

$$\begin{aligned} L_1'(0) &= \lim_{t\to 0} -\xi(t)L_1(t) - \lambda \\ &= -\lambda - \lim_{t\to 0}\left(\left(\frac{n-2+O_3(t)}{t+O_2(t)} + L_1'(t)t + O_1(t)\right)\cdot\left(L_1'(t)t + O_1(t)\right)\right) \\ &= -\lambda - \lim_{t\to 0} L_1'(t)\frac{n-2+O_3(t)}{1+\frac{O_2(t)}{t}} \\ &= -\lambda - (n-2)L_1'(0) \\ L_1'(0) &= \frac{-\lambda}{n-1}. \end{aligned}$$

Thus, by smoothness $L_1(t) < 0$ on $(0,\varepsilon)$ for some $\varepsilon \in (0,T]$. Note that, by simple calculation,

$$L_1' = -K - L_1^2 \tag{15}$$

and hence, $K(0) = -L_1'(0) - L_1(0)^2 = \frac{\lambda}{n-1} > 0$. Before we can analyse the sign of K and L_1 on $(0,T)$ we need to develop an expression for K'. First, note that by using equation (11), equation (15) is equivalent to $K = \xi L_1 + \lambda - L_1^2$. Now, differentiating $K = \xi L_1 + \lambda - L_1^2$ using equation (11) and (15) gives

$$K' = -(n-2)KL_2 - (n-2)L_1L_2^2. \tag{16}$$

We now claim that K is positive on $(0, T)$, moreover we claim that L_1 is negative on $(0, T)$. We will also show K is positive at T.

Suppose for sake of contradiction, there exists a non positive point of $-L_1$ or K on $[0, T)$. Note that as $-L_1(0) = K(0) > 0$ and L_1 and K are smooth on this interval, then by intermediate value theorem at least one of L_1 and K must vanish on $(0, T)$. We consider 2 cases, L_1 does not vanish before K or L_1 vanishes before K.

Suppose that L_1 vanishes before K, that is c_1 is the first point such that $L_1(c_1) = 0$ and $K > 0$ on $[0, c_1]$. But then by equation (15), $L_1'(c_1) = -K(c_1) < 0$, so c_1 is not the first vanishing point of L_1. Indeed, $L_1(c_1 - \varepsilon_0) > 0$ for some small ε_0 and by intermediate value theorem there exists a $c_2 \in (0, c_1 - \varepsilon_0)$ such that $L_1(c_2) = 0$.

Instead suppose L_1 does not vanish before K, then set c_0 as the first point such that $K(c_0) = 0$ which exists by intermediate value theorem. Hence, by assumption, $L_1 < 0, K > 0$ on $[0, c_0)$, so letting $\alpha \in (0, c_0)$, $K(\alpha) > 0$. As $L_1 \leq 0$ on $[0, c_0]$, by equation (16), $K' \geq -(n-2)L_2 K$. Let γ be the solution to the IVP $\gamma' = -(n-2)L_2\gamma$ on the interval $[\alpha, T]$ where $\gamma(\alpha) = K(\alpha) > 0$. Then clearly, $K \geq \gamma$ on $[\alpha, T]$. However, by Lemma 6.1 as L_2 is smooth on $[\alpha, T]$, $K(c_0) \geq \gamma(c_0) > 0$.

Moreover, the same argument shows that $K(T) \geq \gamma(T) > 0$. □

We can now use the above result to obtain a proposition about the eigenvalues of multiplicity $n-2$.

Proposition 6.2. *Both* $-\dfrac{f_2''}{f_2}$ *and* $\dfrac{f_1'f_2'}{f_1f_2}$ *are positive on* $[0, T]$.

Proof. We note that by equation (8) and (9),

$$-\frac{f_2''}{f_2} = -\frac{f_1'f_2'}{f_1f_2} \tag{17}$$

$$f_2'' = L_1 f_2'.$$

So it suffices to show positivity for only one of these eigenvalues at any point. We first show positivity on the interior of the interval. Note that we showed L_1 is smooth on $[0, T)$ and negative on $(0, T)$ in Proposition 6.1. Thus, as $f_2'(0) = 1 > 0$ by the boundary conditions, Lemma 6.1 hence shows $f_2' > 0$ on $[0, T)$. Therefore, as $f_2 > 0$ and $L_1 < 0$ on $(0, T)$,

$$-\frac{f_1'f_2'}{f_1f_2} = -L_1\frac{f_2'}{f_2} > 0 \text{ on } (0, T).$$

We now check the boundaries. At 0 as $f_1'(0) = 0$ and $f_2(0) = 0$ and f_1, f_2, f_1', f_2' are smooth on $[0, T]$ we can use L'Hôpitals rule,

$$\begin{aligned}\lim_{t\to 0} -\frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)} &= \lim_{t\to 0} -\frac{f_1''(t)f_2'(t) + f_1'(t)f_2''(t)}{f_1'(t)f_2(t) + f_1(t)f_2'(t)} \\ &= \lim_{t\to 0} -\frac{f_1''(t)f_2'(t)}{f_1(t)f_2'(t)} = \lim_{t\to 0} K(t) = \frac{\lambda}{n-1} > 0\end{aligned}$$

as per Proposition 6.1. To show the eigenvalues are positive at T we note that by boundary conditions $f_2(T) > 0$, so is suffices to show $f_2''(T) < 0$. As we have shown $f_2''(t) < 0$ on $(0, T)$, by smoothness

$f_2''(T) \leq 0$. Moreover, combining (17) and (10) and rearranging we get an ODE,

$$f_2''(t) = (n-3)\frac{1 - f_2'(t)^2}{2f_2(t)} - \lambda \frac{f_2(t)}{2}.$$

As $f_2(T) > 0$, by uniqueness and existence of 2nd order ODEs, for fixed $\alpha := f_2(T) > 0$, there exists a unique solution on a neighbourhood about T. Suppose $f_2''(T) = 0$, then $f_2(t) = \alpha = \sqrt{\frac{n-3}{\lambda}}$ solves this equation. However, then $f_2'(t) = 0$ near T, but as above $f_2'(t) > 0$ for all $t \in (0, T)$, a contradiction, and thus $f_2''(T) < 0$. □

We now can use Proposition 6.1 and 6.2 to show a third proposition which is closely follows Proposition 2 from [6], to show the final eigenvalue is positive.

Proposition 6.3. $\dfrac{1 - f_2'^2}{f_2^2}$ *is positive on* $[0, T]$.

Proof. For $t \in (0, T]$, it suffices to show that $f_2' \in (-1, 1)$ on $(0, T]$ as $f_2 > 0$ here. In fact at T, $f_2'(T) = 0$ so we only need to investigate the open interval and later 0. We first note that by Proposition 6.2, $f_2' > 0$ on $[0, T)$ so we only need to show $f_2' < 1$. Note by equation (9),

$$-\lambda f_2'(t) = (n-2)\frac{f_2''(t) f_2'(t)}{f_2(t)} + \frac{f_1''(t) f_2'(t)}{f_1(t)}.$$

If we multiply equation (10) by f_2, and differentiate it, using the expression above for $-\lambda f_2'(t)$ we find

$$0 = -f_2'''(t) - \frac{f_1'(t) f_2''(t)}{f_1(t)} + \frac{f_1'(t)^2 f_2'(t)}{f_1(t)^2} + (n-4)\frac{-f_2'(t) f_2''(t)}{f_2(t)} + (n-3)\frac{f_2'(t)(f_2'(t)^2 - 1)}{f_2(t)^2}.$$

Suppose for sake of contradiction $f_2'(c_0) \geq 1$ for some $c_0 \in (0, T)$. As $f_2'(0) = 1$, $f_2'(T) = 0$ and f_2' is smooth on $[0, T]$ it is bounded and has a maximum at $t_0 \in (0, T)$ where $f_2'(t_0) \geq f_2'(c_0) \geq 1$, $f_2''(t_0) = 0$ and $f_2'''(t_0) \leq 0$. Thus substituting in these inequalities we find

$$\begin{aligned} 0 &= -f_2'''(t_0) - \frac{f_1'(t_0) f_2''(t_0)}{f_1(t_0)} + \frac{f_1'(t_0)^2 f_2'(t_0)}{f_1(t_0)^2} + (n-4)\frac{-f_2'(t_0) f_2''(t_0)}{f_2(t_0)} + (n-3)\frac{f_2'(t_0)(f_2'(t_0)^2 - 1)}{f_2(t_0)^2} \\ &\geq \left(\frac{f_1'(t_0)}{f_1(t_0)}\right)^2 > 0 \end{aligned}$$

as $L_1(t_0)^2 > 0$ by Proposition 6.1. Thus, we found a contradiction and hence, $f_2' \in (0, 1)$ on $(0, T)$. Consequently the eigenvalue is positive on $(0, T)$.

Lastly at 0, as $f_2'(0) = 1$ and $f_2(0) = 0$ we use L'Hôpitals rule

$$\lim_{t \to 0} \frac{1 - f_2'(t)^2}{f_2(t)^2} = \lim_{t \to 0} \frac{-2f_2'(t) f_2''(t)}{2f_2(t) f_2'(t)} = \lim_{t \to 0} -\frac{f_2''(t)}{f_2(t)} > 0$$

by Proposition 6.2. □

Together, Propositions 6.1, 6.2 and 6.3 show that the curvature operator is positive and hence 2-positive. Consequently, we can use the Böhm-Wilking rounding theorem [2] stated below.

Theorem 6.2 (Böhm-Wilking rounding theorem)**.** *On a compact manifold the normalised Ricci flow evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.*

We recall that the extension of g to $\mathbb{S}^n$ was $\tilde{g}$ as in equation (3) and Lemma 4.1. Note that using the diffeomorphism, $(\mathbb{S}^n, \tilde{g})$ also has a positive curvature operator. As stated in Section 2, Einstein manifolds are fixed point solutions to the normalised Ricci flow. Thus, as $(\mathbb{S}^n, \tilde{g})$ is Einstein, by the Böhm-Wilking rounding theorem, it must have constant sectional curvature.

It is well know that any compact, simply-connected smooth manifold of dimension n with constant positive sectional curvature is isometric to a round metric on $\mathbb{S}^n$. Therefore, as $n \geq 4 \geq 2$, $\mathbb{S}^n$ is a compact, simply-connected smooth manifold and thus $\tilde{g}$ must be round. We note that the round metric of Ricci curvature $n-1$ on $\mathbb{S}^n$ can be achieved by setting $T = \frac{\pi}{2}$, $f_1(t) = \cos(t)$ and $f_2(t) = \sin(t)$. It is easy to see that the conditions of Lemma 4.1 and equations (8), (9) and (10) are satisfied as expected.

7 Discussion and conclusion

In this report we introduced the relevant notions of manifolds and Riemannian geometry, constructed a cohomogenity one group action and a class of invariant metrics. We then gave constraints such that the manifold was Einstein and showed that it must be round for $n \geq 4$. The Lie group we investigated was $SO(2) \times SO(n-1)$ and we studied the class of doubly warped metrics on $I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ that can be extended to $\mathbb{S}^n$.

It is interesting to note that Böhm was able to find infinitely many Einstein metrics on $\mathbb{S}^m$ for $5 \leq m \leq 9$ that were invariant under any cohomogenity one action of the form $SO(l) \times SO(k)$ where $l, k \geq 3$. The work done above cannot be extended to such a group action as the Einstein equations include a fifth eigenvalue of the curvature operator which allows for non-positive eigenvalues.

Possible further research may include loosening the constraints of Einstein to a Ricci soliton, similar to [6]. Other possible areas of research may include investigating other classes of metrics over spheres or trying to apply the Böhm-Wilking rounding theorem to other 4 dimensional compact manifolds such as $CP_2 \# \overline{CP_2}$.

8 Bibliography

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9 Appendix

9.1 Appendix 1: Smooth Manifolds

In this appendix we present smooth manifolds as in [12].

Definition 9.1. We define a smooth manifold of dimension n (sometimes denoted M^n) as a second countable Hausdorff set M and a collection of injective maps $\varphi_\alpha : \Omega_\alpha \to \mathbb{R}^n$, $\Omega_\alpha \subseteq M$ open for all $\alpha \in I$ an indexing set, which satisfies the following properties:

1. $\bigcup_{\alpha \in I} \Omega_\alpha = M$;
2. if $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$ then $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\Omega_\alpha \cap \Omega_\beta) \to \varphi_\alpha(\Omega_\alpha \cap \Omega_\beta)$ is smooth; and
3. $\{(\varphi_\alpha, \Omega_\alpha)\}$ is maximal.

For terminology, a pair $(\varphi_\alpha, \Omega_\alpha)$ is called a chart and a collection $\{(\varphi_\alpha, \Omega_\alpha) : \alpha \in \Lambda\}$ for indexing set $\Lambda \subseteq I$ such that the collection covers M is called an atlas.

Remark. Any atlas satisfying the first two conditions of Definition 9.1 can be extended to be maximal, that is, such that the collection $\{(\varphi_\alpha, \Omega_\alpha)\}$ is not contained in a larger atlas.

Example 9.1. A trivial example of a smooth manifold of dimension n is simply $\mathbb{R}^n$, where the charts are all of the form (I_d, U), where I_d is the identity and $U \subseteq \mathbb{R}^n$ is an open set.

Example 9.2. Another example of an smooth manifold is $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$. There are many ways to develop charts for $\mathbb{S}^n$, but for this example we use stereographic projection to find an atlas for $\mathbb{S}^2$.

Let $(x_1, x_2, x_3) \in \mathbb{S}^2$, $\Omega_N := \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 < 1\}$, $\Omega_S := \{(x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 > -1\}$ and

$$\varphi_N(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right) \text{ and } \varphi_S(x_1, x_2, x_3) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right),$$

then indeed $\{(\varphi_N, \Omega_N), (\varphi_S, \Omega_S)\}$ is an atlas and it will satisfy conditions 1 and 2 from Definition 2.1 when checked. For example $\varphi_S \circ \varphi_N^{-1}(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ is smooth on $\varphi_S(\Omega_N \cap \Omega_S) = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Using charts and their inverses can allow functions on manifolds to be given locally as maps between Euclidean spaces. As differentiability of functions, $f : \mathbb{R}^m \to \mathbb{R}^n$, is well known, the notion of a differentiability on smooth manifolds is then given ensuring differentiability of the analogous function in Euclidean space.

Definition 9.2. A function $f : M \to N$ from a smooth manifold M^m to another N^n is smooth at $p \in M$ if for all charts $(\varphi_\alpha, \Omega_\alpha)$, $(\varphi_\beta, \Omega_\beta)$ such that $p \in \Omega_\alpha$, $f(p) \in \Omega_\beta$, the function

$$\varphi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(\Omega_\alpha) \to \mathbb{R}^n$$

$\varphi_\alpha(\Omega_\alpha) \subseteq \mathbb{R}^m$, is smooth at $\varphi_\alpha(p)$. The function f is said to be smooth if it is smooth at q for all $q \in M$. The set of all smooth functions from M to $\mathbb{R}$ is denoted by $C^\infty(M)$.

Remark. The function f is smooth at p if such conditions hold in Definition 9.2 for one such chart of M and N as transition maps are smooth by condition 2 of Definition 2.1.

Definition 9.3. Let M^n be a smooth manifold. A tangent vector to M at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule, that is for all $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$ we have:

1. $v(af + bg) = av(f) + bv(g)$; and
2. $v(fg)(p) = v(f)(p) \cdot g(p) + f(p) \cdot v(g)(p)$.

We denote the set of all tangent vectors to M at p as T_pM. The set of tangent vectors is called the tangent bundle denoted $TM := \{(v, p) : p \in M,\ v \in T_pM\}$.

Remark. One can show that for given a given tangent vector v at p, $v(f) = v(g)$ if $f = g$ locally about p. Thus, this definition only requires smoothness on a neighbourhood of p.

Remark. Let M be a smooth manifold, $\varepsilon > 0$ and $\alpha : (-\varepsilon, \varepsilon) \to M$ a smooth function such that $\alpha(0) = p \in M$. Then consider the tangent vector $v = \alpha'(0)$, which acts on functions $f \in C^\infty(M)$ by

$$\alpha'(0)f := \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}.$$

Moreover all tangent vectors can be generated this way. Intuitively, this is the extension of directional derivative to manifolds. That is, the vector $\alpha'(0) = v \in T_pM$ acts on smooth functions by taking their derivative along α at p.

Remark. We can define addition and scalar multiplication as standard: for tangent vectors $v_1, v_2 \in T_pM$, $a \in \mathbb{R}$ and $f \in \mathbb{C}^\infty(M)$, $(v_1 + v_2)(f) := v_1(f) + v_2(f)$ and $(a \cdot v_1)(f) = a \cdot v_1(f)$. Then it is clear that the tangent plane T_pM is a vector space. Moreover, for a fixed chart containing p, $\varphi = (x^1, ..., x^n)$, the collection $\left\{(\partial_{x^i})_p\right\}_{i=1}^n$ forms a basis of the tangent plane at p, where the coordinate maps act as vectors via the previous remark. It is standard notation to denote $\partial_{x^i} = \frac{\partial}{\partial x^i}$ and this will be used below too. Moreover we will use the Einstein summation convention, whereby we implicitly sum over indexed terms in an equation.

In order to analyse tangent vectors over the entire manifold, we define *vector fields.*

Definition 9.4. A vector field X in M, a smooth manifold, is a map $X : M \to TM$ such that $X_p := X(p) \in T_pM$. A vector field is smooth if the map $X : M \to TM$ is smooth and the set of smooth vector fields is denoted as $\mathfrak{X}(M)$.

9.2 Appendix 2:

In this appendix we introduce the tensor product and bivectors. Tensors are used widely and the tensor product is very natural operator. Bivectors are needed to define the curvature operator.

Firstly, we recall the definition of a *tensor.*

Definition 9.5. Given a smooth manifold M, a (p,q) tensor, T, is a multilinear map which maps p vectors and q covectors to $\mathbb{R}$ at each tangent plane. That is for all $x \in M$, then $T_x : (T_xM)^q \times (T_x^*M)^p \to \mathbb{R}$.

We now introduce the *tensor product* which takes two tensors of order (p,q) and (l,k) and gives a tensor of order $(p+l,q+k)$.

Definition 9.6. Given a smooth manifold M, a (p,q) tensor, T_1, and a (l,k) tensor, T_2, then the tensor product is the $(p+l,q+k)$ tensor $T_1 \otimes T_2$ defined by:

$$T_1 \otimes T_2(v_1, ..., v_{q+k}, w_1, ..., w_{p+l}) = T_1(v_1, ..., v_q, w_1, ..., w_p)T_2(v_{q+1}, ..., v_{q+k}, w_{p+1}, ..., w_{p+l})$$

where $v_i \in T_xM$, $w_j \in T_x^*M$.

We now introduce *bivectors*, a common tool used to describe planes. There are many different definitions, but we define them as equivalence classes of tensor products.

Definition 9.7. As vectors naturally are $(1,0)$ tensors, we consider the subset of tensors $S_p = \{(u \otimes v) : u, v \in T_pM\}$. Then we define bivectors as the equivalence classes of S_p, that is, $u \wedge v := [u \otimes v]$ under the equivalence relation $u \otimes v \sim x \otimes y$ if $x \otimes y = -y \otimes x$. The second exterior power of T_pM, denoted $\Lambda_p^2 M$, is the vector field obtained by extending the set of bivectors via addition. We also define the extension of the metric to bivectors as

$$g(X \wedge Y, U \wedge V) := g(X,U)g(Y,V) - g(X,V)g(Y,V).$$

Similarly, $\Lambda^2 M$ is then defined over smooth vector fields.

Remark. Although one can still interpret bivectors as maps, it is more convenient to consider them as bilinear skew symmetric representations of planes. We note that we get bilinearity via the tensor product.

Remark. As the wedge product is a bilinear and antisymmetric map, $\Lambda_p^2 M$ is spanned by $(x_i \wedge x_j)_{i<j}$ where $(x_i)_{i=1}^n$ is a basis of T_pM. Moreover, if $(e_i)_{i=1}^n$ is a local orthonormal basis of $\mathfrak{X}(M)$ then $(e_i \wedge e_j)_{i<j}$ will form a local orthonormal basis for $\Lambda^2 M$.

9.3 Appendix 3:

Here we collate calculations from Section 2 referenced for definitions. First we show that indeed the Lie bracket is a vector field. It is clearly linear, so we show the Leibniz rule. Let $f, g \in C^\infty(M)$, $u = X_p$ and $v = Y_p$ for $p \in M$ then

$$\begin{aligned}[u,v](fg)(p) &= u(v(fg))(p) - v(u(fg))(p) \\ &= u(v(f)\cdot g + f \cdot v(g))(p) - v(u(f)\cdot g + f\cdot u(g))(p) \\ &= u(v(f))(p)\cdot g(p) + u(v(g))(p)\cdot f(p) - v(u(f))(p)\cdot g(p) - v(u(g))(p)\cdot f(p) \\ &= [u,v](f)(p)\cdot g(p) + f(p)\cdot [u,v](g)(p).\end{aligned}$$

We now show that the (1,3) Riemannian curvature tensor is indeed tensorial in all its components. Firstly we note that, by definition of the Levi-Civita connection, $[fX, Y] = f\nabla_X Y - f\nabla_Y X - Y(f)X = f[X, Y] - Y(f)X$. Hence,

$$R(fX, Y)Z = \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z$$

$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z$$

$$= fR(X, Y)Z.$$

Similarly we find $R(X, fY)Z = fR(X, Y)Z$. Lastly,

$$R(X, Y)fZ = \nabla_X\nabla_Y fZ - \nabla_Y\nabla_X fZ - \nabla_{[X,Y]}fZ$$

$$= \nabla_X f\nabla_Y Z + \nabla_X Y(f)Z - \nabla_Y X(f)Z - \nabla_Y f\nabla_X Z - f\nabla_{[X,Y]}Z - [X, Y](f)Z$$

$$= f\nabla_X\nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z - Y(X(f))Z - X(f)\nabla_Y Z$$

$$-Y(f)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z - [X, Y](f)Z$$

$$= fR(X, Y)Z.$$

Lastly, we show that $\mathrm{Ric}^{g_0} = \mathrm{Ric}^{g_t}$ for Example 2.3. That is, the Ricci curvature does not change under a positive scaling of the metric. Firstly, by scaling the metric, $\hat{g} = cg$ for $c > 0$, the Levi-Civita connection $\widehat{\nabla}_X Y = \nabla_X Y$ is unchanged by the Koszul formula. Hence, $\widehat{R}(X, Y)Z = R(X, Y)Z$, $\widehat{R}(X, Y, Z, W) = cR(X, Y, Z, W)$. Given a local orthonormal basis $(e_i)_{i=1}^n$ of smooth vector fields, in g, then $(\hat{e}_i)_{i=1}^n$ is a local orthonormal basis in $\hat{g}$ where $\hat{e}_i = \frac{e_i}{\sqrt{c}}$, and so

$$\widehat{\mathrm{Ric}}(X, Y) = \widehat{R}(X, \hat{e}_i, \hat{e}_i, Y) = R(X, e_i, e_i, Y) = \mathrm{Ric}(X, Y).$$

9.4 Appendix 4:

In this appendix we show that the diffeomorphism given at the end of Section 3 is indeed a diffeomorphism. Recall it was given by,

$$\Phi(t, \theta, s) = \left(\cos\left(\frac{\pi t}{2T}\right)\theta, \sin\left(\frac{\pi t}{2T}\right)s\right)$$

$$= \left(\cos\left(\frac{\pi t}{2T}\right)\theta_1, \cos\left(\frac{\pi t}{2T}\right)\theta_2, \sin\left(\frac{\pi t}{2T}\right)s_1, ..., \sin\left(\frac{\pi t}{2T}\right)s_{n-1}\right)$$

for $t \in (0, T)$, $\theta = (\theta_1, \theta_2) \in \mathbb{S}^1$, $s = (s_1, ..., s_{n-1}) \in \mathbb{S}^{n-2}$. Note that $|\Phi(t, \theta, s)|^2 = \cos^2\left(\frac{\pi t}{2T}\right)|\theta| + \sin^2\left(\frac{\pi t}{2T}\right)|s| = 1$, so $\Phi(I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}) \subseteq \mathbb{S}^n$ and indeed we find $\Phi(I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}) = \mathbb{S}^n \setminus P$, where $P = (\{0\}^2 \times \mathbb{S}^{n-2}) \cup (\mathbb{S}^1 \times \{0\}^{n-1})$.

We now claim that the inverse is given by,

$$\Phi^{-1}(z_1, z_2, ...z_{n+1}) = \left(\frac{2T}{\pi}\arccos(|\theta|), \frac{\theta}{|\theta|}, \frac{s}{|s|}\right)$$

$$= \left(\frac{2T}{\pi} \arccos(\sqrt{z_1^2+z_2^2}), \frac{z_1}{\sqrt{z_1^2+z_2^2}}, \frac{z_2}{\sqrt{z_1^2+z_2^2}}, \frac{z_3}{\sqrt{z_3^2+...+z_{n+1}^2}}, ..., \frac{z_{n+1}}{\sqrt{z_3^2+...+z_{n+1}^2}} \right)$$

for $(z_1, z_2, ..., z_{n+1}) \in \mathbb{S}^n \setminus P$ and $\theta = (z_1, z_2),\ s = (z_3, ..., z_{n+1})$. Note that as we remove the poles from $\mathbb{S}^n$, $0 < |\theta|, |s| < 1$ and arccos is well defined and smooth on this interval. Clearly $\frac{\theta}{|\theta|} \in \mathbb{S}^1$, $\frac{s}{|s|} \in \mathbb{S}^{n-2}$ and indeed we find that the image, $\mathrm{Im}(\Phi^{-1}) = I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ covers M. We do check that it is indeed an inverse map,

$$\begin{aligned} \Phi^{-1}(\Phi(t,\theta,s)) &= \Phi^{-1}\left(\cos\left(\frac{\pi t}{2T}\right)\theta, \sin\left(\frac{\pi t}{2T}\right)s\right) \\ &= (t,\theta,s). \end{aligned}$$

Lastly when we consider the natural structure on $I \times \mathbb{S}^1 \times \mathbb{S}^{n-2}$ and $\mathbb{S}^n$ then we find that Φ and Φ^{-1} are differentiable.

9.5 Appendix 5:

In this appendix we detail some necessary but long calculations to ultimately find the curvature operator in Section 4. We will require definitions not given in Section 2 about the Hessian and distance functions from [13].

Definition 9.8. Let (M, g) be a Riemannian manifold and $r \in C^\infty(M)$ a function. The Hessian of r is a symmetric (0,2) tensor defined by

$$\mathrm{Hess}\, r(X,Y) = g(S(X),Y) = g(\nabla_X \nabla_r, Y)$$

where ∇_r is the gradient vector field of r, defined by the vector field such that $g(Z, \nabla_r) = Z(r)$ for all $Z \in \mathfrak{X}(M)$. The operator $S : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called the shape operator.

Remark. The gradient vector field of a coordinate function x_i for a given chart is $\nabla_{x_i} = \partial_{x_i}$.

Definition 9.9. Given a Riemannian manifold (M, g), a function $r \in C^\infty(M)$ is a distance function on M if $g(\nabla_r, \nabla_r) = 1$ identically. The level sets of such a function are defined as $M_r := \{x \in M : r(x) = r\}$ and its induced metric is denoted g_r.

For convenience in sections below we will adopt the notation $g_{\mathbb{S}^1} := d\theta^2$, $g_{\mathbb{S}^{n-2}} := ds^2$, $g_1 := f_1(t)^2 d\theta^2$, $g_2 := f_2(t)^2 ds^2$ and $g_t = g_1 + g_2$ for fixed $t \in (0, T)$. We also note that $t : M \to I \subset \mathbb{R}$ is a distance function, as $\nabla_t = \partial_t$ and $g(\partial_t, \partial_t) = 1$. We then find that g_t is indeed the induced metric over the level sets of M_t.

Before we can do any calculations, we introduce a lemma, which is analogous to an exercise in [8].

Lemma 9.1. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with Levi-Civita connections ∇^1 and ∇^2 respectively. Then, the Levi-Civita connection on $(M_1 \times M_2, g_1 + g_2)$ for any vector fields $X_1, Y_1 \in \mathfrak{X}(M_1 \times M_2)$ tangent to and only dependant on M_1 and $X_2, Y_2 \in \mathfrak{X}(M_1 \times M_2)$ tangent to and only dependant on M_2 is*

$$\nabla_{Y_1+Y_2}(X_1 + X_2) = \nabla^1_{Y_1} X_1 + \nabla^2_{Y_2} X_2.$$

Remark. We define a vector $v \in T_p(M_1 \times M_2)$ as tangent to M_1, if $v(f) = 0$ for all functions which only depend on M_2. Note that we can extend tangent vectors of M_1 to tangent vectors of $M_1 \times M_2$ tangent to M_1 as below. Let $X \in T_pM_1$, then we can define $\widetilde{X} \in T_p(M_1 \times M_2)$ by $\widetilde{X}(f) = X(\tilde{f})$, where $\tilde{f} \in C^\infty(M_1)$ is $\tilde{f}(x) = f(x, p_2)$ and $p = (p_1, p_2)$. Note that we implicitly used this when constructing the local orthonormal basis in Section 4.

We now can compute the Hessian of t and its covariant derivative as they are in many curvature formulas and will then help in calculations below. We note ∂_t vanishes the shape operator and hence for $X = \partial_t$ or $Y = \partial_t$, the Hessian vanishes. Indeed we calculate

$$\mathrm{Hess}\, t(\partial_t, Y) = g(\nabla_{\partial_t}\nabla_t, Y) = g(\nabla_Y \nabla_t, \partial_t) = g(\nabla_Y \partial_t, \partial_t)$$

$$= \frac{1}{2}(Y(g(\partial_t, \partial_t)) + \partial_t(g(\partial_t, Y)) - \partial_t(g(\partial_t, Y)) + g([Y, \partial_t], \partial_t) - g([Y, \partial_t], \partial_t) - g([\partial_t, \partial_t], Y))$$

$$= \frac{1}{2}Y(1) - \frac{1}{2}g(0, Y) = 0$$

by the Koszul formula. Thus as the Hessian is a tensor we only need to consider the Hessian for vector fields without a ∂_t component (i.e. $g(X, \partial_t) = 0$). Moreover, we only need to consider a basis for vectors tangent to $\mathbb{S}^1$ or $\mathbb{S}^{n-2}$. Set $\hat{e_1} = \frac{e_1}{f_1(t)}$ and $\hat{e_i} = \frac{e_i}{f_2(t)}$ for all $i \geq 2$. Then $(\hat{e_i})_{i=1}^{n-1}$ is a local orthonormal basis for $(\mathbb{S}^1 \times \mathbb{S}^{n-2}, g_{\mathbb{S}^1} + g_{\mathbb{S}^{n-2}})$ and hence not dependant on t. Thus, we create the functions $h_i^X, h_i^Y \in C^\infty(M)$ such that $X = \sum_{i=1}^{n-1} h_i^X \hat{e_i}$ and $Y = \sum_{i=1}^{n-1} h_i^Y \hat{e_i}$ locally. Then, remembering we adopt the Einstein summation convention,

$$\mathrm{Hess}\, t(X, Y) = h_i^X h_j^Y \mathrm{Hess}\, t(\hat{e_i}, \hat{e_j}) = h_i^X h_j^Y g(\nabla_{\hat{e_i}} \partial_t, \hat{e_j})$$

$$= h_i^X h_j^Y \frac{1}{2}\left(\hat{e_i}(g(\partial_t, \hat{e_j})) + \partial_t(g(\hat{e_i}, \hat{e_j})) - \hat{e_j}(g(\hat{e_i}, \partial_t)) + g([\hat{e_i}, \partial_t], \hat{e_j}) - g([\partial_t, \hat{e_j}], \hat{e_i}) - g([\hat{e_i}, \hat{e_j}], \partial_t)\right)$$

$$= h_i^X h_j^Y \left(\frac{1}{2}\partial_t(g(\hat{e_i}, \hat{e_j})) - \frac{1}{2}g([\partial_t, \hat{e_i}], \hat{e_j}) - \frac{1}{2}g([\partial_t, \hat{e_j}], \hat{e_i})\right)$$

$$= h_i^X h_j^Y \frac{1}{2}\partial_t(g(\hat{e_i}, \hat{e_j}))$$

$$= h_i^X h_j^Y \frac{1}{2}\partial_t(f_1(t)^2 g_{\mathbb{S}^1}(\hat{e_i}, \hat{e_j}) + f_2(t)^2 g_{\mathbb{S}^{n-2}}(\hat{e_i}, \hat{e_j}))$$

$$= h_i^X h_j^Y (f_1'(t) f_1(t) g_{\mathbb{S}^1}(\hat{e_i}, \hat{e_j}) + f_2'(t) f_2(t) g_{\mathbb{S}^{n-1}}(\hat{e_i}, \hat{e_j}))$$

$$= h_i^X h_j^Y \left(\frac{f_1'(t)}{f_1(t)} g_1(\hat{e_i}, \hat{e_j}) + \frac{f_2'(t)}{f_2(t)} g_2(\hat{e_i}, \hat{e_j})\right)$$

$$= f_1'(t) f_1(t) g_1(X, Y) + f_2'(t) f_2(t) g_2(X, Y)$$

as $g_{\mathbb{S}^1}$ and $g_{\mathbb{S}^{n-2}}$ are not dependant on t and $g([\hat{e_i}, \hat{e_j}], \partial_t)) = 0$ as $[\hat{e_i}, \hat{e_j}] = \nabla_{\hat{e_i}} \hat{e_j} - \nabla_{\hat{e_i}} \hat{e_j}$ must be tangent to $\mathbb{S}^1 \times \mathbb{S}^{n-2}$ by Lemma 9.1. Similarly, by Lemma 9.1, $[\partial_t, \hat{e_j}] = \nabla_{\partial_t} \hat{e_j} - \nabla_{\hat{e_i}} \partial_t = 0$.

Thus we see that the shape operator is defined by $S(\partial_t) = 0$, $S(e_1) = \frac{f_1'(t)}{f_1(t)} e_1$ and $S(e_j) = \frac{f_1'(t)}{f_1(t)} e_j$ for $j \geq 2$. Therefore $\mathrm{Hess}^2\ t(\cdot, \cdot) = g(S(\cdot), S(\cdot)) = \left(\frac{f_1'(t)}{f_1(t)}\right)^2 g_1(\cdot, \cdot) + \left(\frac{f_2'(t)}{f_2(t)}\right)^2 g_2(\cdot, \cdot)$. This is another important term that appears in curvature equations.

The last term that is important to calculate is the covariant derivative of the Hessian. In order to do this we first note that the covariant derivative acts on tensors by,

$$\nabla_X T(Y_1, ..., Y_p, Z_1, ..., Z_q) =$$

$$X(T(Y_1, ..., Y_p, Z_1, ..., Z_q)) - T(\nabla_X Y_1, ..., Y_p, Z_1, ..., Z_q)) - ... - T(Y_1, ..., Y_p, Z_1, ..., \nabla_X Z_q))$$

Where Y_i are vector fields, and Z_i are dual vector fields. Thus, by property 5 in Definition 2.3, $\nabla_X g(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$. With this information, we can continue to calculate the covariant derivative of the Hessian,

$$\nabla_{\partial_t} \text{Hess } t = \nabla_{\partial_t} \left(\frac{f'(t)}{f_1(t)} g_1 + \frac{f_2'(t)}{f_2(t)} g_2 \right)$$

$$\begin{aligned}
&= \partial_t \left(\frac{f_1'(t)}{f_1(t)} \right) g_1 + \frac{f_1'(t)}{f_1(t)} \nabla_{\partial_t} g_1 + \partial_t \left(\frac{f_2'(t)}{f_2(t)} \right) g_2 + \frac{f_2'(t)}{f_2(t)} \nabla_{\partial_t} g_2 \\
&= \frac{f_1''(t)}{f_1(t)} g_1 - \frac{f_1'(t)^2}{f_1(t)^2} g_1 + \frac{f_2''(t)}{f_2(t)} g_1 - \frac{f_2'(t)^2}{f_2(t)^2} g_2 \\
&= -\text{Hess}^2\ t + \frac{f_1''(t)}{f_1(t)} g_1 + \frac{f_2''(t)}{f_2(t)} g_2.
\end{aligned}$$

We can finally state and use some propositions from [13].

Proposition 9.1. *If r is a distance function then,* $(\nabla_{\partial_r} \text{Hess } r)(X, Y) + \text{Hess}^2\ r(X, Y) = -R(X, \partial_r, \partial_r, Y)$.

Rearranging Proposition 9.1 and using our calculation for the covariant derivative of the Hessian we find,

$$\begin{aligned}
R(X, \partial_t, \partial_t, Y) &= -(\nabla_{\partial_t} \text{Hess } t)(X, Y) - \text{Hess}^2\ t(X, Y) \\
R(X, \partial_t, \partial_t, Y) &= -\frac{f_1''(t)}{f_1(t)} g_1(X, Y) - \frac{f_2''(t)}{f_2(t)} g_2(X, Y).
\end{aligned} \tag{18}$$

We use another proposition from [13] called the mixed and tangential curvature equations.

Proposition 9.2. *If r is a distance function, g_r is the induced metric and R^r is the Riemannian curvature over level sets then :*

1. $g(R(X, Y)V, W) = g_r(R^r(X, Y)V, W) - \Pi(Y, V)\Pi(X, W) + \Pi(X, V)\Pi(Y, W)$; *and*

2. $g(R(X, Y)Z, \partial_r) = -(\nabla_X \Pi(Y, Z)) + (\nabla_Y \Pi(X, Z))$.

where X, Y, Z are vector fields tangent to the level sets M_r.

In our case $r = t$, $\Pi = \text{Hess } t$ and $g_r = g_1 + g_2$. Moreover, X, Y, Z are any vector fields without ∂_t components. We find that the mixed curvature $\nabla_X \Pi$ vanishes and we can then use Proposition 9.1 and 9.2 to yield equations (4), (5), (6) and (7). Indeed, for any $X \in \mathfrak{X}(M)$ tangent to M_t for any fixed $t \in I$,

$$\begin{aligned}
\nabla_X \Pi &= \nabla_X \text{Hess } t \\
&= \nabla_X \left(\frac{f_1'(t)}{f_1(t)} g_1 + \frac{f_2'(t)}{f_2(t)} g_2 \right)
\end{aligned}$$

$$= X\left(\frac{f_1'(t)}{f_1(t)}\right) g_1 + \frac{f_1'(t)}{f_1(t)}\nabla_X g_1 + X\left(\frac{f_2'(t)}{f_2(t)}\right) g_2 + \frac{f_2'(t)}{f_2(t)}\nabla_X g_2 = 0.$$

Note that for the remainder of this appendix the index i will always be in the range $2 \le i \le n-1$. It is important not to confuse it with the Einstein summation convention. We will now use equation (18) to start calculating the terms of the curvature operator. We immediately find,

$$R(e_1, \partial_t, \partial_t, e_1) = -\frac{f_1''(t)}{f_1(t)} g_1(e_1, e_1) = -\frac{f_1''(t)}{f_1(t)},$$

$$R(e_i, \partial_t, \partial_t, e_i) = -\frac{f_2''(t)}{f_2(t)} g_2(e_i, e_i) = -\frac{f_2''(t)}{f_2(t)}, \text{ and}$$

$$R(e_1, \partial_t, \partial_t, e_i) = 0 = R(e_i, \partial_t, \partial_t, e_1).$$

Thus,

$$g(\mathfrak{R}(\partial_t \wedge e_1), \partial_t \wedge e_1) = R(\partial_t, e_1, e_1, \partial_t) = R(e_1, \partial_t, \partial_t, e_1) = -\frac{f_1''(t)}{f_1(t)}.$$

Similarly,

$$g(\mathfrak{R}(\partial_t \wedge e_i), \partial_t \wedge e_i) = -\frac{f_2''(t)}{f_2(t)}$$

and

$$g(\mathfrak{R}(\partial_t \wedge e_i), \partial_t \wedge e_1) = 0 = g(\mathfrak{R}(\partial_t \wedge e_i), \partial_t \wedge e_1).$$

Now, using Equation 2 in Proposition 9.2, for $1 \le l < m \le n-1$ and $1 \le j \le n-1$,

$$g(\mathfrak{R}(\partial_t \wedge e_j), e_l \wedge e_m) = -R(e_m, e_l, e_j, \partial_t) = \nabla_{e_m}\Pi(e_l, e_j) - \nabla_{e_l}\Pi(e_m, e_j) = 0.$$

Combining this with previous calculations we achieve equation (4) and (5)

$$\mathfrak{R}(\partial_t \wedge e_1) = -\frac{f_1''(t)}{f_1(t)}\partial_t \wedge e_1 \text{ and } \mathfrak{R}(\partial_t \wedge e_i) = -\frac{f_2''(t)}{f_2(t)}\partial_t \wedge e_i.$$

We continue by using Equation 1 from Proposition 3.4 and our calculation for the Hessian,

$$g(\mathfrak{R}(e_1 \wedge e_i), e_1 \wedge e_i) = R(e_1, e_i, e_i, e_1)$$

$$= g_t(R^t(e_1, e_i)e_i, e_1) - \Pi(e_i, e_i)\Pi(e_1, e_1) + \Pi(e_1, e_i)\Pi(e_i, e_1)$$
$$= -\left(-\frac{f_2'(t)}{f_2(t)} g_2(e_i, e_i) \cdot -\frac{f_1'(t)}{f_1(t)} g_1(e_1, e_1)\right)$$
$$= -\frac{f_1'(t) f_2'(t)}{f_1(t) f_2(t)}$$

as $R^t(e_1, e_i)e_i$ must be tangent to $\mathbb{S}^{n-2}$. Indeed, as t is fixed and $g_t = g_1 + g_2$, we can apply Lemma 9.1 and find

$$R^r(e_1, e_i)e_i = \nabla_{e_1}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_1}e_i - \nabla_{[e_1, e_i]}e_i = -\nabla_{g([e_1, e_i], e_j)e_j}e_i = X$$

where X is tangent to $\mathbb{S}^{n-2}$. Similarly, we use Equation 1 from Proposition 3.4 for the off diagonal terms. That is, for $2 \le j < k \le n-1$,

$$g(\mathfrak{R}(e_1 \wedge e_i), e_j \wedge e_k) = g_t(R^t(e_1, e_i)e_k, e_j) - \Pi(e_i, e_k)\Pi(e_1, e_j) + \Pi(e_1, e_k)\Pi(e_i, e_j)$$

$$= -g_t(R^t(e_k, e_j)e_i, e_1) = 0$$

where $R^t(e_k, e_j)e_i$ must also be tangent to $\mathbb{S}^{n-2}$ by Lemma 9.1.

Thus as there are no contributing $\partial_t \wedge e_j$ terms as per above calculations, we have achieved equation (6),

$$\mathfrak{R}(e_1 \wedge e_i) = -\frac{f_1'(t)f_2'(t)}{f_1(t)f_2(t)} e_1 \wedge e_i.$$

Lastly, as g_2 is the metric of curvature $\frac{1}{f_2(t)^2}$ on $\mathbb{S}^{n-2}$ and $(\hat{e_i})_{i=2}^{n-1}$ defined above is an orthonormal basis of $(\mathbb{S}^{n-2}, g_{\mathbb{S}^{n-2}})$, for $2 \leq j < k \leq n-1$, since the curvature operator is the identity on $(\mathbb{S}^{n-2}, g_{\mathbb{S}^{n-2}})$,

$$\begin{aligned} g(\mathfrak{R}(e_j \wedge e_k), e_j \wedge e_k) &= g_t(R^t(e_j, e_k)e_k, e_j) - \Pi(e_k, e_k)\Pi(e_j, e_j) + \Pi(e_j, e_k)\Pi(e_k, e_j) \\ &= f_2(t)^2 g_{\mathbb{S}^{n-2}}(R^{\mathbb{S}^{n-2}}(e_j, e_k)e_k, e_j) - \left(\frac{f_2'(t)}{f_2(t)} g_2(e_k, e_k) \cdot \frac{f_2'(t)}{f_2(t)} g_2(e_i, e_i)\right) \\ &= \frac{1}{f_2(t)^2} g_{\mathbb{S}^{n-2}}(\hat{e_j} \wedge \hat{e_k}, \hat{e_j} \wedge \hat{e_k}) - \frac{f_2'(t)^2}{f_2(t)^2} = \frac{1 - f_2'(t)^2}{f_2(t)^2}. \end{aligned}$$

Also, for $2 \leq j < k \leq n-1$, $2 \leq l < m \leq n-1$ and $l \neq j$ or $m \neq k$,

$$g(\mathfrak{R}(e_j \wedge e_k), e_l \wedge e_m) = R(e_j, e_k, e_m, e_l)$$

$$\begin{aligned} &= g_t(R^t(e_j, e_k)e_m, e_l) - \Pi(e_k, e_m)\Pi(e_j, e_l) + \Pi(e_j, e_m)\Pi(e_k, e_l) \\ &= \frac{1}{f_2(t)^2} g_{\mathbb{S}^{n-2}}(\tilde{e_j} \wedge \tilde{e_k}, \tilde{e_l} \wedge \tilde{e_m}) \\ &= 0. \end{aligned}$$

Thus, as there are no other contributing terms from above, we achieve equation (7)

$$\mathfrak{R}(e_j \wedge e_k) = \frac{1 - f_2'(t)^2}{f_2(t)^2} e_j \wedge e_k.$$

9.6 Appendix 6:

In this appendix we calculate the Einstein equations under the given transformation required for Proposition 6.1. Firstly we note that

$$R_i' = L_2 R_2 \text{ and}$$
$$L_i' = \frac{f_2''}{f_2} - L_i^2$$

by simple calculations. Then,

$$\begin{aligned} \xi' &= L_1' + (n-2)L_2' \\ &= \frac{f_1''}{f_1} - L_1^2 + (n-2)\frac{f_2''}{f_2} - (n-2)L_2^2 \\ &= -\lambda - L_1^2 - (n-2)L_2^2 \end{aligned}$$

by equation (8),

$$\begin{aligned} L_1' &= \frac{f_1''}{f_1} - L_1^2 + \xi L_1 - \xi L_1 \\ &= \frac{f_1''}{f_1} - L_1^2 + L_1^2 + (n-2)L_1 L_2 - \xi L_1 \end{aligned}$$

$$= \frac{f_1''}{f_1} + (n-2)\frac{f_1' f_2'}{f_1 f_2} - \xi L_1$$
$$= -\lambda - \xi L_1$$

by equation (9) and

$$L_2' = \frac{f_2''}{f_2} - L_2^2 + \xi L_2 - \xi L_2 + (n-3)R_2^2 - (n-3)R_2^2$$
$$= \frac{f_2''}{f_2} - L_2^2 + L_1 L_2 + (n-2)L_2^2 - (n-3)R_2^2 - \xi L_2 + (n-3)R_2^2$$
$$= \frac{f_2''}{f_2} + \frac{f_1' f_2'}{f_1 f_2} - (n-3)\frac{(f_2')^2 - 1}{f_2^2} - \xi L_2 + (n-3)R_2^2$$
$$= -\lambda - \xi L_2 + (n-3)R_2^2$$

by equation (10).