AMSI VACATIONRESEARCH SCHOLARSHIPS 2021–22

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Optimal Stopping Differential Equations

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1 Abstract

In many applications, we continually get new information while the future remains unknown and must find the optimal time to take a particular action in order to maximise the expected reward. Such problems are known as optimal stopping problems. They can be used to model and assist investment decision making, and is used in the study of real options. Optimal stopping methods help investors make decisions when dealing with long-term projects which face a substantial amount of uncertainty, such as unexpected events that cause a severe disruption to the market. In the context of such optimal stopping problems, differential equations arise with corresponding free boundary value problems. These can be solved using analytical and numerical methods. From these solutions, we aim to analyse market behaviour to assist real options decision making.

2 Introduction

We are often faced with the dilemma of deciding the most ideal time to make a particular choice in order to maximise our expected reward, i.e. what is the best time to do something in order to get the best out of it? In scenarios where data is successively coming in but it's unknown what that data is going to look like, these decisions are referred to as optimal stopping problems (Peskir & Shiryaev 2006). They involve finding when to pause the data flow and optimise one's situation, even though we do not know if the future will be more or less rewarding. A key example within a financial context is finding the optimal time to pull out of an investment in order to maximise profit. Using statistical tools such as stochastic processes and descriptive operators, these problems can be modelled as differential equations, and relevant boundary conditions can be used to solve the problem (Gapeev & Lerche 2011).

In this report, we will discuss two optimal stopping problems in particular. The first problem is maximising the profit from an American put option, which presents a free boundary problem to be solved analytically. In the second problem, we will analyse a more sophisticated differential equation model (Nunes et al. 2019) which is not readily solved analytically. We will then develop a numerical algorithm in MATLAB to solve these types of free boundary value problems using finite difference methods.

2.1 Statement of Authorship

This report was written by Patrick Daley under the supervision of Christopher Lustri and Georgy Sofronov. All theoretical knowledge is well-established and no new analytical methods have been developed. The numerical algorithm was produced in MATLAB and was developed by Christopher Lustri and Patrick Daley. All sources of information are contained in the reference list.

3 Background

3.1 Optimal Stopping Problems

In various applications, we obtain new information sequentially while the future remains unknown and must identify the optimal time to engage in a certain action such that the expected reward is maximised. These problems are known as optimal stopping problems (Peskir & Shiryaev 2006).

They are often applied in the study of real options, providing a framework for models that assist investment decision making. They are particularly useful in scenarios where investors are dealing with long term projects, since optimal stopping methods are capable of dealing with the substantial amounts of uncertainty in the future for such projects.

3.2 Stochastic Processes

To describe the probability of the market value behaviour and how it may change over time, a mathematical tool is required such as a *stochastic process*. This is defined as a parameterised collection of random variables: $\{X_t\}_{t\in T}$. The parameter space T refers to the time interval at which the stopping action can take place, which is generally the halfline $[0,\infty)$ (Oksendal 2013). This could be interpreted in a financial modelling context as the market value at time t.

3.2.1 Brownian Motion

Brownian Motion is the stochastic process that describes randomness or uncontrolled motion (Oksendal 2013). This can be written as $\{B_t\}_{t\geq 0}$. Random trajectories of Brownian motion can be simulated, as seen in Figure 1. This visually demonstrates how the stochastic process represents random behaviour with no clear, underlying trend.

3.2.2 Geometric Brownian Motion

For financial applications, we generally use a more advanced stochastic process named *Geometric Brownian Motion* $X = \{X_t\}_{t \geq 0}$ which solves the following differential equation (Gapeev & Lerche 2011):

$$dX_t = rX_t dt + \sigma X_t dB_t. \tag{1}$$

This can be described as an extension to Brownian Motion, that is achieved by applying the exponential function. The first term represents the general trend of the process and the second term represents the random noise. The advantages of this stochastic process is that we no longer obtain negative values and and there is a general exponential growth trend. Hence, this makes the process relevant to the application of financial problems as market prices follow these properties. This is demonstrated in the random trajectories in Figure 1:

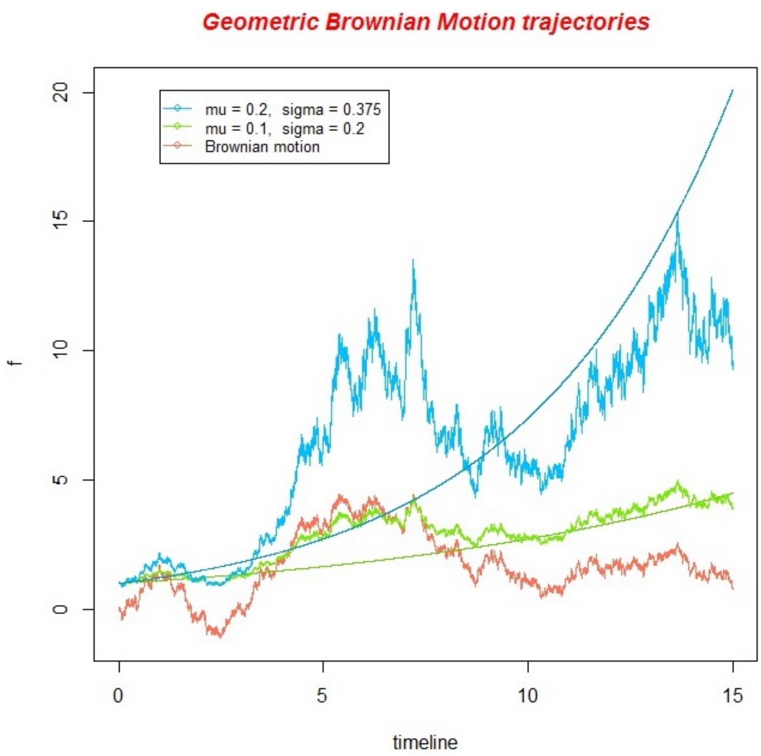

Figure 1: Random trajectories of Brownian Motion and Geometric Brownian Motion (*A Random Walk Through Mathematics* 2013).

3.3 Differential Equations

Using these stochastic processes and descriptive operators, we are able to model the behaviour of the market and develop differential equations that describe optimal stopping problems. By establishing particular boundary conditions relevant to the context of the problem, we can attempt to find a solution to these equations and hence the broader questions at hand.

Often this leads to a *free boundary problem*. This involves not only solving the unknown function, but also solving the unknown domain. These problems are generally very challenging, so whilst some may be solved analytically, there is often the need to develop numerical solutions.

4 Financial Derivatives: Put and Call Options

An option is a financial derivative which allows the owner to buy or sell an asset sometime in the future at a pre-determined price. A put option refers to when an investor locks in a price to sell an asset at a later date. The strategy is ideal if they believe the asset's value will go down as they will make profit selling an over-priced asset. A call option refers to when an investor locks in a price to buy an asset instead, and hence is an ideal strategy if they believe it's value will increase.

Optimal stopping problems often arise when analysing these investment strategies. One needs to determine at which time they should exercise their option, i.e. buy or sell their asset, in order to maximise their profit.

We will consider an optimal stopping problem in the context of an American put option with an infinite horizon. This means the owner of the option can exercise at any time from the execution date with no expiration period. Furthermore, the strike price is set and so the pre-determined value of the asset will not change. Hence, the problem at hand is what is the most ideal time to exercise the option in order to maximise profit?

So, the arbitrage-free price of the put option is given by:

$$V(x) = \sup_{\tau} E_x(e^{-rt}(K - X_t)^+). \tag{2}$$

We should note from this function that:

- V represents the value of the firm and x represents the price to invest.

- The supremum means we take the maximum expected value in the given time span $\tau = [0, \infty)$, at each time t.
- The function $(K - X_t)^+ = max(0, K - X_t)$ represents our profit size, with parameter K representing the strike price. The option will only be exercised if we achieve a positive profit, and will not be exercised under any circumstance if it results in a loss.
- The parameter r is the interest rate.
- The exponential decay term represents the need to eventually stop somewhere, it is not optimal to continue forever.

Using the relevant statistical operator, we can form the following differential equation that models this optimal stopping problem.

$$Dx^2V'' + rxV' - rV = 0, \quad D = \frac{\sigma^2}{2}. \tag{3}$$

where V represents the value of the option and x represents the price of the asset. Also, σ is a parameter from the stochastic process (1).

We must now establish some boundary conditions. Firstly, when the investor sells an asset of value x at price K, they will make a profit margin of $K - x$. Secondly, we require the function to be a continuous, smooth-fitting curve at the optimal stopping point b. Finally, the profit must always be positive and can't exceed the sell price of K. Applying this logic, we can develop the following boundary conditions for the problem:

$$V(x) = (K - x)^+ \ \text{ for } \ x = b, \tag{4}$$

$$V'(x) = -1 \ \text{ for } \ x = b, \tag{5}$$

$$0 \leq V(x) \leq K \ \text{ for all } \ x > 0. \tag{6}$$

These boundary conditions describe a free boundary problem. These are difficult to solve as there are two key components:

1. Finding the arbitrage-free price, i.e. the unknown value function (2).
2. Finding the optimal exercise time, an unknown point b.

However, these boundary conditions uniquely specify the solution. This is evident from the numerical outputs in section 6.2, with values either side of the correct value of b diverging as seen in figures 5 and 6.

Equation (3) follows the structure of an Euler-Cauchy equation, to which there are well known methods to find a solution as outlined in (Peskir & Shiryaev 2006). Hence, the solution to the problem set is:

$$b = \frac{K}{1 + D/r}, \tag{7}$$

$$V(x) = \begin{cases} \frac{D}{r}(\frac{K}{1+D/r})^{1+r/D} x^{-r/D}, & x \in [b, \infty), \\ K - x, & x \in (0, b]. \end{cases} \tag{8}$$

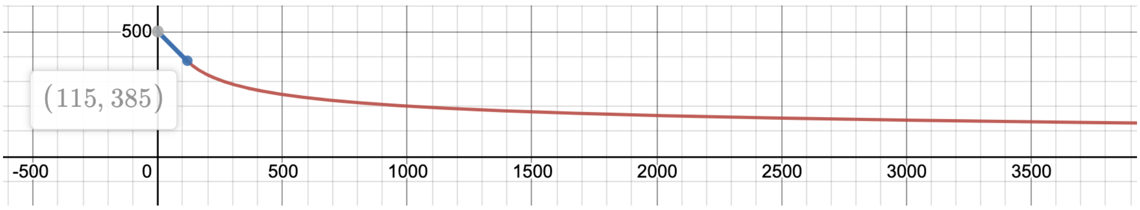

Figure 2: Plot of $V(x)$ with $D = 0.5$, $r = 0.15$, $K = 500$.

5 Jump-Diffusion Processes

There are often external, unexpected events which cause a severe disruption to the market, resulting in booms and crashes. These occur quite regularly and are responsible for the volatility of the market. Examples of factors that may cause such effects range from government legislation to technological improvements.

A recent example is the impact of the COVID-19 pandemic on the economy and stock market. The unprecedented social isolation saw many business failing to receive consumer demand (Craven et al. 2020), resulting in significant drops in the value of these firms. In particular, the oil market saw an extreme downturn following the coronavirus outbreak in China in early 2020 which caused supply and demand issues with exports/imports (Dutta et al. 2020).

It is of great importance to incorporate the potential influence of such "jumps" when modelling market values. Whilst Geometric Brownian Motion effectively models random behaviour, it does not account for any external effects on the system which may have sudden and significant impacts. Thus, we require a new stochastic process if we want to incorporate these effects into our model.

These spontaneous impacts can be modelled via the *jump-diffusion process*. These are more advanced then the previous stochastic processes, having an extra term to account for the probability of these jumps occurring.

A differential equation that models the optimal stopping time in a jump-diffusion process is presented in the paper (Nunes et al. 2019). The class of the stochastic process is a one–dimensional jump–diffusion, and it solves the following stochastic differential equation:

$$\frac{dX(t)}{X(t^-)} = \mu dt + \sigma dW(t) + \kappa dN(t). \tag{9}$$

Applying the relevant operators, this results in a differential equation model of the form:

$$x^2V'' + \frac{r}{D}xV' - \frac{r}{D}V = Ax^{\alpha}(\ln x)^n, \tag{10}$$

where again V represents the value of the firm and x represents the price to invest. Also, parameters A and α are non-zero real numbers while parameter n is a positive integer.

The equation (10) arises in settings with more complicated free boundary problems, that are not always readily solved analytically as we saw before. Thus, it is important to utilise other efficient methods to solve these problems such as numerical algorithms.

6 Numerical Methods

A *numerical method* refers to the process of solving a mathematical problem by utilising computer programs. They are an incredibly powerful tool, and can provide an alternative and more efficient solution to problems with analytical solutions. Often, they can still generate a solution quickly even if the problem is impossible to solve analytically.

6.1 Finite Differences

The *finite difference method* is an approximation method used to numerically solve partial differential equations (Zhou 2012). It may be implemented to solve a diverse range of problems with different types of boundary conditions, and so can be incorporated when solving free boundary problems.

The finite difference method is a relatively simple numerical method and the development of technology has lead to faster computers capable of various techniques. However, it still holds relevance today due to the ease of its application to various problems.

The method involves approximating the derivatives in a differential equation using the finite difference formulas, which allows us to transform the problem into a system of equations which may be solved using linear algebra.

This is achieved through *discretisation*; breaking down a continuous variable into many small, discrete components. This can be done by taking the interval outlined by the boundary conditions to solve the equation and dividing it into n equal subintervals of length h, as visualised in Figure 3.

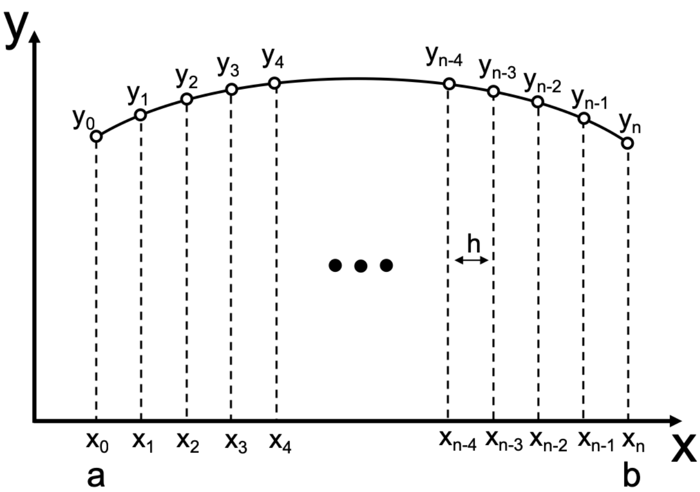

Figure 3: Visualisation of the division of the function into subintervals (Kong et al. 2020).

By then applying an understanding of the first principle definition of a derivative and extending these via Taylor expansions, we can derive the finite difference formulas. The central difference formulas are often selected as they provide more stable solutions and are useful when values are fixed on either side of the interval. These formulas are as follows:

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-i}}{2h}, \tag{11}$$

$$\frac{d^2y}{dx^2} = \frac{y_{i-1} + 2y_i + y_{i+1}}{h^2}. \tag{12}$$

By substituting these formulas into the differential equation for each n function values, we obtain a system of $n+1$ algebraic equations which can be readily solved numerically (Kong et al. 2020). Hence, this technique can be used to develop numerical algorithms to solve equations such as (3) and (10) that are found in optimal stopping problems.

6.2 Solving the Free Boundary Problem

The following numerical algorithm was developed to solve equations in free boundary problems, and will be applied to the problem in Section (4). This algorithm is a shooting method, which involves guessing the solution and repeatedly trialling simulations with an adjusted guess until the solution is found. It uses the finite differences method to solve the equation.

Several variables were declared to represent the parameters and establish the problem's domain. The first interval of the problem can be quickly computed, and the finite difference method is then applied to solve the second interval.

To solve the free boundary problem, we utilised the fact that as $x \to \infty, \ \ V \to 0$. We selected a larger value of x, which we will label x_∞. We also selected a value of b, with $b < K$, which is the "guess".

Next, we solved equation (3) on $[0, b]$, with boundary conditions $V(0) = K$ and $V(b) = K - b$. We then solved equation (3) on $[b, x_\infty]$ using the finite difference method. This is done by dividing $V(x)$ into a large number of subintervals, such that V_j is the function value for the j-th subinterval, and substituting formulas (11) and (12) into equation (3) and rearranging V_j in terms of V_{j-1} and V_{j-2}. V_1 was able to be calculated using boundary condition (4), and V_2 was calculated using the value for V_1 and boundary condition (5). From there, we could compute the remaining V_j values.

Finally, we checked if $V(x_\infty) = 0$. If this condition was met, within a margin of error tolerance, we had found our solution. However, if the condition was not met, the guess for b had to be adjusted and another iteration of the algorithm was conducted. This process repeated until the solution was computed.

Figure 4 shows the final output after computing the correct b value. Figure 5 demonstrates when shooting from b values larger than the solution, the function decreases indefinitely and we must reiterate with smaller values of b. Similarly, when shooting from smaller b values, we must increase our guess for b each iteration until the graph no longer grows indefinitely.

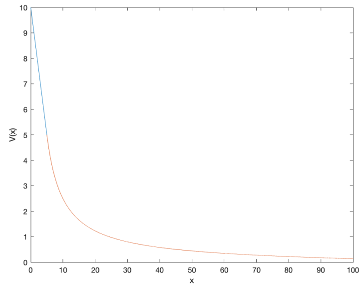

Figure 4: A plot of the function with $D = 0.5$, $r = 0.5$, $K = 10$ and $b = 5$.

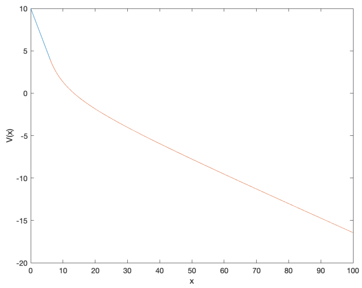

Figure 5: A plot of the function with $D = 0.5$, $r = 0.5$ and $K = 10$, shooting from $b = 6$.

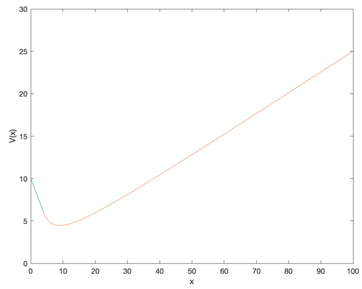

Figure 6: A plot of the function with $D = 0.5$, $r = 0.5$ and $K = 10$, shooting from $b = 4$.

7 Discussion and Conclusion

There are various applications where optimal stopping problems arise, particularly in a financial context. Using stochastic processes, the behaviour of financial markets can be modelled via developing differential equations to help solve such problems. We have explored some specific examples of these problems, developing some analytical solutions to (3) and (10) using relevant boundary conditions. This led to the realisation of the importance of other efficient methods, and so a numerical algorithm was developed to solve such free boundary problems.

This research could be extended by considering the optimal stopping time differential equation models in settings with multiple actors by applying game theoretic principles. It would extend the models to allow for the presence of another party who can also invest in the market (i.e. the second actor), which will result in a system of differential equations to analyse. The numerical algorithm could also be applied to different problems, allowing for potential to extend it upon the given framework and generalise it for wider applications. Furthermore, the algorithm's sophistication could be developed by improving the parameter adjustment between trials by applying Newton's method, allowing for more effective iterations to lower computing time.

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