UNIFORM DIOPHANTINE APPROXIMATION

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Key words and phrases. Continued Fraction, Hausdorff dimension, metric Diophantine approximation.

Contents

Abstract. Let the continued fraction expansion of a real number r be $r = [a_1(r), a_2(r), ...]$. The growth of partial quotients is related with sets which are improvements to Dirichlet's theorem. We have calculated the Hausdorff dimension of the sets

$$\Lambda := \{(x, y) \in [0,1]^2 : \max\{a_n(x)a_{n+1}(x), a_n(y)a_{n+1}(y)\} \to \infty \quad \text{as} \quad n \to \infty\}$$

$$\Lambda(\Phi) := \{(x, y) \in [0,1]^2 : \max\left\{a_{un}(x)a_{un+1}(x), a_{vn}(y)a_{vn+1}(y)\right\} \geq \Phi(n) \text{ for all } n \geq 1\}.$$

Where $\Phi : \mathbb{N} \to (1, \infty)$ is a function such that $\Phi(n) \to \infty$ as $n \to \infty$.

1. Introduction

To begin we look at the properties of Continued Fraction expansions of real numbers, which can be defined as the following Gauss transformation $T : [0,1) \to [0,1)$:

$$T(0) := 0, \;\; T(x) := \tfrac{1}{x} \pmod 1, \;\; \text{for } x \in (0,1).$$

Let $a_1(x) = \lfloor x^{-1} \rfloor$. (where $\lfloor . \rfloor$ is the floor function or integer part of x^{-1}) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$. This allows every real number between [0,1) to have a unique continued fraction expansion of the form:

$$x = [a_1(x), a_2(x), a_3(x), ...] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}$$

where x is the real number being represented, $[a_1(x), a_2(x), a_3(x), ...]$ is the shorthand using the partial quotients $a_n(x)|_{n \geq 1}$. $\frac{p_n(x)}{q_n(x)} = [a_1(x), ..., a_n(x)]$ is defined as the nth convergent of x obtained by approximating x to n terms. Where $p_n(x)$ is the numerator and $q_n(x)$ is the denominator of the nth convergent.

The interest we have in continued fractions is using their convergents as approximations of irrational numbers as rational numbers. as summarised by:

$$\frac{1}{(2 + a_{n+1}(x))q_n^2(x)} \leq |x - \frac{p_n(x)}{q_n(x)}| \leq \frac{1}{(a_{n+1}(x))q_n^2(x)}$$

Recently it has been proven that the product of partial quotients is related with improvements to Dirichlet's theorem. Specifically building on the work of Davenport-Schmidt [3], Kleinbock-Wadleigh [11] considered the set:

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{c} \exists\, N : \text{ the system } |qx - p| < \psi(t), \; |q| < t \\ \text{has a non trivial integer solution for all } t > N \end{array} \right\}$$

Which they called the set of ψ-Dirichlet improvable numbers.

The nth convergents are the best rational approximation for real numbers, which shows that the continued fractions approach is very useful in analysing the properties of rational approximations of sets of real numbers. However since continued fractions are not applicable in higher dimensions, there has yet to be a higher dimensional analog to the guass map of continued fractions such that it captures all the features of approximation that it does in one dimension.

Lü-Zhang [14] attempted to use the Continued Fraction algorithm to compute Hausdorff dimension of a set with the partial quotients in their continued fraction of sets of points in the plane with certain growth conditions. To be precise, they considered the following set. For any positive integers s and t, define:

$$E = \{(x, y) \in [0, 1)^2 : \max\{a_{sn}(x), a_{tn}(y)\} \to \infty \text{ as } n \to \infty\}.$$

and calculated its Hausdorff dimension. To be precise, they obtained the following result

Theorem 1.1 (Lü-Zhang)**.**

$$\dim_{\mathcal{H}}(E) = \frac{3}{2}.$$

In this project, we extend their idea to address the corresponding problem by considering the growth of the product of consecutive partial quotients. We consider the following two sets and calculate their Hausdorff dimension.

$$\Lambda := \{(x, y) \in [0, 1]^2 : \max\{a_n(x)a_{n+1}(x), a_n(y)a_{n+1}(y)\} \to \infty \quad \text{as} \quad n \to \infty\}$$

Theorem 1.2.

$$\dim_{\mathcal{H}}(\Lambda) = 3/2.$$

Let $\Phi : \mathbb{N} \to (1, \infty)$ be a function such that $\Phi(n) \to \infty$ as $n \to \infty$. For any $u, v \in \mathbb{N}$. We define the set:

$$\Lambda(\Phi) := \{(x, y) \in [0, 1]^2 : \max\left\{a_{un}(x)a_{un+1}(x), a_{vn}(y)a_{vn+1}(y)\right\} \geq \Phi(n) \text{ for all } n \geq 1\}.$$

Theorem 1.3. *Let Φ be a positive function. Then*

$$\dim_{\mathcal{H}}(\Lambda(\Phi)) = 1 + \frac{1}{1 + \tau^\alpha} = \frac{2 + \tau^\alpha}{1 + \tau^\alpha}$$

where

$$\alpha = \frac{1}{\max\{u, v\}} \quad \textit{and} \quad \log \tau = \limsup_{n\to\infty} \frac{\log \log \Phi(n)}{n}$$

2. Preliminaries

2.1. **Hausdorff measure and dimension.** Let $0 < s \in \mathbb{R}^n$ let $E \subset \mathbb{R}^n$. Then, for any $\rho > 0$ a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $\operatorname{diam}(B_i) \leq \rho$ such that $E \subset \bigcup_i B_i$ is called a ρ-cover of E. Let

$$\mathcal{H}^s_\rho(E) = \inf \sum_i \operatorname{diam}(B_i)^s,$$

where the infimum is taken over all possible ρ-covers $\{B_i\}$ of E. It is clear that $\mathcal{H}^s_\rho(E)$ increases as ρ decreases and so approaches a limit as $\rho \to 0$. This limit will be zero, infinity or a finite positive value. Accordingly, the *s-Hausdorff measure* $\mathcal{H}^s$ of E is defined as:

$$\mathcal{H}^s(E) = \lim_{\rho \to 0} \mathcal{H}^s_\rho(E).$$

For any subset E one can verify that there exists a unique value of s such that $\mathcal{H}^s(E)$ instantly goes from infinity to zero. This value of s for which this jump occurs is called the *Hausdorff dimension of* E and is denoted by $\dim_{\mathcal{H}} E$:

$$\dim_{\mathcal{H}} E := \inf\{s \in \mathbb{R}_+ \ : \mathcal{H}^s(E) = 0\}$$

2.2. **Continued fractions and Diophantine approximation.** Recall for $x \in [0,1)\backslash\mathbb{Q}$ has continued fraction expansion $x = [a_1, a_2, ...]$, from Section 1, we have $a_n(x) = [1/T^{n-1}(x)]$ for each $n \geq 1$. Recall the sequences $p_n = p_n(x),\ q_n = q_n(x)$, discussed previously has the recursive relation:

$$p_{n+1} = a_{n+1}(x)p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}(x)q_n + q_{n-1}, \quad n \geq 0. \tag{2.1}$$

Thus $p_n = p_n(x),\ q_n = q_n(x)$ are determined by the partial quotients $a_1, ..., a_n$, so we may write $p_n = p_n(a_1, ..., a_n), \quad q_n = q_n(a_1, ..., a_n)$. When it is clear which partial quotients are involved we denote them as $p_n,\ q_n$ for simplicity.

We define the cylinder of order n as the set of all real numbers whose continued fraction expansion begins with partial quotients $(a_1, ..., a_n)$, i.e.:

For any integer vector $(a_1, ..., a_n) \in \mathbb{N}^n$ with $n \geq 1$, write

$$I_n := I_n(a_1, ..., a_n) := \{x \in [0,1) : a_1(x) = a_1, ..., a_n(x) = 1_n\}$$

We will frequently be using the follow properties of continued fraction expansions. They are explained in the standard texts [9, 10].

Proposition 2.1. For any positive integers $a_1, ..., a_n$, let $p_n = p_n(a_1, ..., a_n)$ and $q_n = q_n(a_1, ..., a_n)$ be defined recursively by (1). Then:

(P$_1$)

$$I_n = \begin{cases} [\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}) & \text{if } n \text{ is even;} \\ (\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}] & \text{if } n \text{ is odd.} \end{cases}$$

Thus, the length of the cylinder of order n is given by

$$\frac{1}{2q_n^2} \leq |I_n| = \frac{1}{q_n(q_n+q_{n-1})} \leq \frac{1}{q_n^2} \leq \left(\prod_{i=1}^{n} a_i\right)^{-2},$$

since

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n, \text{ for all } n \geq 1.$$

(P$_2$) For any $n \geq 1, q_n \geq 2^{(n-1)/2}$ and

$$1 \leq \frac{q_{n+m}(a_1, ..., a_n, b_1, ..., b_m)}{q_n(a_1, ..., a_n) \cdot q_m(b_1, ..., b_m)} \leq 2.$$

(P$_3$)

$$\prod_{i=1}^{n} a_i \leq q_n \leq \prod_{i=1}^{n}(a_i+1) \leq 2^n \prod_{i=1}^{n} a_i.$$

(P$_4$)

$$\frac{1}{(a_{n+1}(x)+2)q_n^2(x)} < \left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{1}{q_n(x)(q_{n+1}(x)+T^{n+1}(x)q_n(x))} < \frac{1}{a_{n+1}q_n^2(x)}.$$

(P$_5$) There exista a constant $K > 1$ such that for almost all $x \in [0, 1)$,

$$q_n(x) \leq K^n, \text{ for all} n \text{sufficiently large.}$$

Let μ be the Gauss measure given by

$$d\mu = \frac{1}{(1+x)\log 2}dx$$

The next proposition concerns the position of a cylinder in [0,1).

Proposition 2.1 ([10])**.** *Let $I_n = I_n(a_1, ..., a_n)$ be a cylinder of order n, which is partitioned into sub-cylinders $\{I_{n+1}(a_1, ..., a_n, a_{n+1}) : a_{n+1} \in \mathbb{N}\}$. Where n is odd, these sub-cylinders are positioned from left to right, as a_{n+1} increases from 1 to ∞; when n is even, there are positioned from right to left.*

The following result is due to Luczak[12].

Lemma 2.2 ([12])**.** *For any $b, c > 1$, the sets*

$$\{x \in [0,1) : a_n(x) \geq c^{b^n} \textit{ for infinitely many } n \in \mathbb{N}\},$$

$$\{x \in [0,1) : a_n(x) \geq c^{b^n} \textit{ for all } n \geq 1\},$$

have the same Hausdorff dimension $\frac{1}{b+1}$.

The following Lemma was proved by Good [5].

Lemma 2.3 ([5])**.**

$$\dim_{\mathcal{H}}\{x \in [0,1) : a_{n+1}(x) \to \infty \textit{ as } n \to \infty\} = \frac{1}{2}.$$

The following lemma proved by Marstrand [13] is well-known in the field.

Lemma 2.4 ([13])**.** *For any measurable sets A and B,*

$$\dim_{\mathcal{H}}(A \times B) \geq dim_{\mathcal{H}}A + dim_{\mathcal{H}}B.$$

3. Proofs of Theorems 1.2 and 1.3

3.1. **Proof of Theorem 1.2.** Recall that:

$$\Lambda := \{(x,y) \in [0,1]^2 : \max\{a_n(x)a_{n+1}(x), a_n(y)a_{n+1}(y)\} \to \infty \quad \text{as} \quad n \to \infty\}$$

3.1.1. *Lower Bound:* We begin by defining a set

$$\lambda = \{x : a_n(x)a_{n+1}(x) \to \infty \ \text{ as } \ n \to \infty\}.$$

Then trivially $\lambda \times [0,1) \subseteq E$. We also know λ contains set $\{x : a_n(x) \to \infty$ as n $\to \infty\}$. Then Lemma 2.3,

$$\dim_{\mathcal{H}}(\{x : a_n(x) \to \infty \text{ as } n \to \infty\}) = \frac{1}{2}.$$

This tells us that: $\dim_{\mathcal{H}}(\lambda) \geq \frac{1}{2}$.

Which tells us that the minimum Hausdorff dimension for $\lambda \times [0,1) = 1 + \frac{1}{2} \Rightarrow \dim_{\mathcal{H}}(\lambda \times [0,1)) \geq \frac{1}{2} + 1$. Hence

$$\dim_{\mathcal{H}}(\Lambda) \geq \frac{3}{2}.$$

3.1.2. *Upper Bound:* We start off by defining a function Λ_M,

$$\Lambda_M = \bigcap_{N=1}^{\infty} \{(x,y) \in \mathbb{I}^2 : \max\{a_n(x)a_{n+1}(x), a_n(y)a_{n+1}(y)\} \geq M, \quad \forall n \geq N\}.$$

It follows that:

$$\Lambda_M \subset \{(x,y) \in \mathbb{I}^2 : \max\{a_n(x)a_{n+1}(x), a_n(y)a_{n+1}(y)\} \geq M, \quad \forall n \geq 1\}.$$

Which implies that for any $(x,y) \in \Lambda_M$:

$$\text{either } a_n(x)a_{n+1}(x) \geq M \text{ for all } n \in \Omega$$
$$\text{or } a_n(y)a_{n+1}(y) \geq M \text{ for all } n \in \mathbb{N}\backslash\Omega$$

Where Ω is an infinite subset. From this we know that the following set contains Λ_M and function for use of a covering argument.

$$\Lambda_M \subseteq [\bigcup_{n\in\Omega} \bigcup_{a_1,a_2,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} I_n(a_1,...,a_n)\times[0,1)]$$
$$\bigcup [\bigcup_{n\in N\setminus\Omega} \bigcup_{a_1,a_2,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} [0,1)\times I_n(a_1,...,a_n)]$$

Here we begin to form a cover, using squares of side length $|I_n|$, to cover we need to use a number of squares equal to $|I_n|^{-1}$, forming the cover:

$$\mathcal{H}^s(\Lambda_M) \ll \liminf_{n\longrightarrow\infty} \sum_{a_1,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} (|I_n|^s|I_n|^{-1})$$

Note that, $|I_n| = (\prod_{i=1}^{n} a_i)^{-2}$. Therefore

$$\mathcal{H}^s(\Lambda_M) \ll \liminf_{n\longrightarrow\infty} \sum_{a_1,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} \prod_{i=1}^{n} a_i^{-2(s-1)}$$

We then transition this to a logarithm form:

$$\mathcal{H}^s(\Lambda_M) \ll \liminf_{n\longrightarrow\infty} \sum_{a_1,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} e^{-2(s-1)\sum_{i=1}^{n} log(a_i)}$$

We know:

$$\sum_{i=1}^{n} \log(a_i) = \log(a_1) + \log(a_2) + \cdots + \log(a_n)$$

and

$$a_1a_2 \geq M, a_2a_3 \geq M, ..., a_na_{n+1} \geq M.$$

Which implies $\sum_{i=1}^{n} log(a_i) \geq \frac{(n-1)}{2} log(M)$ Therefore:

$$\mathcal{H}^s(\Lambda_M) \ll \liminf_{n\longrightarrow\infty} \sum_{a_1,\ldots,a_{n+1}:a_r a_{r+1}\geq M;\ 1\leq r\leq n} e^{-(s-1)(n-1)\log(M)}. \tag{3.1}$$

Next, we define a family of probability measure on the unit interval [0,1] for each $t > 1$ and $(a_1, ..., a_n) \in \mathbb{N}^n$, define

$$\mu_t(I_n) = e^{-n\,P(t)-t\sum_{i=1}^{n}\log(a_i)}$$

Where $p(t) = \log \sum_{j=1}^{2} \frac{1}{j^t} < \infty$.

It follows that

$$\sum_{a_n+1} \mu_t(I_{n+1}) = \mu_t(I_n) \quad \text{and} \quad \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \mu_t(I_n) = 1.$$

Next we fix $s > \frac{3}{2}$, and let $t = s - \frac{1}{2}$ as $t > 1$. We choose M sufficiently large such that

$$0 \leq P(t) \leq (\frac{2s-3}{n}) \log(M)$$

It follows that:

$$(-s+1)(n-1)\log(M) \leq -nP(t) - t\sum_{i=1}^{n} \log(a_i)$$

$(-s+1)(n-1)\log(M) \leq -nP(t) - (s-\frac{1}{2})\frac{(n-1)}{2}\log(M)$
$0 \leq P(t) \leq (\frac{2s-3}{n})\log(M) \implies s > \frac{3}{2}$
This tells us that the RHS of (3.1) converges to zero when $s > \frac{3}{2}$. Thus by the definition of Hausdorff dimension, we have $\dim_{\mathcal{H}}(\Lambda_M) \leq 3/2$, and as a consequence $\dim_{\mathcal{H}}(\Lambda) \leq 3/2$.

3.2. **Proof of Theorem 1.3.** Recall that $\Phi : \mathbb{N} \to (1, \infty)$ is a function such that $\Phi(n) \to \infty$ as $n \to \infty$. For any $u, v \in \mathbb{N}$. We define the set:

$$\Lambda(\Phi) := \{(x, y) \in [0,1]^2 : \max\left\{a_{un}(x)a_{un+1}(x), a_{vn}(y)a_{vn+1}(y)\right\} \geq \Phi(n) \text{ for all } n \geq 1\}.$$

Theorem 1.3 Let Φ be a positive function. Then

$$\dim_{\mathcal{H}}(\Lambda(\Phi)) = 1 + \frac{1}{1+\tau^\alpha} = \frac{2+\tau^\alpha}{1+\tau^\alpha}$$

where

$$\alpha = \frac{1}{\max\{u, v\}} \quad \text{and} \quad \log\tau = \limsup_{n\to\infty} \frac{\log\log\Phi(n)}{n}$$

To prove this theorem we consider different cases for τ. When $\tau < 1$, trivially it follows that $\Lambda(\Phi) = \mathbb{I}^2$.

3.2.1. $\tau = 1$.

In this case for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $\Phi(n) \leq e^{(1+\epsilon)^n}$. Then

$$\begin{aligned}\Lambda(\Phi) &\supset \{x \in [0,1) : a_{un}(x)a_{un+1}(x) \geq e^{(1+\epsilon)^n} \text{ for all } n \geq n_0\} \times [0,1] \\ &\supset \{x \in [0,1) : a_n(x) \geq e^{(1+\epsilon)^n} \text{ for all } n \geq n_0\} \times [0,1].\end{aligned}$$

Using Lemmas 2.2 and 2.4, we have:

$$\dim_{\mathcal{H}}\Lambda(\Phi) \geq \lim_{\epsilon\to 0} \frac{1}{1+(1+\epsilon)} + 1 = \frac{3}{2}.$$

The upper bound follows from Theorem 1.2 as $\Lambda(\Phi) \subseteq \Lambda$. Proving the theorem for case $\tau = 1$.

3.2.2. $1 < \tau < \infty$.

Let $1 < c < \tau$. By definition of τ, there exists infinitely many n in an infinite subset $\Omega \subset \mathbb{N}$ such that:

$$\log c \leq \frac{\log\log\Phi(n)}{n} \quad \text{i.e.} \quad \Phi(n) \geq e^{c^n} \quad \forall n \in \Omega.$$

Thus for every $n \in \Omega$ either $a_{un}(x)a_{un+1}(x) \geq e^{c^n}$ or $a_{vn}(y)a_{vn+1}(y) \geq e^{c^n}$ We define:

$$\Lambda_1 := \left\{x \in [0,1] : a_{un}(x) \geq e^{c^n} \text{ for i.m. } n \in \mathbb{N}\right\}$$

and

$$\Lambda_2 := \left\{y \in [0,1] : a_{vn}(y) \geq e^{c^n} \text{ for i.m. } n \in \mathbb{N}\right\}$$

Hence

$$\Lambda(\Phi) \subseteq (\Lambda_1 \times [0,1]) \bigcup ([0,1] \times \Lambda_2)$$

By Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
\dim_{\mathcal{H}}(\Lambda(\Phi)) &\leq 1 + \max\left\{\lim_{c\to\tau} \frac{1}{1+c^{\frac{1}{u}}}, \lim_{c\to\tau} \frac{1}{1+c^{\frac{1}{v}}}\right\} \\
&\leq 1 + \lim_{c\to\tau} \frac{1}{1+c^{\frac{1}{\max\{u,v\}}}} \\
&\leq 1 + \frac{1}{1+\tau^{\frac{1}{\max\{u,v\}}}}
\end{aligned}$$

The lower bound for this case is proven by a similar argument to the upper. We fix $c > \tau$, then $\Phi(n) \leq e^{c^n}$ holds for all $n \geq n_0$. Therefor if we choose $u < v$, we then have the inclusion

$$\begin{aligned}
\Lambda(\Phi) &\supseteq \left\{x \in [0,1] : a_{un}(x)a_{un+1}(x) \geq e^{c^n} \text{ for all } n \geq n_0\right\} \times [0,1] \\
&\supseteq \left\{x \in [0,1] : a_{un}(x) \geq e^{c^n} \text{ for all } n \geq n_0\right\} \times [0,1]
\end{aligned}$$

Therefor, By Lemmas 2.2 and 2.4 we know:

$$\dim_{\mathcal{H}}(\Lambda(\Phi)) \geq 1 + \lim_{c\to\tau} \frac{1}{1+c^{\max\{u,v\}}} = 1 + \frac{1}{1+\tau^{\max\{u,v\}}}.$$

3.2.3. $\tau = \infty$. This case is quickly solved from the above argument that

$$\dim_{\mathcal{H}}(\Lambda(\Phi)) = 1 + \lim_{\tau\to\infty} \frac{1}{1+\tau^{\max u,v}} = 1 + 0 = 1.$$

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