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Properties of Brownian Motion

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Abstract

In this report, we study Brownian motion and some of its general properties. Brownian motion is used to model the erratic, random movement of particles due to collisions with one another. We begin with its general properties, including its definition, before going on to derive the distribution of the maximum across an interval and the distribution of hitting times. We then explore certain martingales of Brownian motion, including one of the implications as a part of Lévy's characterisation of Brownian motion. Lastly, we look at the analytical properties of Brownian motion, including the non-differentiability, the unbounded variation and the non-zero quadratic variation, as well as the consequences of these properties, before formulating a framework of integration that circumvents these consequences, the Itô integral.

Contents

1 Introduction

Brownian motion was first observed by Jan Ingenhousz in 1785, when he noticed the irregular motion and behaviour of coal dust particles on the surface of alcohol. It was named after a botanist, Robert Brown, who observed pollen grains suspended in water under a microscope moving along an irregular, jittery motion. There are many models and theories for describing this motion in physics, but in mathematics, it is described by the Wiener process, a continuous-time stochastic process named after Norbert Wiener. The Brownian motion is synonymous with the Wiener process in mathematical literature.

Brownian motion is defined by three mathematical properties, the independence of increments, the normality of increments and the continuity of paths. As it is a stochastic process, these defining properties have implications on many of its probabilistic properties, which have been studied extensively, but it also has significant association to analysis and its analytical properties, essentially forming the cornerstone of stochastic analysis. This report hopes to explore these probabilistic and analytical properties in finer detail than what an undergraduate course would usually cover.

The report is written under the assumption that the reader is familiar with the theory of σ-algebras, conditional expectation and martingales.

2 Statement of Authorship

The details included in this report are purely theoretical, all of which are well-established in mathematical literature, thus no new theorems have been developed. The purpose of this project was to study a field of mathematics not generally covered in undergraduate mathematics and the interesting results studied in the process have been included in this report by Pu Ti Dai. The information in this report has been primarily sourced from Fima Klebaner's Introduction to Stochastic Calculus with Applications and other resources provided by supervisors Gregory Markowsky and Kaustav Das.

3 Brownian Motion and its General Properties

3.1 Definition of Brownian Motion

Brownian motion has three defining mathematical properties (Klebaner 2005):

1. The independence of increments. For all times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the increments $B_{t_n} - B_{t_{n-1}}$, $B_{t_{n-1}} - B_{t_{n-2}}$, ..., $B_{t_2} - B_{t_1}$ are independent random variables.

2. Normally-distributed increments. An increment of the process, i.e. $B_t - B_s$, where $s < t$, is Normally distributed with mean 0 and variance $t - s$, i.e.,

$$B_t - B_s \sim N(0, t - s) \text{ for } t > s.$$

3. Continuity of paths. The function $t \mapsto B_t$ for $t \geq 0$ is a continuous function of t with probability 1.

We call $\{B_t : t \geq 0\}$ a standard Brownian motion if $B_0 = 0$, i.e. if the Brownian motion starts at 0. All references to Brownian motion in this report are assumed to be standard Brownian motion, unless specified otherwise.

3.2 Space and Time Homogeneity of Brownian Motion

It is quite easy to see that the starting point of a one-dimensional Brownian motion is the mean of the process. In other words, a Brownian motion can be translated through varying its mean. If we let B_t^x denote a Brownian motion where $B_0 = x$, then we can say

$$B_t^x = B_t^0 + x.$$

This means that Brownian motion is space-homogeneous. In other words, its distribution does not change with a shift in space.

Brownian motion is also time-homogeneous, where its distribution does not change with a shift in time. We can see this with the independent increments of Brownian motion, which all follow a Normal distribution with a mean 0 and a variance equivalent to the length of the increment. As such, as long as two increments of a Brownian motion occur over the same length, regardless of where they occur, they are distributed identically.

3.3 Covariance Function of Brownian Motion

Theorem 1 (Covariance Function of Brownian Motion)**.** The covariance function of B_s, i.e. $\text{Cov}(B_s, B_t)$, is equal to $\min(s, t)$. If $s < t$, then $\text{Cov}(B_s, B_t) = s$.

Proof. By the definition of the covariance function, we have

$$\text{Cov}(B_s, B_t) = \mathbb{E}[B_s B_t] \text{ as } B_0 = 0.$$

If $s < t$, then $B_t = (B_t - B_s) + B_s$.

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s^2] + \mathbb{E}[B_s(B_t - B_s)] = s.$$

We can do the same when $s > t$ and show that $\text{Cov}(B_s, B_t) = t$. Thus $\text{Cov}(B_s, B_t) = \min(s, t)$. □

3.4 Scaling Invariance of Brownian Motion

Theorem 2 (Scaling Invariance of Brownian Motion)**.** Let B_t be a standard Brownian motion and let $a \in \mathbb{R}$. The process X_t where X_t is defined as $\frac{1}{a} B_{a^2 t}$ is also a standard Brownian motion (Mörters and Peres 2010).

Proof. The independence of increments, the normality of increments and the continuity of paths remain unaffected under scaling. As such, to show that X_t is a standard Brownian motion, the increments only need to be shown to follow the same Normal distribution as a standard Brownian motion.

$$\begin{aligned}\mathbb{E}\left[\frac{1}{a}B_{a^2t} - \frac{1}{a}B_{a^2s}\right] &= \frac{1}{a}(\mathbb{E}[B_{a^2t}] - \mathbb{E}[B_{a^2s}]) \\ &= 0.\end{aligned}$$

$$\begin{aligned}\text{Var}\left[\frac{1}{a}B_{a^2t} - \frac{1}{a}B_{a^2s}\right] &= \frac{1}{a^2}\text{Var}[B_{a^2t} - B_{a^2s}] \\ &= \frac{1}{a^2}(a^2t - a^2s) \\ &= t - s.\end{aligned}$$

□

3.5 Recurrence Property of Brownian Motion

Definition 1 (Stopping Time)**.** A random time T is called a stopping time if it is possible to determine whether T has occurred at time t already by observing the information available up to time t. For a more rigorous definition, for any t, the sets $\{T \leq t\} \in \mathcal{F}_t$, the σ-field (which may be generated by the random process) up to t (Klebaner 2005).

Definition 2 (Hitting Time of Brownian Motion)**.** A hitting time of Brownian motion, T_x, is the first time a Brownian motion reaches a certain level, x, hence it is also a stopping time. More formally

$$T_x = \inf\{t > 0 : B_t = x\}.$$

Definition 3 (Exit Time of Brownian Motion)**.** An exit time of Brownian motion, τ, is the time for a Brownian motion that starts in an interval (a, b) to exit the interval. It is also a stopping time. It is denoted by

$$\tau = \min(T_a, T_b).$$

Theorem 3. Let $a < x < b$ and $\tau = \min(T_a, T_b)$. Then $\mathbb{P}(\tau < \infty | B_0 = x) = 1$ and $\mathbb{E}_x[\tau | B_0 = x] < \infty$.

An implication of this theorem is that a Brownian motion will eventually reach all values of x, almost surely, if we allow the process to last forever. This also means that given an exit time τ,

$$\mathbb{P}(\tau < \infty) = 1.$$

So we can expect a Brownian motion to eventually exit an interval, regardless of the width of the interval or the location of the upper and lower bounds.

This can also be interpreted as we can expect a Brownian motion to hit a level a an infinite number of times. This is known as the recurrence property of Brownian motion and is formalised below.

Theorem 4.

$$\mathbb{P}(T_b < \infty | B_0 = a) = 1.$$

3.6 Maximum and Minimum of Brownian Motion

It is possible to derive a probability distribution for the maximum of a Brownian motion within a time interval, but first, we must introduce the *reflection principle*.

Theorem 5 (Reflection Principle)**.** Let T be a stopping time. Define $\hat{B}_t = B_t$ for $t \leq T$ and $\hat{B}_t = 2B_T - B_t$ for $t > T$. Then $\hat{B}_t$ is also a Brownian motion.

Theorem 6 (Maximum of Brownian Motion)**.** Let M_t denote the maximum value that a Brownian motion starting at $x = 0$ achieves in the interval $[0, t]$, i.e. $M_t = \max_{0 \leq s \leq t} B_s$. For any $x > 0$,

$$\mathbb{P}(M_t \geq x) = 2\mathbb{P}(B_t \geq x) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right),$$

where $\Phi(x)$ is the standard Normal distribution function.

Proof. Let T_x denote the hitting time for x. If we have $M_t > x$, it is implied that our process hits x before time t i.e. $T_x < t$. Hence we have

$$\mathbb{P}(M_t \geq x) = \mathbb{P}(T_x \leq t, B_t \geq x) + \mathbb{P}(T_x \leq t, B_t \leq x).$$

By the reflection principle, there is equal probability for the process to be greater than x and less than x at t, i.e. $\mathbb{P}(T_x \leq t, B_t \geq x) = \mathbb{P}(T_x \leq t, B_t \leq x)$. Hence

$$\mathbb{P}(M_t \geq x) = 2\mathbb{P}(T_x \leq t, B_t \geq x).$$

However, as Brownian motion is a continuous process, $B_t > x$ implies the process must reach x, which can also be written as $T_x < t$. This gives the final result

$$\mathbb{P}(M_t \geq x) = 2\mathbb{P}(B_t \geq x).$$

□

Theorem 7 (Minimum of Brownian Motion)**.** Let m_t denote the minimum value that a Brownian motion starting at $x = 0$ achieves in the interval $[0, t]$, i.e. $m_t = \min_{0 \leq s \leq t} B_s$. For any $x < 0$,

$$\mathbb{P}(m_t \leq x) = 2\mathbb{P}(B_t \geq -x) = 2\mathbb{P}(B_t \leq x).$$

This can be easily proven using the reflection principle and the maximum of a Brownian motion found earlier.

Using the maximum of Brownian motion found before, we can now derive the distribution of hitting times.

Theorem 8 (Distribution of Hitting Times)**.** The probability distribution of T_x is given by

$$f_{T_x}(t) = \frac{|x|}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{2t}},$$

which is the Lévy distribution with a location parameter of 0 and a scale parameter of x^2, which also happens to be a special case of the Inverse Gamma distribution. Moreover, $\mathbb{E}[T_x] = +\infty$.

Proof. Let $x > 0$. The events $M_t \geq x$ and $T_x \leq t$ are the same, so

$$\begin{aligned}\mathbb{P}(T_x \leq t) &= \mathbb{P}(M_t \geq x)\\ &= 2\mathbb{P}(B_t \geq x)\\ &= \int_x^\infty \sqrt{\frac{2}{\pi t}} e^{-\frac{y^2}{2t}}\, dy.\end{aligned}$$

The formula for the density of T_x is obtained by differentiation after the change of variables $u = \frac{y}{\sqrt{t}}$ in the integral. Finally,

$$\mathbb{E}[T_x] = \frac{|x|}{\sqrt{2\pi}} \int_x^\infty t^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}\, dt = \infty, \text{ since } t^{-\frac{1}{2}} e^{-\frac{x^2}{2t}} \sim \frac{1}{\sqrt{t}},\ t \to \infty.$$

□

We can use the time-homogeneity of Brownian motion to generalise this formula to an arbitrary starting level a and an arbitrary hitting level b, as the distribution of $T_b - T_a$ is equivalent to the distribution of T_{b-a} if $B_0 = 0$. This gives

$$f_{T_b - T_a}(t) = \frac{|b-a|}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{(b-a)^2}{2t}}.$$

4 Martingale Properties of Brownian Motion

4.1 Definition of a Martingale

Before we can look at the martingale properties of Brownian motion, we must first define what a *filtration* and a *martingale* is.

Definition 4 (Filtration)**.** A filtration, $\{\mathcal{F}_t : t \geq 0\}$, is a family of increasing sub-σ-algebras of $\mathcal{F}$, i.e.

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}.$$

Definition 5 (Martingales)**.** A stochastic process X_t, $t \geq 0$ is a martingale if for any t it is integrable ($E|X_t| < \infty$), and for any $s > 0$

$$\mathbb{E}(X_{t+s}|\mathcal{F}_t) = X_t, \text{ almost surely,}$$

where $\mathcal{F}_t$ is a filtration up to time t.

An intuitive interpretation of a martingale is a fair game. If we let Y_t denote our earnings from the game at time t, then by playing the game one more time, our expected earnings will not change as it is a fair game, i.e. Our expected earnings at time $t+1$ is the same as our earnings at t.

An discrete example of this is tossing a fair coin, where a heads results in the gain of \$1 and a tail results in the loss of \$1. Assuming our total loss or gain after n tosses is S_n, or expected loss or gain after another toss can be expressed as

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n).$$

As we know the next toss can be a heads or a tails with equal probability, we do not expect to gain anything from the additional game, so we would not expect any change in our total loss or gain. It is equivalent to S_n, and thus is a martingale

4.2 Important Martingales of Brownian Motion

Theorem 9 (Important Martingales of Brownian Motion)**.** If B_t is a Brownian motion, then the following are martingales:

1. B_t.
2. $B_t^2 - t$.
3. $e^{uB_t - \frac{u^2}{2}t}$ for any u (also known as the exponential martingale of Brownian motion).

Proof. B_t follows a Normal distribution of mean 0 and variance t by definition, so it is integrable.

$$\begin{aligned}\mathbb{E}[B_{t+s}|\mathcal{F}_t] &= \mathbb{E}[B_t + (B_{t+s} - B_t)|\mathcal{F}_t] \\ &= \mathbb{E}[B_t|\mathcal{F}_t] + \mathbb{E}[B_{t+s} - B_t|\mathcal{F}_t] \\ &= B_t + \mathbb{E}[B_{t+s} - B_t] \text{ due to the independence of increments} \\ &= B_t.\end{aligned}$$

Hence B_t is a martingale.

As B_t has a mean 0 and variance t, $\mathbb{E}[B_t^2] = t$. Therefore B_t^2 is integrable. t is also integrable.

$$\begin{aligned}\mathbb{E}[B_{t+s}^2 - (t+s)|\mathcal{F}_t] &= \mathbb{E}[(B_t + (B_{t+s} - B_t))^2 - (t+s)|\mathcal{F}_t] \\ &= \mathbb{E}[B_t^2 + 2B_t(B_{t+s} - B_t) + (B_{t+s} - B_t)^2 - (t+s)|\mathcal{F}_t] \\ &= \mathbb{E}[B_t^2|\mathcal{F}_t] + \mathbb{E}[2B_t(B_{t+s} - B_t)|\mathcal{F}_t] + \mathbb{E}[(B_{t+s} - B_t)^2|\mathcal{F}_t] - (t+s) \\ &= B_t^2 + 2B_t\mathbb{E}[(B_{t+s} - B_t)|\mathcal{F}_t] + s - t - s \\ &= B_t^2 - t.\end{aligned}$$

Hence $B_t^2 - t$ is a martingale.

The moment generating function of B_t, $\mathbb{E}[e^{uB_t}]$, is $e^{tu^2/2}$, which implies the integrability of $e^{uB_t - \frac{u^2}{2}t}$ for any u.

$$\begin{aligned}\mathbb{E}[e^{uB_{t+s} - \frac{u^2}{2}(t+s)}|\mathcal{F}_t] &= \mathbb{E}[e^{u(B_t + (B_{t+s} - B_t)) - \frac{u^2}{2}(t+s)}|\mathcal{F}_t] \\ &= \mathbb{E}[e^{uB_t - \frac{u^2}{2}t} e^{u(B_{t+s} - B_t) - \frac{u^2}{2}s}|\mathcal{F}_t] \\ &= e^{uB_t - \frac{u^2}{2}t}\mathbb{E}[e^{u(B_{t+s} - B_t) - \frac{u^2}{2}s}|\mathcal{F}_t] \\ &= e^{uB_t - \frac{u^2}{2}t}\mathbb{E}[e^{u(B_{t+s} - B_t)}e^{-\frac{u^2}{2}s}] \text{ (due to independence of increments)} \\ &= e^{uB_t - \frac{u^2}{2}t}\mathbb{E}[e^{uW_s}]e^{-\frac{u^2}{2}s}, \text{ where } W_s \sim N(0, t-s) \\ &= e^{uB_t - \frac{u^2}{2}t} \text{ as the MGF of } W_s \text{ is } e^{\frac{u^2}{2}s}.\end{aligned}$$

Hence $e^{uB_t - \frac{u^2}{2}t}$ is a martingale for any u. □

The third martingale is known as the exponential martingale of Brownian motion. It has important applications in financial mathematics, such as for modelling the discounted stock price in the so-called 'risk-neutral' Black-Scholes model. It is also used in Girsanov's theorem for changing measures, which also has applications in financial mathematics.

The first two martingales, B_t and $B_t^2 - t$, bear some resemblance to Hermite polynomials. It can be shown that all two-variable Hermite polynomials of Brownian motion and time are in fact martingales.

The second martingale provides a characterisation (Lévy's characterisation) of Brownian motion.

4.3 Lévy's Characterisation of Brownian Motion

Before we can provide Lévy's characterisation of Brownian motion, we must introduce and prove another theorem which will be used in proving Lévy's characterisation.

Theorem 10 (Conditional Expectation of Itô's Lemma)**.** Let X_t be a continuous process that is a continuous-time martingale, where $X_t^2 - t$ is a martingale as well, and let f be a bounded function with bounded first and second derivatives (Nielsen 2010). For all $0 \leq s \leq t$, we have

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = X_s + \frac{1}{2}\int_s^t \mathbb{E}(f''(X_u)|\mathcal{F}_s)du.$$

Proof. Let $\Pi = (t_k)_{k=0}^n$ be a partition of the interval $[s,t]$ such that $s = t_0 < t_1 < t_2 < \cdots < t_n = t$, then by Taylor's formula, we have

$$\begin{aligned} f(X_t) &= f(X_s) + \sum_{k=1}^n (f(X_{t_k}) - f(X_{t_{k-1}})) \\ &= f(X_s) + \sum_{k=1}^n f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^n f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2 + R_\Pi. \end{aligned}$$

Taking conditional expectations on both sides and using the Tower property, we obtain

$$\begin{aligned} \mathbb{E}[f(X_t)|\mathcal{F}_s] = f(X_s) &+ \sum_{k=1}^n \mathbb{E}[\mathbb{E}[f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_s] \\ &+ \frac{1}{2}\sum_{k=1}^n \mathbb{E}[\mathbb{E}[f''(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}})^2|\mathcal{F}_{t_{k-1}}]|\mathcal{F}_s] + \mathbb{E}[R_\Pi|\mathcal{F}_s] \\ &= f(X_s) + \frac{1}{2}\sum_{k=1}^n \mathbb{E}[f''(X_{t_{k-1}})|\mathcal{F}_s](t_k - t_{k-1}) + \mathbb{E}[R_\Pi|\mathcal{F}_s]. \end{aligned}$$

Using the continuity of X_t, it can be shown that $R_\Pi \to 0$ in L^2 when the length $|\Pi|$ tends to 0. Hence also, $\mathbb{E}[R_\Pi|\mathcal{F}_s] \to 0$ in L^2. Since the function $u \to \mathbb{E}[f''(X_u|\mathcal{F}_s)]$ is continuous a.s, we get that

$$\sum_{k=1}^n \mathbb{E}[f''(X_{t_{k-1}})|\mathcal{F}_s](t_k - t_{k-1}) \to \int_s^t \mathbb{E}(f''(X_u)|\mathcal{F}_s)du \text{ a.s.}$$

When $|\Pi| \to 0$ and since f'' is bounded, the bounded convergence theorem gives that the convergence is also in L^2. Combining the above, we get

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = X_s + \frac{1}{2}\int_s^t \mathbb{E}(f''(X_u)|\mathcal{F}_s)du.$$

□

Theorem 11 (Lévy's Characterisation of Brownian Motion)**.** Given a continuous process X_t, X_t is a Brownian motion is if both X_t and $X_t^2 - t$ are martingales.

Proof. A continuous process is a Brownian motion if we can show that the characteristic function of its intervals corresponds to the characteristic function of an independent Normal distribution.

$$\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t] = e^{-\frac{1}{2}u^2(t-s)} \text{ for any } u \text{ and } 0 \leq s \leq t.$$

Taking the conditional expectation of Itô's Lemma, where our function is $f(x) = e^{iux}$ for all $x \in \mathcal{R}$, we obtain

$$\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t] = 1 - \frac{1}{2}u^2 \int_s^t \mathbb{E}[e^{iu(X_v - X_s)}|\mathcal{F}_t]dv.$$

Since the integrand on the right hand side is continuous, the left side can be differentiated with respect to t

$$\frac{d}{dt}\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t] = -\frac{1}{2}u^2\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t].$$

This can be written as a separable ODE

$$g'(t) = -\frac{1}{2}u^2 g(t) \text{ where } g(t) = \mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t]$$

with initial condition $g(s) = 1$. This can be solved to find that

$$g(t) = e^{-\frac{1}{2}u^2(t-s)}.$$

Hence

$$\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t] = e^{-\frac{1}{2}u^2(t-s)} \text{ for any } u \text{ and } 0 \leq s \leq t$$

which implies that X_t has independent increments due to the non-randomness of the right side.
Taking the expectation, we get

$$\begin{aligned}\mathbb{E}[e^{iu(X_t - X_s)}] &= \mathbb{E}[\mathbb{E}[e^{iu(X_t - X_s)}|\mathcal{F}_t]] \\ &= e^{-\frac{1}{2}u^2(t-s)}\end{aligned}$$

which corresponds to the characteristic function of a Normal distribution with mean 0 and variance $t - s$. Thus X_t is a Brownian motion. □

5 Analytical Properties of Brownian Motion

We now explore the analytical properties of Brownian motion, which were primarily retrieved from Klebaner (2005).

5.1 Non-differentiability of Brownian Motion

Theorem 12 (Non-differentiability of Brownian Motion)**.** Almost all trajectories of Brownian Motion are not differentiable for any t.

Proof. We present a heuristic proof for the non-existence of the derivative for specific t. If we attempt to differentiate B_t by first principles, we have

$$\lim_{\Delta \to 0} \frac{B_{t+\Delta} - B_t}{\Delta}.$$

As the increments of Brownian motion follow a Normal distribution, we have

$$\lim_{\Delta \to 0} \frac{\sqrt{\Delta} Z}{\Delta} = \lim_{\Delta \to 0} \frac{Z}{\sqrt{\Delta}}.$$

where Z is a standard Normal variable. This limit converges to ∞ in distribution as $\lim_{\Delta \to 0} \mathbb{P}(|\frac{Z}{\sqrt{\Delta}}| > K) = 0$ for all K. □

5.2 Infinite Variation of Brownian Motion and Implications

An implication of the non-differentiability of Brownian motion is that it has unbounded *variation*. This is defined as follows.

Definition 6 (Variation of a Function)**.** The variation of a function of real variable, g, over the interval $[a, b]$ is defined as

$$V_g([a, b]) = \sup \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|,$$

where the supremum is taken over partitions $a = t_0^n < t_1^n < \cdots < t_n^n = b$.

Definition 7. g is of finite variation if $V_g([0, t]) < \infty$ for all t. g is of bounded variation if $\sup_t V_g([0, t]) < \infty$. In other words, if for all t, $V_g([0, t]) < C$, a constant independent of t.

Functions of bounded variation over an interval $[0, T]$ are differentiable almost everywhere on $[0, T]$. By showing that Brownian motion is nowhere differentiable a.s, we have essentially shown that it possesses unbounded variation.

An intuitive interpretation of the unbounded variation of Brownian motion is that it is infinitely-bumpy. We can use the scaling invariance property to show that if we 'zoom' in on a standard Brownian motion in a certain way, it is still a standard Brownian motion. As such, no matter how much we zoom in, the process will not be smooth. If we do the same with a function of finite variation, we will eventually end up with a straight line, which is clearly not the case here.

An implication of the unbounded variation of Brownian motion is that the *Riemann integral* and its generalisation, the *Riemann-Stieltjes integral*, fail when trying to integrate with respect to Brownian motion.

Definition 8 (Riemann Integral)**.** The Riemann Integral of f over interval $[a, b]$ is defined as the limit of Riemann sums

$$\int_a^b f(t)dt = \lim_{\delta \to 0} \sum_{i=1}^{n} f(\xi_i^n)(t_i^n - t_{i-1}^n),$$

where the t_i^n's represent partitions of the interval

$$a = t_0^n < t_1^n < \cdots < t_n^n = b,\ \delta = \max_{1 \le i \le n}(t_i^n - t_{i-1}^n), \text{ and } t_{i-1}^n \le \xi_i^n \le t_i^n.$$

Definition 9 (Riemann-Stieltjes Integral)**.** The Riemann-Stieltjes integral of f with respect to a function of finite variation g over an interval $(a, b]$ is defined as

$$\int_a^b f(t)dg(t) = \lim_{\delta \to 0} \sum_{i=1}^{n} f(\xi_i^n)(g(t_i^n) - g(t_{i-1}^n)),$$

with the quantities defined the same as the Riemann integral.

Theorem 13. Let $\delta_n = \max_i(t_i^n - t_{i-1}^n)$ denote the largest interval in the partition of $[a, b]$. If

$$\lim_{\delta \to 0} \sum_{i=1}^{n} f(t_i^n)(g(t_i^n) - g(t_{i-1}^n))$$

exists for any continuous function f, then g must be of finite variation on $[a, b]$.

Brownian motion directly violates this theorem due to infinite variation, so the Riemann-Stieltjes integral does not exist when integrating with respect to Brownian motion.

5.3 Quadratic Variation of Brownian Motion

Furthermore, Brownian motion possesses non-zero *quadratic variation*, which is not normally observed in common continuous functions as they all possess bounded variation.

Definition 10 (Quadratic Variation of a Function)**.** The quadratic variation of a function of real variable, g, over the interval $[a, b]$ is defined as

$$[g](t) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (g(t_i^n) - g(t_{i-1}^n))^2$$

where the limit is taken in probability over partitions $a = t_0^n < t_1^n < \cdots < t_n^n = b$ with $\delta_n = \max_{1 \le i \le n}(t_i^n - t_{i-1}^n)$.

Theorem 14. If g is a continuous function with finite variation, then its quadratic variation is zero.

Proof.

$$\begin{aligned}[g](t) &= \lim_{\delta_n \to 0} \sum_{i=1}^{n} (g(t_i^n) - g(t_{i-1}^n))^2 \\ &\le \lim_{\delta_n \to 0} \max_i |g(t_i^n) - g(t_{i-1}^n)| \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)| \\ &\le \lim_{\delta_n \to 0} \max_i |g(t_i^n) - g(t_{i-1}^n)| V_g([0, t]).\end{aligned}$$

Since g is continuous, it is uniformly continuous on $[0, t]$, hence $\lim_{\delta_n \to 0} \max_i |g(t_i^n) - g(t_{i-1}^n)| = 0$ and the result follows. □

Theorem 15 (Quadratic Variation of a Brownian Motion)**.** The quadratic variation of a Brownian motion over $[0, t]$ is t.

Proof. We give the proof for a sequence of partitions, for which $\sum_n \delta_n < \infty$. Let $T_n = \sum_i |B(t_i^n) - B(t_{i-1}^n)|^2$. It is easy to see that

$$\mathbb{E}[T_n] = \mathbb{E}\left[\sum_i |B(t_i^n) - B(t_{i-1}^n)|^2\right] = \sum_{t=1}^{n}(t_i^n - t_{i-1}^n) = t - 0 = t.$$

By using the fourth moment of $N(0, \sigma^2)$ distribution is $3\sigma^4$, we obtain the variance of T_n

$$\begin{aligned}\text{Var}[T_n] &= \text{Var}[\sum_i |B(t_i^n) - B(t_{i-1}^n)|^2] = \sum_i \text{Var}[(t_i^n) - B(t_{i-1}^n)^2] \\ &= \sum_i 3(t_i^n - t_{i-1}^n)^2 \leq 3\max(t_i^n - t_{i-1}^n)t = 3t\delta_n.\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \text{Var}[T_n] < \infty$. Using the monotone convergence theorem, we find

$$\sum_{n=1}^{\infty} \text{Var}[T_n] = \sum_{n=1}^{\infty} \mathbb{E}[(T_n - \mathbb{E}[T_n])^2] = \mathbb{E}\left[\sum_{n=1}^{\infty}(T_n - \mathbb{E}[T_n])^2\right] < \infty.$$

This implies the series inside the expectation converges almost surely. Hence its terms converge to zero, and $T_n - \mathbb{E}[T_n] \to 0$ a.s., consequently $T_n \to t$ a.s. □

The non-zero quadratic variation of Brownian motion also implies unbounded variation for all trajectories of Brownian motion.

5.4 Construction of Itô's Integral

To compensate for the fact that the Riemann-Stieltjes integral does not exist when integrating with respect to Brownian motion, we must define an alternate method of integration, the Itô integral. We will only introduce the Itô integral when integrating simple processes.

We can begin by identifying properties that we would like

$$\int_0^T X(t)dB_t$$

to have. The integral should have the property that if $X(t) = 1$, then $\int_0^T dB_t = B_T - B_0$. At the same time, if the integrand is a constant function (i.e. $X(t) = c$), we would like $\int_0^T X(t)dB_t = c(B_T - B_0)$. The integral over $(0, T]$ should also be the sum of integrals over sub-intervals $[0, a]$ and $[a, T]$, so $\int_0^T X(t)dB_t = \int_0^a X(t)dB_t + \int_a^T X(t)dB_t$. That way, $\int_0^T X(t)dB_t$ can be easily defined if $X(t)$ takes two different values over the interval. As such, the integral is defined for simple processes, processes that are constant over finitely many intervals. This can then be extended through the limiting procedure to define more general processes.

Definition 11 (Simple Non-random Processes)**.** A simple non-random process $X(t)$ is a process for which there exist times $0 = t_0 < t_1 < \cdots < t_n = T$ and constants $c_0, c_1, ..., c_{n-1}$, such that

$$X(t) = c_0 I_0(t) + \sum_{i=0}^{n-1} c_i I_{(t_i, t_{i+1}]}(t),$$

where $I_A(t)$ is the indicator function, taking the value 1 if $t \in A$ and 0 otherwise. A simple non-random process is essentially a piecewise function.

The Itô integral of a simple non-random process is defined as a sum

$$\int_0^T X(t)dB_t = \sum_{i=0}^{n-1} c_i(B_{t_{i+1}} - B_{t_i}).$$

Due to the independence of increments for Brownian motion, the sum is a Gaussian random variable with mean 0 and variance

$$\begin{aligned}\text{Var}\left[\int_0^T X(t)dB_t\right] &= \text{Var}\left[\sum_{i=0}^{n-1} c_i(B_{t_{i+1}} - B_{t_i})\right] \\ &= \sum_{i=0}^{n-1} \text{Var}[c_i(B_{t_{i+1}} - B_{t_i})] \\ &= \sum_{i=0}^{n-1} c_i^2(t_{i+1} - t_i).\end{aligned}$$

We can next define simple random processes, but before that, the concept of being *adapted to a filtration* must be introduced.

Definition 12. A process $X(t)$ is called adapted to a filtration $\mathcal{F} = (\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for all t.

We can now define what a simple adapted process is.

Definition 13 (Simple Adapted Processes)**.** A process $X = \{X(t), 0 \leq t \leq T\}$ is called a simple adapted process if there exist times $0 = t_0 < t_1 < \cdots < t_n = T$ and random variables $\xi_0, \xi_1, ..., \xi_{n-1}$, such that ξ_0 is a constant, ξ_i is $\mathcal{F}_{t_i}$-measurable (depends on the values of B_t for $t \leq t_i$, but not on values of B_t for $t > t_i$), and $\mathbb{E}[\xi_i^2] < \infty$, $i = 0, ..., n-1$; such that

$$X(t) = \xi_0 I_0(t) + \sum_{i=0}^{n-1} \xi_i I_{(t_i, t_{i+1}]}(t).$$

The main difference between simple non-random processes and simple adapted processes is that the c_i's in simple non-random processes are constants, while the ξ_i's in simple adapted processes are random variables.

The Itô integral for simple adapted processes is defined as such

$$\int_0^T X(t)dB_t = \sum_{i=0}^{n-1} \xi_i(B_{t_{i+1}} - B_{t_i}).$$

The Itô integral of simple adapted processes has four main properties

1. Linearity. If $X(t)$ and $Y(t)$ are simple processes and α and β are constants, then

$$\int_0^T \alpha X(t) + \beta Y(t) dB_t = \alpha \int_0^T X(t)dB_t + \beta \int_0^T Y(t)dB_t.$$

2. For the indicator function over an interval $(a, b]$ ($I_{(a,b]}$),

$$\int_0^T I_{(a,b]} dB_t = B_a - B_b, \quad \int_0^T I_{(a,b]} X(t)dB_t = \int_a^b X(t)dB_t.$$

3. Zero mean property. $\mathbb{E}[\int_0^T X(t)dB_t] = 0$.

4. Isometry property.

$$\mathbb{E}\left[\left(\int_0^T X(t)dB_t\right)^2\right] = \int_0^T \mathbb{E}[X^2(t)]dB_t.$$

We will not be providing proofs for these properties for now.

6 Discussion and Conclusions

The report explores the three defining properties of Brownian motion, as well as their general implications on the recurrence property, maximums and minimums over a time interval and hitting time distributions of Brownian motion. It then looks at a few important martingales that arise from Brownian motion, as well as how Brownian motion can be defined differently through Lévy's characterisation. Finally, we look at the analytical properties of Brownian motion, such as the non-differentiability, the unbounded variation and the non-zero quadratic variation, as well as the consequences on integration. The report does not explore the arcsine law of Brownian motion, which looks at the probability distribution of zeroes of Brownian motion, or any applications of the exponential martingale of Brownian motion, such as Girsanov theorem. Lastly, it only constructs the Itô itegral for simple adapted processes and not general adapted processes, without providing proof of the properties either. These are all potential areas of further study.

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Appendix

Properties of Conditional Expectation

These include some common properties of conditional expectation used throughout the report, as stated by Williams (1991). Let all X's satisfy $\mathbb{E}[|X|] < \infty$ and let $\mathcal{G}$ and $\mathcal{H}$ denote sub-σ-algebras of $\mathcal{F}$.

- If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$, a.s.
- Linearity. $\mathbb{E}[a_1X_1 + a_2X_2|\mathcal{G}] = a_1\mathbb{E}[X_1|\mathcal{G}] + a_2\mathbb{E}[X_2|\mathcal{G}]$, a.s.
- Tower property. If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}], \text{ a.s.}$$

- 'Taking out what is known'. If Z is $\mathcal{G}$-measurable and bounded, then

$$\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}], \text{ a.s.}$$

- Role of independence. If $\mathcal{H}$ is independent of $\sigma(\sigma(X), G)$, then

$$\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] = \mathbb{E}[X|\mathcal{G}], \text{ a.s.}$$

  In particular, if X is independent of $\mathcal{H}$, then $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$, a.s.