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<Random Walks (Generic) and Their Applications>

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1 Prelude

1.1 Acknowledgements

It is a pleasure to thank my supervisors, A/Prof. Andrea Collevecchio and Prof. Kais Hamza, for their encouragements, advices, and guidances throughout the project.

1.2 Abstract

This paper first introduces random walks and the concept of (discrete) harmonic functions. We discuss how random walks relate to the Dirichlet problem for harmonic functions and provide its unique solution. Afterwards, we introduce the heat equation in a discrete set-up and analyse it in a probabilistic way by finding its solution with random walks. In continuous space and continuous time, the heat equation can be interpreted by Brownian motion, and its solution can be expressed in terms of the density function of Brownian motion. Conversely, by choosing the delta function as the initial condition, we can use the solution of the heat equation to study the random process.

1.3 Statement of Authorship

The research is based on the results and ideas in the book "Random Walk and the Heat Equation" written by Gregory Lawler. I have summarised the information and added extra details to fulfill the results and unproven ideas. I benefited greatly from the discussion with my supervisors, A/Prof. Andrea Collevecchio and Prof Kais Hamza.

2 Introduction

Random walk is one of the fundamental stochastic processes that is widely studied. It satisfies Markov property, in which future behaviour is independent of past history. Heat equation is a partial differential equation (PDE) that contains time variable and position variable and certain of its partial derivative. There are many methods to solve the heat equation, such as Separation of variables and Fourier transforms. It is interesting to investigate both discrete and continuous heat equations from a probabilistic perspective. Furthermore, harmonic functions and boundary value problems are discussed to facilitate understanding. The solutions to boundary value problems and heat equations can be found by imagining particles doing the random walk and Brownian motion. The explicit equations of solutions are presented in terms of properties and results of random walk and Brownian motion.

3 Discrete case

3.1 Random walk

Random walks are stochastic processes. There are simple random walks that present a path which consists of independent steps towards one of all possible lattice directions with a length of 1 at each time step. Random walk on the integer lattice $\mathbb{Z}^d$ can be imagined as a walker that starts at a point and moves to an adjacent point at each time step. In this paper, the relationship between the simple symmetric random walk and heat equation shall be investigated. A simple symmetric random walk is a random walk such that the walker has an equivalent probability of moving to one of its immediate neighbours.

Definition 3.1.1 [Simple random walk in $\mathbb{Z}^d$] *Suppose that $X_1, X_2, ..., X_n$ is a sequence of independent and identically distributed random variables such that X_n has equalling probability $\frac{1}{2d}$ of each unit vector with one component of absolutely value 1 in $\mathbb{Z}^d$. Random walk started at point $S_0 \in \mathbb{Z}^d$ and*

$$S_n = S_0 + X_1 + X_2 + ... + X_n \tag{1}$$

X_n is referred to as independent step of the simple random walk at each time step. S_n is a sequence/discrete stochastic process and presents simple random walk in d-dimensional integer lattice.

3.2 Harmonic functions

Before exploring the heat equation, the harmonic function and boundary value problems must first be introduced here. We define two linear operators $\mathbf{Q}, \mathcal{L}$ on functions $F : \mathbb{Z}^d \to \mathbb{R}$ by

$$\mathbf{Q}F(x) = \frac{1}{2d} \sum_{y \in \mathbb{Z}^d, |x-y|=1} F(y), \quad x \in \mathbb{Z}^d \tag{2}$$

$$\mathcal{L}F(x) = (\mathbf{Q} - \mathbf{I})F(x) = \frac{1}{2d} \sum_{y \in \mathbb{Z}^d, |x-y|=1} F(y) - F(x) \tag{3}$$

The linear operator $\mathcal{L}$ is called the *(discrete) Laplacian.* If we let S_n be a simple random walk in $\mathbb{Z}^d$. Then we can find,

$$\mathcal{L}F(x) = \frac{1}{2d} \sum_{y \in \mathbb{Z}^d, |x-y|=1} F(y) - F(x) = \mathbb{E}[F(S_{n+1}) - F(S_n)|S_0, S_1, ..., S_{n-1}, S_n = x] \tag{4}$$

Definition 3.2.1 *A function $F : \mathbb{Z}^d \to \mathbb{R}$ is harmonic at x if $\mathcal{L}F(x) = 0$.*

Definition 3.2.2 *Given a set $A \subset \mathbb{Z}^d$, a function $F : A \to \mathbb{R}$ is harmonic on A if $\mathcal{L}F(x) = 0$ for all $x \in A$.*

3.3 Boundary value problems

The Dirichlet problem for harmonic functions is stated as a boundary value problem and will be explored in this section. Dirichlet problem indicates the boundary condition is Dirichlet type such that assigns a specific value to function F at the boundary. Let's first consider the case when A is finite subset, where a boundary and closure of set are defined for the following discussion.

For the subset $A \subset \mathbb{Z}^d$, the *boundary* of A (∂A) can be defined as

$$\partial A = \left\{ z \in \mathbb{Z}^d \backslash A : dist(z, A) = 1 \right\} \tag{5}$$

The *closure* of A, denoted $\bar{A}$, is the union of A and ∂A.

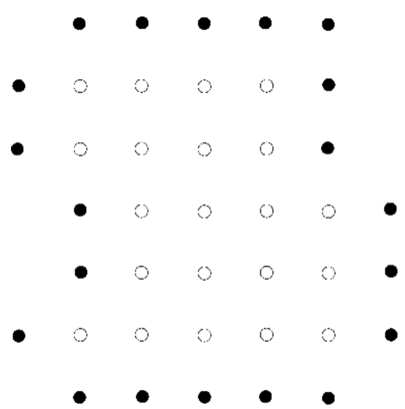

Figure 1. White points are set A, black points are boundary of A

Note that, for any finite set A in d-dimensional integer grid, its boundary point set ∂A is well defined as each finite set has a unique boundary set. Given that the function $F : \partial A \to \mathbb{R}$ is same as assigning a specific real number to each boundary point, we should be able to find the solution to the boundary value problem.

Theorem 3.3.1 *If A is a finite subset of $\mathbb{Z}^d$, for any function $F : \partial A \to \mathbb{R}$, there is an unique extension F^* to $\bar{A}$ such that F^* is harmonic on A and $F^*(x) = F(x), x \in \partial A$. It is given by*

$$F^*(x) = E[F(S_{T_A})|S_0 = x] = \sum_{y\in\partial A} \mathbb{P}\{S_{T_A} = y|S_0 = x\} F(y) \tag{6}$$

where T_A is the stopping time of the random walk which starts at point x in A, $T_A = min\{n \geq 0, S_n \in \partial A\}$. S_n is a simple random walk in $\mathbb{Z}^d$

Before we prove this theorem, we will define a martingale with respect to a sequence.

Definition 3.3.2 *A sequence of random variable $M_0, M_1, ...,$ is martingale with respect to the sequence $X_0, X_1, ...,$ if for all $n \geq 0$, the following holds.*

1. M_n is a function of $X_0, X_1, ...X_n$
2. $\mathbb{E}[|M_n|] < \infty, n \geq 0$
3. $\mathbb{E}[M_{n+1}|X_0, ..., X_n] = M_n$

Proposition 3.3.3 *Suppose $F : \mathbb{Z}^d \to \mathbb{R}$ satisfies $\mathcal{L}F(x) = 0$, Then $M_n = F(S_n)$ is a martingale respect to random walk S_n.*

Proof.

1. M_n is a function of $S_0, S_1, ..., S_n$
2. $\mathbb{E}[|F(S_n)|] \leq n \times max_{1\leq i\leq n}(|F(S_i)|) < \infty$
3. $\mathcal{L}F(x) = \mathbb{E}[F(S_{n+1}) - F(S_n)|S_0, S_1, ..., S_n = x] = 0$

$$\begin{aligned}
\mathbb{E}[F(S_{n+1})|S_0, ..., S_n] &= \mathbb{E}[F(S_{n+1}) - F(S_n) + F(S_n)|S_0, ..., S_n] \\
&= \mathbb{E}[F(S_{n+1}) - F(S_n)|S_0, ..., S_n] + F[S_n] \\
&= 0 + F(S_n) \\
&= F(S_n)
\end{aligned}$$

Thus, $M_n = F(S_n)$ is a martingale respect to random walk □

Theorem 3.3.4 Consider a martingale M_n and a stopping time T, both respect to $X_0, X_1, ..., X_n$. The stopped martingale as following is martingale respect to sequence $X_0, ..., X_n$,

$$\begin{cases} M_n & \text{if } n \leq T \\ M_T & \text{if } n > T \end{cases} \tag{7}$$

Proof.

$M_{n\wedge T} = M_{n-1\wedge T} + \mathbb{1}_{T\geq n}(M_n - M_{n-1})$

If $t \geq n$ then $M_{n\wedge T} = M_n$, $M_{n-1\wedge T} = M_{n-1}$, $\mathbb{1}_{T\geq n} = 1$

If $t < n$ then $M_{n\wedge T} = M_T$, $M_{n-1\wedge T} = M_T$, $\mathbb{1}_{T<n} = 0$

Let $B_k = \mathbb{1}_{T\geq k}$, $M_{n\wedge T} = \sum_{k=1}^{n} B_k(M_k - M_{k-1}) + M_0$

1. B_k is a function of $X_0, ..., X_{k-1}$. Event $\{T \geq k\} = \bigcap_{i=1}^{k-1} \{T \neq i\}$ which only depends on the information of $X_0, ..., X_{k-1}$. Thus B_k is a function : $\{X_0, ..., X_{k-1}\} \to \{0, 1\}$. Meanwhile, M_n is function of $X_0, ..., X_n$ as M_n is martingale. As a result, $M_{n\wedge T}$ is a function of $X_0, ..., X_n$

2.

$|M_{n\wedge T}| \leq \max_{0\leq k\leq n} |M_k| \leq |M_0| + ... + |M_n|$

$$\begin{aligned}
\mathbb{E}[|M_{n\wedge T}|] &\leq \mathbb{E}[|M_0| + ... + |M_n|] \\
&\leq \mathbb{E}[|M_0|] + ... + \mathbb{E}[|M_n|] \\
&< \infty && (\mathbb{E}[|M_0|] < \infty)
\end{aligned}$$

3.

$$\begin{aligned}
\mathbb{E}[M_{n+1\wedge T}|X_0, ..., X_n] &= \mathbb{E}[M_{n\wedge T} + \mathbb{1}_{T\geq n}(M_{n+1} - M_n)|X_0, ..., X_n] \\
&= M_{n\wedge T} + \mathbb{1}_{T\geq n}\mathbb{E}[M_{n+1} - M_n|X_0, ..., X_n] \\
&= M_{n\wedge T}
\end{aligned}$$

□

Suppose F is harmonic on A, and there is a random walk starts at x, $S_0 = x \in \bar{A}$ with stopping time $T_A = min\{n \geq 0 : S_n \notin A\}$. From Proposition 3.3.3 and Theorem 3.3.4 we can see that $M_n = F(S_{n\wedge T_A})$ is a martingale. By taking expectation on both side of $\mathbb{E}[M_{n+1}|X_0, ..., X_n] = M_n$ yields $E(X_{n+1}) = E(X_n), n \geq 0$. Hence, we can conclude that, $\mathbb{E}[M_n] = \mathbb{E}[M_{n-1}] = ... = \mathbb{E}[M_0]$. Looking back to Theorem 3.3.1, we have

$$F(S_0) = F(x) = M_0 = \mathbb{E}[M_0] = \mathbb{E}[M_n] = \sum_{y\in\bar{A}} \mathbb{P}\{S_{n\wedge T_A} = y|S_0 = x\} F(y) \tag{8}$$

As A is a finite set, a random walk on A is identical to a absorbing markov chain. Thus, a random walker will finally end up at the boundary point in finite time such that $T < \infty$ with probability 1. We can take the limits of equation (8) to get,

$$F(x) = \lim_{n\to\infty} \sum_{y\in\bar{A}} \mathbb{P}\{S_{n\wedge T_A} = y|S_0 = x\} F(y) = \sum_{y\in\partial A} \mathbb{P}\{S_{T_A} = y|S_0 = x\} F(y) \tag{9}$$

To complete the proof of Theorem 3.3.1, uniqueness is proved below. Suppose F_1 and F_2 are two harmonic functions on A with the same boundary condition. Since ($\mathcal{L}$ is linear operator on F), difference of two harmonic functions is also harmonic. Thus, $F_1 - F_2$ is a harmonic function on A with boundary conditions of value 0, hence $(F_1 - F_2)(x) = 0$ for $x \in \bar{A}$. Thus, $F_1 = F_2$.

We can extend the Theorem 3.3.1 into the infinite set with two extra constraints. The explicit theorem is defined below,

Theorem 3.3.5 *Suppose A is a proper subset of $\mathbb{Z}^d$ such that for all $x \in \mathbb{Z}^d$,*

$$\lim_{n\to\infty} \mathbb{P}\{T_A > n | S_0 = x\} = 0$$

and function $F : \partial A \to \mathbb{R}$ is bounded, then there is an unique extension of F to $\bar{A}$ as F^* such that F^* is harmonic in A and $F^*(x) = F(x), x \in \partial A$. It is given by

$$F^*(x) = E[F(S_{T_A})|S_0 = x] = \sum_{y\in\partial A} \mathbb{P}\{S_{T_A} = y|S_0 = x\} F(y) \tag{10}$$

T_A is the stopping time of the random walk which starts at point x in A, $T_A = min\{n \leq 0, S_n \in \partial A\}$.

Proof. Let S_n be a simple random walk started at $S_0 = x, x \in \bar{A}$, with $T_A = min\{n \leq 0, S_n \in \partial A\}$. The same argument shows that $M_n = F(S_{n\wedge T_A})$ is martingale and

$$F(x) = \mathbb{E}[M_n] = \sum_{y\in\bar{A}} \mathbb{P}\{S_{n\wedge T_A} = y|S_0 = x\} F(y) \tag{11}$$

As we known, $\lim_{n\to\infty} \mathbb{P}\{T_A > n|S_0 = x\} = 0$, this implies that $\sum_{x\in\partial A} P\{S_{n\wedge T_A} = x\}$ tends to 1 and $T < \infty$, i.e.,

$$\lim_{n\to\infty} \sum_{y\in A} \mathbb{P}\{S_{n\wedge T_A} = y\} = 0$$

As F is bounded, thus

$$\lim_{n\to\infty} \sum_{y\in A} F(y)\mathbb{P}\{S_{n\wedge T_A} = y\} = 0$$

By taking the limits of equation (11), we can conclude that

$$F(x) = \lim_{n\to\infty} \sum_{y\in\bar{A}} \mathbb{P}\{S_{n\wedge T_A} = y|S_0 = x\} F(y) = \sum_{y\in\partial A} \mathbb{P}\{S_{T_A} = y|S_0 = x\} F(y) = \mathbb{E}[F(S_{T_A})|S_0 = x] \tag{12}$$

□

Example 1. Now we consider the example when subset $A \in \mathbb{Z}$ such that $A = \{1, ..., N-1\}$ and $\partial A = \{0, N\}$.

Let $F(0) = a$ and $F(N) = b$. By Theorem 3.3.1, we can find the solution

$$\begin{aligned} F(x) &= \mathbb{P}\{S_T = 0|S_0 = x\} F(0) + \mathbb{P}\{S_T = N|S_0 = x\} F(0) \\ &= (1 - \mathbb{P}\{S_T = N|S_0 = x\})F(0) + \mathbb{P}\{S_T = N|S_0 = x\} F(0) \\ &= F(0) + \frac{x}{N}[F(N) - F(0)] \\ &= a + \frac{x(b-a)}{N} \end{aligned}$$

Here $\mathbb{P}\{S_T = N|S_0 = x\} = \frac{x}{N}$ is result from the Gambler's Ruin Problem.

3.4 Heat equation

We will first introduce the general view of the mathematical model of heat flow. Suppose we have a finite set $A \subset \mathbb{Z}^d$ with boundary ∂A. The temperature at the boundary is always constant with an initial temperature condition at $x \in A$ to be $p_0(x)$. At each integer time unit n, the heat at x, denoted $p_n(x)$, is spread evenly among its 2d nearest neighbours. If the heat is transferred to the boundary, it is lost forever. A more probabilistic view of heat transfer is given by modelling this by random walks. Imagine there are a large number of "heat particles" on set A, which represents the initial heat condition. They perform the simple random walk on set A. Once they hit the boundary, they are killed. The temperature at point x on A at time n, $p_n(x)$ is given by the density of heat particles at point x.

We will discuss the case that the boundary condition is 0 at first. The Heat equation states: At any time n, the temperature at $x \in A$, $p_n(x)$ is given by the heat transformed from the neighbouring points.

$$p_{n+1}(x) = \frac{1}{2d} \sum_{|y-x|=1} p_n(y) = \mathbf{Q} p_n(x) \tag{13}$$

Introduce notation change of temperature at x respect to time n $\partial_n p_n(x) = p_{n+1}(x) - p_n(x), x \in A$, we get the *heat equation*

$$\partial_n p_n(x) = \mathcal{L} p_n(x), x \in A \tag{14}$$

Where $\mathcal{L}$ is (discrete) Laplacian as before. The initial temperature condition is given,

$$p_0(x) = f(x), x \in A \tag{15}$$

The boundary condition is fixed for every time unit n

$$p_n(x) = 0, x \in \partial A \tag{16}$$

As we known, $\mathbf{Q}$ is the linear operator which is linear transformation in this case. We can order the elements of set A and imagine each $F(x)$ on finite set A as an identical vector of $\mathbb{R}^k$ with $k =$ number of elements of set $A = |A|$. We can think of $\mathbf{Q}$ as $k \times k$ matrices and write $\mathbf{Q} = [Q(x,y)]_{x,y \in A}$ where $Q(x,y) = \frac{1}{2d}$ if $|x - y| = 1$ and otherwise $Q(x,y) = 0$. Matrix $\mathbf{Q}$ is symmetry. We define the entries $Q_n(x,y)$ of $\mathbf{Q}^n = [Q_n(x,y)]$. Then $Q_n(x,y)$ is the probability that at time n, a random walk started at x has not left the set A and at site y.

$Q_n(x, y) = Q_n(y, x)$ as $\mathbf{Q}^n$ is symmetrical.

Given any initial condition f, it is easy to obtain the $p_n(x)$ which satisfies Equation (13) - (16). If $x \in \partial A$, $p_n(x) = 0$ for all $n \geq 0$. If $x \in A$, we can set $p_0(x) = f(x)$ and $p_n(x) = \mathbf{Q}^n f(x)$ which is vector representation of function. The obtained function is unique. Suppose we have $a_n(x)$ and $b_n(x)$ that both satisfy equation (13) - (16) then $a_n(x) - b_n(x) = 0$ if $x \in \partial A$ and $a_n(x) - b_n(x) = 0 = \mathbf{Q}^n(a_0(x) - b_0(x)) = \mathbf{Q}^n(f(x) - f(x)) = 0$ if $x \in A$. So, $a_n(x) = b_n(x)$.

As A is a finite subset and $\mathbf{Q}$ is $k \times k$ real symmetric matrix, then we can find k orthogonal $k \times 1$ eigenvectors v_j with k corresponding eigenvalues λ_j such that $\mathbf{Q}v_j = \lambda_j v_j$. We know that they are orthogonal and hence linearly independent. Thus, any vector $v \in \mathbb{R}^k$ can be represented by the linear combination of k orthogonal eigenvectors v_j.

Given any initial condition $f(x)$ on A with k elements, we can present it by

$$f(x) = \sum_{j=1}^{k} c_j v_j$$

According to $p_n(x) = \mathbf{Q}^n f(x)$ and $\mathbf{Q}v_j = \lambda_j v_j$ we can get

$$p_n(x) = \sum_{j=1}^{k} c_j \lambda_j^n v_j$$

The function which satisfies the equation (13) - (16) with given initial condition is founded, we will summarise here.

Theorem 3.4.1 *Suppose A is a finite subset of $\mathbb{Z}^d$ with k elements. Given the initial condition $f(x)$, the solution to heat equation (13) - (16) is given by*

$$p_n(x) = \sum_{j=1}^{k} c_j \lambda_j^n \phi_j(x), x \in A \tag{17}$$

$\phi_j(x)$ are k linearly independent function that satisfies $\mathbf{Q}\phi_j(x) = \lambda_j \phi_j(x)$ with corresponding eigenvalue λ_j, c_j are chosen below

$$f(x) = \sum_{j=1}^{k} c_j \phi_j(x) \tag{18}$$

Temperature at boundary $p_n(x)$ is constant

$$p_n(x) = 0, x \in \partial A \tag{19}$$

Example 2 We now consider the case when $A = \{1, ..., N - 1\}$ and boundary of A, $\partial A = \{0, N\}$. From the Theorem 3.4.1, The crucial step is finding the eigenvectors and eigenvalues of $\mathbf{Q}$. We know the sum rule for sine, $sin((x \pm 1)\theta) = sin(x\theta)cos(\theta) \pm cos(x\theta)sin(\theta)$ According to sum rule for sine, we can find that

$$\begin{aligned}\mathbf{Q}\{\sin(x\theta)\} &= \frac{1}{2}sin((x+1)\theta) + \frac{1}{2}sin((x-1)\theta)\\ &= sin(x\theta)cos(\theta) + \frac{1}{2}cos(x\theta)sin(\theta) - \frac{1}{2}cos(x\theta)sin(\theta)\\ &= sin(x\theta)cos(\theta)\\ &= \lambda sin(x\theta)\end{aligned}$$

Here we have $\{\sin(x\theta)\}$ denotes the vector with dimension $N-1\times 1$ and components associated to $x\in A$ is $sin(\theta x)$. The corresponding eigenvalue to each eigenvector is $cos(\theta)$. By taking $\theta = \frac{j\pi}{N}$, we have $sin\frac{0j\pi}{N} = sin(0) = sin(j\pi) = sin\frac{Nj\pi}{N} = 0$. Thus we get eigenfunction $\phi_j(x) = \sin(\frac{j\pi x}{N})$. Since they are eigenvectors of symmetric matrix Q and have different eigenvalues, they are pairwise orthogonal and hence linearly independent.

Hence, we can write the initial condition function f on A in the way,

$$f(x) = \sum_{j=1}^{N-1} c_j sin(\frac{j\pi x}{N})$$

The temperature at x in A at time n can be written as:

$$p_n(x) = \sum_{j=1}^{N-1} c_j[cos(\frac{j\pi}{N})]^n sin(\frac{j\pi x}{N})$$

If we let the temperature at boundary points be any real number instead of 0, $F:\partial A\to\mathbb{R}$ equation (13) - (15) does not change in this case, but equation (16) changes to

$$p_n(x) = F(x), x\in\partial A \tag{20}$$

When the boundary condition is 0, it is straightforward to see that the temperature on set A will become 0 as time n goes to infinity $p_n(x)\to 0$. Because the heat particle will reach the boundary will be lost eventually. When there is heat on the boundary points, we can expect the heat equation to be at equilibrium and harmonic on each point when the time approaches infinity. The temperature at each point will stay the same and equal to the solution of the boundary value problem. Thus we can separate this problem into two problems, and write

$$p_n(x) = U_n(x) + V_n(x) \tag{21}$$

Where $U_n(x)$ is the transient response which is the solution of heat equation with the new initial condition and value 0 boundary condition. $V_n(x)$ is a steady-state response which is the solution of the Dirichlet problem for harmonic functions with a given boundary value problem $F:\partial A\to\mathbb{R}$. $V_n(x)$ is fixed according to time. The new initial condition for $U_n(x)$ is $I(x) = f(x) - V_n(x): A\to\mathbb{R}$. $U_n(x)$ is the solution to the heat equation with initial condition $I(x)$ and a boundary temperature of 0. $V_n(x)$ can be obtained from Theorem 3.3.1 and $U_n(x)$ can be founded using Theorem 3.4.1. They are both unique, which has already been shown. The solution to the heat equation with a non zero boundary condition is given by equation (21).

4 Continuous Case

4.1 Brownian Motion

Brownian motion can be imagined as the limit of random walk with increments of time and space approach to 0. However, the detail of construction of Brownian motion is ignored here. We will introduce the definition of Brownian motion for the following sections.

Definition 4.1.1 *A process is* $\{W_t, t \geq 0\}$ *is called standard one dimensional Brownian motion if following properties hold,*

(i) $W_0 = 0$.

(ii) It has normal distributed increments, for all $s < t$ the random variable $W_t - W_s \sim N(0, t-s)$.

(iii) Its increments are independent of the past, i.e., the random variable $W_t - W_s$ is independent of $W_r : r \leq s$.

(iv) All sample paths of W are continuous function of t with probability 1.

Standard Brownian motion in R^d consists of d independent one dimensional standard Brownian motion in each direction. Let $W_t = (W_t^1, ..., W_t^d)$ denotes the standard Brownian motion in R^d, then density of W_t equals to,

$$\phi(x_1, ..., x_d) = (2\pi t)^{-\frac{d}{2}} exp\left\{\frac{x_1^2 + ... + x_d^2}{2t}\right\} \tag{22}$$

As we can see the density function is radially symmetric. We will use the rotation invariant property of d-dimensional Brownian motion later.

4.2 Harmonic Functions

In the discrete case, the function F on $\mathbb{Z}^d$ is harmonic on $x \in \mathbb{Z}^d$, if $F(x)$ equals the average of the function value on its nearest points. In the continuous case, if U is an open subset of R^d, we would expect f to be harmonic on x if it equals an average of function values on any sphere with centre x in set U.

Notation 4.2.1 *A subset* U *of* $\mathbb{R}^d$ *is open if every point in U is center of an open ball contained in U. Open ball with center* x *and radius* r *is defined by*

$$B_r(x) = \{y : |y - x| < r\}$$

Definition 4.2.2 *Suppose* U *is an open subset of* $\mathbb{R}^d$*, a function* $f : U \to \mathbb{R}$ *is harmonic in U, if and only if it is continuous and satisfies the following mean value property: for every* $x \in U$*, and every* $0 < \epsilon < dist(x, \partial U) = inf\,\{d(x, u) : u \in \partial U\}$

$$f(x) = MV(f; x, \epsilon) = \int_{|y-x|=\epsilon} f(y) ds(y) \tag{23}$$

The function s *is the surface measure on the sphere of radius* ϵ *at center* x *normalised such that*

$$\int_{|y-x|=\epsilon} 1 ds(y) = 1$$

The function $MV(f; x, \epsilon)$ equals the mean value of f on the sphere of radius ϵ about x

4.3 Boundary value problems

The boundary value problems in the continuous case is similar to the discrete case, but some conditions are edited for the continuous space. Given the continuous function on the boundary of a bounded open subset A $F : \partial U \to R$, we are going to find the harmonic functions on U. We will summarise the problem to make it more clear, Given a bounded open subset $U \subset \mathbb{R}^d$ and a continuous function $F : \partial U \to \mathbb{R}$, we try to find an extension of F to the closure of U, which is harmonic such that:

$$F : \bar{U} \to \mathbb{R} \; is \; continuous \tag{24}$$

$$F(x) = MV(F; x, \epsilon), x \in U \tag{25}$$

Theorem 4.3.1 Suppose W_T is Brownian motion starts from x and $T = min\{t \geq 0 : W_t = y, |y - x| = \epsilon\}$ distribution of W_T is uniformly distributed on the sphere of radius ϵ at x.

Proof. Given W_t as the multidimensional-Brownian motion, and W'_t as the rotation of W_t according to the started point, as we know, Brownian motion is rotationally invariant. So W'_t is also multidimensional-Brownian motion, such that W_t and W'_t has the same distribution. This implies that W_T and W'_T have the same distribution and are rotationally invariant. Meanwhile, both W_T and $W'_T \in$ sphere of radius ϵ at x. Thus, the distribution of W_T is uniformly distributed on the sphere of radius ϵ. Thus, $MV(f, x, \epsilon) = E(f(W_T)|W_0 = x)$ □

Theorem 4.3.2 Given a bounded open subset $U \subset \mathbb{R}^d$ and a function $F : \partial U \to \mathbb{R}$, an extension of F to closure of U which is harmonic in U is given by:

$$F(x) = \mathbb{E}[F(W_{T_U})|W_0 = x] \tag{26}$$

Proof. Given $T = \{t \geq 0 : W_t = y, |y - x| = \epsilon\}$, for every $0 < \epsilon < dist(x, \partial U)$

$$\mathbb{E}[W_{T_U}|W_0 = x] = \mathbb{E}[\mathbb{E}[W_{T_U}|W_0 = x, W_T = y]|W_0 = x]$$

From the strong markov property, we know that W_{T+t} is Brownian motion started at W_T which is independent to the past. Thus, $\mathbb{E}[W_{T_U}|W_0 = x, W_T = y] = \mathbb{E}[W'_{T_U}|W'_0 = W_T]$. W'_t is a brownian motion starts at $W_T = y$.

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[W_{T_U}|W_0 = x, W_T = y]|W_0 = x] &= \mathbb{E}[\mathbb{E}[W'_{T_U}|W'_T = W_T]|W_0 = x] \\
&= \mathbb{E}[F(W_T)|W_0 = x] \\
&= MV(F; x, \epsilon)
\end{aligned}$$

□

The function is also unique and continuous, both are not proven here and should be proven after obtaining required knowledge in the future.

4.4 Heat equation

We will consider the mathematical model of heat transfer on a subset of $\mathbb{R}^d$ with diffusion coefficient $\frac{1}{2}$. Let $u(t,x), t \geq 0, x \in \mathbb{R}^d$ denote the temperature at point x at time t. We obtain the *heat equation* $\partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x)$. Firstly, we will consider the domain is R^d and discuss the bounded subset later.

Given an initial temperature condition as bounded and continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we find the solution to $u(t,x)$. Like the discrete case, we can imagine there is a large number of heat particles initially and doing Brownian motion as the heat transfer process. The temperature at each point x at time t is the density of heat particle over x at t. On a very heuristic level, we can imagine that there are $f(y)$ particles that starts at y at time 0. The fraction of particle that starts at y will be at point x at time t is the probability that a Brownian motion starts at y and has moved to x at time t. Thus, the temperature at point x at time t equals to the sum of fraction of heat particles from all points at time t. According to the density function of d-dimensional Brownian motion, we would expect the probability that a particle moves from x to y in time t is the same as from y to x. As a result, we get,

$$u(t,x) = \mathbb{E}[f(W_t)|W_0 = x] \tag{27}$$

W_t is a d-dimensional Brownian motion starting at x, the probability density function at fixed t is given by

$$p(t,x,y) = \frac{1}{(2\pi t)^{(d/2)}} e^{-\frac{|y-x|^2}{2t}} \tag{28}$$

p(t,x,y) is a function of y with fixed x and t.

Instead of expectation expression, we can write equation (27) as

$$u(t,x) = \int_{\mathbb{R}^d} f(y)p(t,x,y)dy_1...dy_d = \int_{\mathbb{R}^d} f(y)\frac{1}{(2\pi t)^{(d/2)}} e^{-\frac{|y-x|^2}{2t}} dy_1...dy_d \tag{29}$$

At time $t = 0$, $u(0,x) = u(0^+,x) = \lim_{t\to 0^+} u(t,x) = f(x)$ as time t approach o from positive side, the density function turns into Dirac delta function.

Notation 4.4.1 *On heuristic level, the Dirac delta function is a function on real line which satisfies*

$$\delta(x) = \begin{cases} +\infty \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases} \tag{30}$$

and also

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{31}$$

Given continuous and bounded function f, and let temperature at point x at time t $u(t,x)$ be equation (29), we can find it satisfies the heat eqaution.

Proof.

$$\begin{aligned}\partial_t u(t,x) = &= \int_{\mathbb{R}^d} f(y)\frac{\partial}{\partial t}(\frac{1}{(2\pi t)^{d\backslash 2}} e^{-\frac{|y-x|^2}{2t}})dy_1...dy_d \\ &= \int_{\mathbb{R}^d} f(y)(-\frac{d}{4\pi}t^{-(d+2)\backslash 2}e^{-\frac{|y-x|^2}{2t}} - \frac{1}{2\pi}t^{-d\backslash 2}\frac{|y-x|^2}{2t^2}e^{-\frac{|y-x|^2}{2t}})dy_1...dy_d\end{aligned}$$

$$\partial_j u(t,x) = \frac{d}{2\pi} t^{-d\backslash 2} \frac{y_i - x_i}{t} e^{-\frac{|y-x|^2}{2t}}$$

$$\partial_{jj} u(t,x) = \frac{d}{2\pi} t^{-d\backslash 2} \frac{-1}{t} e^{-\frac{|y-x|^2}{2t}} + \frac{d}{2\pi} t^{-d\backslash 2} \frac{(y_i - x_i)^2}{t^2} e^{-\frac{|y-x|^2}{2t}}$$

$$\begin{aligned}\Delta_x u(t,x) = \sum_{j=1}^{d} \partial_{jj} u(t,x) &= \int_{\mathbb{R}^d} f(y) \sum_{j=1}^{d} \frac{\partial}{\partial x_{jj}} (\frac{1}{(2\pi t)^{d\backslash 2}} e^{-\frac{|y-x|^2}{2t}}) dy_1 ... dy_d \\ &= \int_{\mathbb{R}^d} f(y) (-\frac{d}{2\pi} t^{-(d+2)\backslash 2} e^{-\frac{|y-x|^2}{2t}} - \frac{1}{2\pi} t^{-d\backslash 2} \frac{|y-x|^2}{t^2} e^{-\frac{|y-x|^2}{2t}}) dy_1 ... dy_d\end{aligned}$$

Thus, $\partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x)$ which satisfies the heat equation with given initial condition.

Solution $u(t,x) = \int_{\mathbb{R}^d} f(y)p(t,x,y)dy_1...dy_d$ is unique as if there are two solutions $u_1(t,x), u_2(t,x)$ both satisfy the problem,

$$v(t,x) = u_1(t,x) - u_2(t,x)$$

$$\partial t v(t,x) = \partial t(u_1(t,x) - u_2(t,x)) = \frac{1}{2}\Delta_x(u_1(t,x) - u_2(t,x)) = \frac{1}{2}\Delta_x v(t,x)$$

$$v(0,x) = u_1(0,x) - u_2(0,x) = f(x) - f(x) = 0, \forall x \in \mathbb{R}^d$$

It is a heat equation problem with zero initial condition, we can find $v(t,x) = 0$ for all $x \in \mathbb{R}^d$ and $t \geq 0$

$$v(t,x) = \int_{\mathbb{R}^d} 0 \times p(t,x,y)dy_1...dy_d = 0, \forall x \in \mathbb{R}^d, \forall t \in [0,\infty]$$

Thus, $u_1(t,x) = u_2(t,x)$ and the solution is unique. □

Theorem 4.4.2 *Given continuous and bounded function on $\mathbb{R}^d$, $f : \mathbb{R}^d \to R$ which provides the initial condition on each point in $\mathbb{R}^d$. Then u(t,x) as defined below satisfies the heat equation*

$$u(t,x) = \int_{\mathbb{R}^d} f(y) \frac{1}{(2\pi t)^{(d/2)}} e^{-\frac{|y-x|^2}{2t}} dy_1...dy_d \tag{32}$$

$$\partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x), t > 0 \tag{33}$$

With initial condition

$$u(0,x) = u(0^+,x) = f(x) \tag{34}$$

Δ_x is the Laplacian of x variable.

Furthermore, there is an extra boundary condition when the domain is bounded instead of all of $\mathbb{R}^d$. We will consider the boundary condition that always has zero temperature. We still have the same interpretation of heat transfer, but heat particles are destroyed once they hit the boundary. Suppose we have bounded open subset $U \subset \mathbb{R}^d$ and ∂U is the boundary of U. Let W_t be Brownian motion and $T = T_U = inf\{t : W_t \in \partial U\}$ be stopping time. We have

$$u(t,x) = \mathbb{E}[f(W_t)\mathbb{1}\{T > t\}|W_0 = x]$$

Let $p(t,x,y,U)$ denote the density function such that particle doing Brownian motion is at $y \in U$ at time t with $W_0 = x, x \in U$ and the particle has not reached boundary by time t. This is identical to the function which satisfies, $\mathbb{P}\{W_t \in V, T > t|W_0 = x\} = \int_V p(t,x,y,U)dy_1...dy_d$. We can write $u(t,x)$ as

$$u(t,x) = \int_U f(y)p(t,x,y,U)dy_1...dy_d \tag{35}$$

p(t,x,y,U) = p(t,y,x,U) as every path from x to y staying in U is also a path from y to x in U.
We have obtained the solution of the heat equation in a probabilistic way by using Brownian motion. Conversely, we can use the solutions of the heat equation to study Brownian motion. The heat equation is a famous partial differential equation and there are multiple ways to find the solution of the heat equation. We can use the solution of heat equation to get Brownian motion's density function.

Theorem 4.4.3 *The solution of bounded heat equation with 0 boundary condition and initial condition $f : U \to R$ as the delta function $\delta(|y - x|)$ is the density function p(t,x,y,U) of particle doing Brownian motion is at $y \in U$ at time t with $W_0 = x, x \in U$ and the particle has not reached boundary by time t.*

Proof.

$$\begin{aligned} u(t,x) &= \int_U \delta(|y - z|)p(t,x,z,U)dz_1...dz_d \\ &= p(t,x,y,U) \end{aligned}$$

□

5 Conclusion

This work has been heavily focused on building a connection between random processes and deterministic equations. The discrete and continuous heat equations resemble the random walk and Brownian motion's distribution function, respectively. The work can be further used to understand the connection between processes and other diffusion equations. Conversely, we can try to use the heat equation to study stochastic processes.

6 References

Lawler, G.F., 2010. Random walk and the heat equation (Vol. 55). American Mathematical Soc..