Chern-Weil Theory in Geometry and Algebraic Topology

Jamie Bell

Supervised by A/Prof. Adam Rennie & Dr. Alex Mundey
University of Wollongong

Abstract

This summer research project will explore the profound relationship between the geometry and topology of manifolds, as encoded by vector bundles. Known as Chern-Weil theory, this approach to studying the interplay of geometry and topology has been foundational to the modern study of geometry, to the development of gauge theories of particle physics (such as the standard model of particle physics), and to the ongoing development of noncommutative geometry. The project will involve learning about the language of vector bundles, connections and curvature, as well as de Rham cohomology. All these topics can be phrased using differential geometry, in particular exterior calculus.

Contents

1 Introduction

This summer research project explores a remarkable link between the geometry and topology of manifolds, known as Chern Weil Theory. In order to appreciate this correspondence, the first aim of this project was to become proficient with the topological and geometrical foundations of the theory. This was accomplished primarily through the close study of specific examples, which feature prominently in this report. In Section 2, we study a calculus-based approach to algebraic topology known as de Rham cohomology. This provides a method of computing topological invariants using the language of exterior algebra. Section 3 then introduces vector bundles, which are the mathematical objects which encode the geometric and topological information in which we are interested. Again, emphasis is placed on computations and specific examples, most notably the Hopf bundle which has numerous applications in both mathematics and physics. Geometry is then introduced in Section 4 through the study of connections and curvature. Roughly speaking, connections generalise the derivative to the vector bundle setting, and curvature gives a measure of the anti-commutativity of this derivative. We also construct examples of connections on vector bundles, and compute their curvature. Finally, in Section 5, we demonstrate how to extract topological information from the curvature via characteristic classes.

1.1 Statement of Authorship

This report is the original work of its author, Jamie Bell. The results presented in this report were inspired by notes provided by, and discussions with, A/Prof. Adam Rennie and Dr. Alex Mundey, as well as from various sources from the academic literature, which are cited throughout the report.

1.2 Acknowledgements

My sincerest thanks go to my supervisors Adam Rennie and Alex Mundey for their invaluable assistance during this research project. Not only did they help me to grasp many difficult concepts, they also trained me to be a better mathematician. I would also like to acknowledge the support and input of Angus Alexander, Ada Masters, Abraham Ng and Alex Paviour.

I extend my appreciation to the Australian Mathematical Sciences Institute for their organisation and funding of the summer research scholarship, as well as the School of Mathematics and Applied Statistics at the University of Wollongong for supplemental funding and their ongoing support of my studies.

2 de Rham Cohomology

Cohomology is a general tool in algebraic topology which describes topological invariants of spaces in algebraic terms. Roughly speaking, de Rham cohomology uses calculus to describe holes in manifolds. Let us start by recalling some facts and definitions from exterior algebra. Most of this discussion is adapted from [4], which contains further details for the interested reader.

2.1 Exterior Calculus

Definition 2.1. Let M be a smooth manifold, $U \subset M$ an open neighbourhood of $x \in U$ and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma(0) = x$. Then the *tangent vector to* γ *at* x is the function $v : C^\infty(U) \to \mathbb{R}$ which satisfies

$$vf = \gamma'(0)f = \frac{d(f \circ \gamma)}{dt}\Big|_{t=0}$$

for all $f \in C^\infty(U)$. The collection of all tangent vectors to some curve γ such that $\gamma(0) = x$ is called the *tangent space at* x, denoted T_xM.

Suppose $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ is a choice of coordinates at $x \in M$. Then we can construct a basis for the tangent vectors in T_xM by considering the tangent vectors to the coordinate curves $t \mapsto (x^1_\alpha(0), \ldots, x^i_\alpha(0) + t, \ldots, x^n_\alpha(0))$. The resulting tangent vectors are denoted $\frac{\partial}{\partial x^i} = \partial_i$. Since T_xM is a vector space, this basis induces another basis in the so-called *dual space*. Indeed, given a basis $\{v_1, \ldots, v_n\}$ for V, the dual basis $\{v^1, \ldots, v^n\}$ of $V^* = \{\phi : V \to \mathbb{R} : \phi \text{ linear}\}$ consists of the maps satisfying

$$v^j(v_k) = \delta^j_k.$$

The *cotangent space* at $x \in U$ is defined to be $(T_xM)^*$, that is the n-dimensional vector space of linear forms on the tangent space at x. Elements of this vector space are called cotangent vectors at x. Then $\{dx^1, \ldots, dx^n\}$ is the dual basis to $\{\partial_1, \ldots, \partial_n\}$. If ω is an assignment of an element $\omega(x) \in (T_xM)^*$ to each $x \in U$, there exist components $h_i : U \to \mathbb{R}$ such that

$$\omega(x) = h_1(x)dx^1(x) + \cdots + h_n(x)dx^n(x).$$

We usually abbreviate this to

$$\omega = h_1dx^1 + \cdots + h_ndx^n.$$

Definition 2.2. A map $\omega : M \to T^*M$ is called a *differential 1-form* if $w(x) \in T^*_xM$ and if, when expressed in the standard basis $w = h_1dx^1 + \cdots + h_ndx^n$, the h_i are all smooth.

Differential 1-forms should be thought of as the *duals to vector fields.* This notion generalises to p-forms for all $1 \le p \le n$.

Definition 2.3. A differential *p-form* is a map $\omega : M \to \Lambda^p(T^*M)$ such that $\omega(x) \in \Lambda^p(T^*_xM)$ for all $x \in M$ and such that the coordinate functions are smooth with respect to any local coordinates. We call p the degree of ω and denote the space of all p-forms on $U \subseteq M$ by $\Omega^p(U)$. For $p = 0$, the 0-forms are just smooth functions, and so $\Omega^0(U) = C^\infty(U)$.

It is possible to combine a p-form ω and q-form η to yield a $(p+q)$-form $\omega \wedge \eta$ (provided $p + q \le n$) using the exterior product (we refer those not familiar with the exterior product to [4, Chapter 1]). It is defined as one would expect, as

$$(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x).$$

As such, the rules of the exterior product carry over to this setting:

$$\eta \wedge \omega = (-1)^{pq} \omega \wedge \eta.$$

The most important operation we do on differential forms is the so-called *exterior derivative*, a certain generalisation of the usual derivative to the setting of exterior algebra.

Theorem 2.4. *There exists a unique linear map $d : \Omega^p(U) \to \Omega^{p+1}(U)$ for $p = 0, 1, \ldots, n-1$ such that for all $f \in C^\infty(U)$, $\omega, \eta \in \Omega^p(U)$,*

1. $df(x)(X(x)) = (Xf)(x)$
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)}(\omega \wedge d\eta)$
3. $d(d\omega) = d^2\omega = 0$

where for $f \in \Omega^0(U)$, we define $f \wedge \eta = f\eta$. We call d the exterior derivative. *Generally, the exterior derivative of a p-form $\omega = \sum_I g_I(dx^{i_1} \wedge \cdots \wedge dx^{i_p})$, where $I = (i_1, \ldots, i_p)$ is a strictly ascending multi-index, is given by*

$$d\omega = \sum_I \left(\frac{\partial g_I}{\partial x^1} dx^1 + \cdots + \frac{\partial g_I}{\partial x^n} dx^n \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Proof. [4, Theorem 2.5.1]. □

Remark. The first condition is a generalisation of the product rule, while the identity $d^2 = 0$ can be interpreted as a generalisation of the equality of mixed partial derivatives.

Definition 2.5. Let M and N be smooth manifold and $\varphi = (\varphi_1, \ldots, \varphi_k) : M \to N$ a smooth map. Then define the *pullback of f* (by φ) as the map $\varphi^* : \Omega^p(N) \to \Omega^p(M)$ given by

$$\varphi^*(f_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_I (f_I \circ \varphi) d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$$

where we sum over strictly increasing indices $I = (i_1, \ldots, i_k)$.

Lemma 2.6. Let $\varphi^* : N \to M$ be the pullback map as defined in Definition 2.5 and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ the exterior derivative. Then $\varphi^* \circ d = d \circ \varphi^*$.

Proof. By linearity of the pullback and exterior derivative, it suffices to check the property holds for an element of the form $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(M)$. We calculate

$$\begin{aligned} \varphi^* \circ d\big(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) &= \varphi^*\big(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \\ &= d(f \circ \varphi) \wedge d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k} \\ &= d\big((f \circ \varphi) d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}\big) \\ &= d \circ \varphi^*\big(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big). \end{aligned}$$

□

Definition 2.7. A form $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$ and *exact* if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.

Remark. Observe that since $d^2 = 0$, any exact form is automatically closed. The converse is not typically true, however. This is going to be *very important* soon as the extent to which this fails will give us a tool to extract topological information.

2.2 de Rham Classes and Homotopy Invariance

We now introduce some of the algebraic aspects of cohomology. The main reference for this discussion was [2].

Definition 2.8. A *cochain* complex is a sequence of vector spaces $\{C^k\}_{k\in\mathbb{Z}}$ together with vector space homomorphisms $d^k \colon C^k \to C^{k+1}$ such that $d^{k+1} \circ d^k = 0$. We often denote a cochain complex by

$$\cdots \longrightarrow C^{-2} \xrightarrow{d^{-2}} C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow \cdots$$

Remark. The prefix in *co*chain derives from the fact they are dual (in some appropriate sense) to chain complexes, which are defined analogously but with $d_k \colon C_{k+1} \to C_k$. Loosely speaking, these correspond to the theory of *homology*, which is also dual to *co*homology. In our definition, it is possible to replace *vector space* with a whole variety of different algebraic objects, for instance abelian groups, modules or commutative rings. Finally, it is common to suppress the superscript from the d^k's and instead define $d : C^\bullet \to C^\bullet$ such that $d|_{C^k} = d^k$ where $C^\bullet = \bigoplus_{k\in\mathbb{Z}} C^k$.

Definition 2.9. A map $f : A \to B$ between two cochain complexes is a *chain map* if it commutes with the differential operators of A and B. That is, $f \circ d_A = d_B \circ f$.

Example 2.10. Lemma 2.6 shows that the pullback map is a chain map. We will use this fact several times.

There are many different cochain complexes (notably simplicial, singular and Čech complexes). See, for instance, [5] for an introductory treatment of simplicial and singular cohomology theory. We restrict our attention to just one type of cohomology; de Rham cohomology.

Definition 2.11. Let M be a smooth n-manifold. Then the *de Rham complex* is a cochain complex such that

$$C^k = \begin{cases} \Omega^k(M) & k \geq 0 \\ 0 & k < 0 \end{cases}$$

and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative. That is, the cochain complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots$$

Remark. In reality, the arrows only increase up to $k = n$ because beyond that the C^k are all identically 0. This follows from the fact $\Lambda^k(T^*M) = 0$ for $k > n$.

Definition 2.12. The kth cohomology group of a cochain complex $(C^\bullet, d)$ is defined to be the vector space

$$H^k(C^\bullet) = \frac{\ker(d^k)}{\operatorname{im}(d^{k-1})}.$$

In particular, the *kth de Rham cohomology group* is

$$H^k_{dR}(M) := H^k(\Omega^\bullet(M)).$$

Remark. For the rest of this document, $H^k(M)$ will always refer to the kth de Rham cohomology class. For suitably nice manifolds M, one can show that $H^k(M)$ is a finite dimensional vector space [2, p.43].

It turns out that $H^\bullet(M)$ carries a natural multiplication given by the so-called *cup product*, giving it a ring structure. In our case, this reduces to the usual wedge product on forms.

Proposition 2.13. For $[\omega] \in H^k(M)$ and $[\eta] \in H^l(M)$ define

$$[\omega] \wedge [\eta] = [\omega \wedge \eta] \in H^{k+l}(M).$$

Then $(H^\bullet(M), +, \wedge)$ is a ring.

Proof. First we show $\wedge$ is well-defined. Suppose $\omega' = \omega + d\alpha$ and $\eta' = \eta + d\beta$. Then one easily computes $\omega' \wedge \eta' = (\omega \wedge \eta) + d\nu$ where $\nu = \alpha \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge \beta + \alpha \wedge d\beta$. Thus $[\omega' \wedge \eta'] = [(\omega + d\alpha) \wedge (\eta + d\beta)]$ and so $\wedge$ is well-defined. Furthermore, if ω and η are both closed (that is $d\omega = 0$ and $d\eta = 0$) then by the antiderivation formula we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta = 0$$

and so $\omega \wedge \eta$ is closed. Thus $\omega \wedge \eta$ respresents a class in $H^{k+l}(M)$. □

Example 2.14. Consider $M = \mathbb{R}$. Then we have

$$\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R}); \quad \Omega^1(\mathbb{R}) = \{f dx : f \in C^\infty(\mathbb{R})\}; \quad \Omega^k(\mathbb{R}) = 0 \quad k > 1.$$

Hence

$$H^0_{dR}(\mathbb{R}) = \frac{\ker(d^0)}{\operatorname{im}(d^{-1})} = \ker(d^0) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ constant}\} \cong \mathbb{R}.$$

Since $\Omega^2(\mathbb{R}) = 0$, clearly $\ker(d^1) = \Omega^1(\mathbb{R})$. We claim that $d^0 : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ is surjective. To see this, fix $\omega = f dx \in \Omega^1(\mathbb{R})$. Then

$$g(x) = \int_0^x f(x) dx$$

satisfies $d^0 g = f dx$ by the Fundamental Theorem of Calculus. Thus

$$H^1_{dR}(\mathbb{R}) = \frac{\Omega^1(\mathbb{R})}{\Omega^1(\mathbb{R})} = 0.$$

This justifies our claim that de Rham cohomology measures the extent to which the Fundamental Theorem of Calculus fails on other manifolds. In fact, it tells us much more than that. For example, $\dim(H^0_{dR}(M))$ tells us the number of connected components of M.

It is useful to keep track of the dimensions of each of the H^k_{dR} in the following

Definition 2.15. Let M be a manifold. The kth *Betti number*, $k \geq 0$, is defined to be

$$b_k(M) := \dim(H^k_{dR}(M)).$$

Betti numbers have a geometric interpretation as the number of k-dimensional 'holes' in our manifold M, where 0-dimensional holes are considered connected components.

There is an intriguing connection between Betti numbers and the well-known Euler characteristic. Indeed,

Definition 2.16. The *Euler-Poincaré characteristic* of an $n-$manifold M is defined as

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M).$$

Remark. In mathematics, the 'Euler characteristic' may refer to many different things. For polyhedra, it refers to the observation that Vertices $-$ Edges $+$ Faces $= 2$. For closed 2-manifolds $b_0 - b_1 + b_2 = 2 - 2g$ where g is the genus. Also $b_0 = b_2 = 1$ so that $\dim H^1 = 2g$. Definition 2.16 is a further generalisation in a certain sense. For a nice account of how all of these ideas relate, see [9].

Explicit calculation of cohomology classes quickly get out of hand and so we want to develop some better tools to do this.

Proposition 2.17. Let $\varphi : M \to N$ be a smooth map between manifolds. Then $\varphi^* : H^k(N) \to H^k(M)$ defined by

$$\varphi^*([\omega]) = [\varphi^*\omega]$$

is a well-defined $\mathbb{R}$-linear map.

Proof. Suppose $\omega, \eta \in \ker(d)$ and $\omega - \eta = d\tau$ for some τ (i.e. $[\omega] = [\eta]$). Then by Lemma 2.6,

$$\varphi^*(\omega - \eta) = \varphi^*(d\tau) = d(\varphi^*\tau) \in \operatorname{im}(d)$$

and so $[\varphi^*(\omega - \eta)] = 0$ in $H^k(M)$. Thus $\varphi^* : H^k(N) \to H^k(M)$ is well-defined. It is clearly $\mathbb{R}$-linear since the usual pullback is. □

Corollary 2.18. Given smooth maps $\varphi : M \to N$ and $\psi : N \to K$ between manifolds, there are maps $\varphi^* : H^k(M) \to H^k(N)$ and $\psi^* : H^k(K) \to H^k(N)$ such that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proof. We calculate, for $[\omega] \in H^k(K)$,

$$(\psi \circ \varphi)^*([\omega]) = \Big[(\psi \circ \varphi)^*(\omega)\Big] = \Big[(\varphi^* \circ \psi^*)(\omega)\Big] = (\varphi^* \circ \psi^*)([\omega]).$$ □

Taking $\psi = \varphi^{-1}$ in Corollary 2.18 gives us the following

Corollary 2.19. If $\varphi\colon M \to N$ is a diffeomorphism, then $\varphi^*\colon H^k(N) \to H^k(M)$ is an isomorphism.

Remark. While simple to prove, this result is profound. It tells us that two manifolds are not equal (up to diffeomorphism) if their de Rham cohomology classes differ.

Lemma 2.20. If M and N are manifolds, then for all $k \in \mathbb{Z}$, $H^k(M \sqcup N) \cong H^k(M) \oplus H^k(N)$.

Proof. For $k \leq 0$ the result is trivial. For $k > 0$, any form $\omega \in \Omega^k(M \sqcup N)$ restricts to forms $\omega|_M \in \Omega^k(M)$ and $\omega|_N \in \Omega^k(N)$ and so $[\omega] \mapsto ([\omega|_M], [\omega|_N])$ is an isomorphism. □

Homotopy Invariance

We have shown that de Rham cohomology is preserved under diffeomorphism. In fact, we can weaken this assumption significantly by defining the notion of *homotopy*.

Definition 2.21. Two smooth maps $f, g : M \to N$ between manifolds are (smoothly) *homotopic* (written $f \sim_H g$) if there exists a smooth map $F : M \times \mathbb{R} \to N$ such that

$$F(x,t) = \begin{cases} f(x) & t \geq 1 \\ g(x) & t \leq 0. \end{cases}$$

Definition 2.22. A (smooth) map $f : M \to N$ is a *homotopy equivalence* if there exists smooth $g : N \to M$ such that

$$f \circ g \sim_H \mathrm{id}_N \text{ and } g \circ f \sim_H \mathrm{id}_M$$

We say that M and N are homotopy equivalent or homotopic.

Example 2.23. Consider $M = S^1$ and $N = \mathbb{R}^2 \setminus \{0\}$. We claim $S^1 \sim_H \mathbb{R}^2 \setminus \{0\}$. To see this, take the inclusion map

$$\iota : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$$

and the retraction map

$$r : \mathbb{R}^2 \setminus \{0\} \to S^1; \quad x \mapsto \frac{x}{|x|}.$$

Then $r \circ \iota = \mathrm{id}_{S^1}$ already so there is nothing to prove. On the other hand, we claim $\iota \circ r \sim_H \mathrm{id}_{\mathbb{R}^2\setminus\{0\}}$. Define $F : (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ by

$$F(x,t) = \begin{cases} x & t \leq 0 \\ (1-t)x + \dfrac{tx}{|x|} & t \in (0,1) \\ \dfrac{x}{|x|} & t \geq 1. \end{cases}$$

Then F is a homotopy, showing $S^1 \sim_H \mathbb{R}^2 \setminus \{0\}$.

Definition 2.24. A manifold M is said to be *contractible* if $M \sim_H$ point.

Example 2.25. Consider $M = \mathbb{R}$. Then the inclusion map $\iota : \text{point} \hookrightarrow \mathbb{R}$ together with the zero map $r : \mathbb{R} \to \text{point}$ given by $x \mapsto 0$ yields a homotopy equivalence. Thus the real line is contractible.

In order to prove that de Rham cohomology is preserved under homotopy, we need the following technical result, for which we will omit the proof.

Proposition 2.26. Let M be a manifold and let $\pi : M \times \mathbb{R} \to M$ be the projection map onto the first factor. Let $s_0 : M \to M \times \mathbb{R}$ be the zero section $x \mapsto (x, 0)$. Then $\pi^* : H^k(M) \to H^k(M \times \mathbb{R})$ and $s_0^* : H^k(M \times \mathbb{R}) \to H^k(M)$ are mutually inverse isomorphisms.

Proof. [2, p. 34-35] □

One consequence of Proposition 2.26 is that we can now compute the de Rham cohomology of $\mathbb{R}^n$ for any n.

Corollary 2.27 (Poincaré Lemma)**.** We have, for all $n \geq 0$,

$$H^k(\mathbb{R}^n) = H^k(\text{point}) = \begin{cases} \mathbb{R}, & k = 0 \\ \mathbb{R}, & k > 0. \end{cases}$$

Moreover, Proposition 2.26 gives us the main result we wanted to establish.

Corollary 2.28 (Homotopy Invariance for de Rham Cohomology)**.** If $f, g : M \to N$ are homotopic maps then $f^* = g^* : H^k(N) \to H^k(M)$.

Proof. Let $F : M \times \mathbb{R} \to M$ be a homotopy such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Let $s_0, s_1 : M \to M \times \mathbb{R}$ be the 0-section and 1-section, respectively, i.e., $s_1(x) = (x, 1)$, then

$$f = F \circ s_0, \quad g = F \circ s_1 \implies f^* = s_0^* \circ F^*, \quad g^* = s_1^* \circ F^*.$$

Since s_1^* and s_0^* both invert the isomorphism π^*, they must be equal (uniqueness of inverses). Hence $f^* = g^*$. □

2.3 The Mayer Vietoris Sequence

The Mayer Vietoris sequence allows us to calculate the cohomology classes of manifolds by piecing together the cohomology of smaller pieces of the manifold. The main algebraic tool we will need are *exact sequences.*

Definition 2.29. A sequence of vector spaces

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

is called *exact* if, for all i, the kernel of f_i is equal to the image of f_{i-1}.

A particularly important special case of exact sequences are the *short exact sequences*.

Definition 2.30. A *short exact sequence* is an exact sequence of vector spaces

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

We now begin the construction of the Mayer Vietoris sequence. Consider the decomposition of a manifold $M = U \cup V$ into open subsets U and V. Now define maps

$$\iota_U : U \cap V \xhookrightarrow{\text{via } U} U \sqcup V$$
$$\iota_V : U \cap V \xhookrightarrow{\text{via } V} U \sqcup V$$
$$g : U \sqcup V \longrightarrow M.$$

The maps ι_U and ι_V are simply inclusion maps, while g is a 'gluing' map. By Lemma 2.20, we may now define maps ι_U^*, ι_V^* and g^* as follows.

$$\iota_U^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V); \qquad (\omega, \eta) \mapsto \omega\big|_{U\cap V}$$
$$\iota_V^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V); \qquad (\omega, \eta) \mapsto \eta\big|_{U\cap V}$$
$$g^* : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V); \qquad \omega \mapsto (\omega\big|_U, \omega\big|_V).$$

Define $\Delta := \iota_U^* - \iota_V^*$. Then we have the following

Proposition 2.31. There is a short exact sequence of vector spaces given by

$$0 \longrightarrow \Omega^k(M) \xrightarrow{g^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\Delta} \Omega^k(U \cap V) \longrightarrow 0.$$

Proof. First suppose $g^*(\omega) = 0$. Then $(\omega\big|_U, \omega\big|_V) = 0$ which implies $\omega = 0$ since $M = U \cup V$. So $\ker(g^*) = 0$ and the sequence is exact at $\Omega^k(M)$. Now fix $(\omega, \eta) \in \ker(\Delta)$. Then $\omega\big|_{U\cap V} = \eta\big|_{U\cap V}$ and $\tau \in \Omega^k(M)$ given by

$$\tau(x) = \begin{cases} \omega(x), & x \in U \\ \eta(x), & x \in V \end{cases}$$

is well-defined on $U \cap V$. Then $g^*(\tau) = (\omega, \tau)$ and hence the sequence is exact at $\Omega^k(U) \oplus \Omega^k(V)$. Finally, if $\omega \in \Omega^k(U \cap V)$, take a partition of unity $\{\phi_U, \phi_V\}$ subordinate to the cover $\{U, V\}$. Then

$$\Delta(\phi_U \omega, -\phi_V \omega) = \phi_U \omega + \phi_V \omega = \omega.$$

Thus the sequence is exact. □

This short exact sequence extends to a long exact sequence known as the *Mayer Vietoris sequence*.

Theorem 2.32 (Mayer Vietoris)**.** *Let M be a manifold and let g^* and Δ be as defined in Proposition 2.31. Then there exists a map $\partial : H^k(U \cap V) \to H^{k+1}(M)$ for $k = 0, 1, \dots$ such that the sequence in Figure 1 is exact.*

$$
\begin{array}{ccccc}
H^0(M) & \xrightarrow{g^*} & H^0(U) \oplus H^0(V) & \xrightarrow{\Delta^*} & H^0(U \cap V) \\
& & \swarrow \partial & & \\
H^1(M) & \xrightarrow[g^*]{} & H^1(U) \oplus H^1(V) & \xrightarrow{\Delta^*} & H^1(U \cap V) \\
& & \swarrow \partial & & \\
H^2(M) & \xrightarrow[g^*]{} & H^2(U) \oplus H^2(V) & \xrightarrow{\Delta^*} & H^2(U \cap V) \\
& & \swarrow \partial & & \\
& & \vdots & &
\end{array}
$$

Figure 1: The Mayer Vietoris Sequence.

The proof of Mayer Vietoris uses a general technique usually known as *diagram chasing*. For the details of this proof, we refer the reader to [2, Proposition 2.3] or [5, Theorem 2.16].

Example 2.33. We will use the Mayer Vietoris sequence to compute the de Rham cohomology classes of the circle. We cover S^1 with two open sets $U = S^1 \setminus \{N\}$ and $V = S^1 \setminus \{S\}$ where N and S are the north and south pole of S^1, respectively. Clearly $U \cup V = S^1$, and $U \cap V = A_1 \sqcup A_2$ is the disjoint union of two line segments which are homeomorphic to the real line. The associated Mayer Vietoris sequence is given by

$$
\begin{array}{ccccc}
H^0(S^1) & \longrightarrow & H^0(U) \oplus H^0(V) & \longrightarrow & H^0(A_1) \oplus H^0(A_2) \\
& & \swarrow & & \\
H^1(S^1) & \longrightarrow & H^1(U) \oplus H^1(V) & \longrightarrow & H^1(A_1) \oplus H^1(A_2) \\
& & \swarrow & & \\
H^2(S^1) & \longrightarrow & \cdots & &
\end{array}
$$

Figure 2: The Mayer Vietoris sequence for S^1.

Since both $\mathbb{R}$ and S^1 are connected, we have $H^0(S^1) = H^0(U) = H^0(V) = H^0(A_1) = H^0(A_2) = \mathbb{R}$. Moreover, the Poincaré Lemma tells us that $H^k(\mathbb{R}) = 0$ for $k > 0$. Thus our sequence becomes

$$\mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\partial} H^1(S^1) \to 0 \to 0 \to H^2(S^1) \to 0.$$

Note that the map from ∂ is surjective by exactness, and $\ker(\delta) = \operatorname{im}(\Delta^*)$. But Δ^* is the pullback of the subtraction map $\Delta = \iota_U^* - \iota_V^*$. Therefore, since the difference of two constant functions is a constant function, its image is the set of constant functions, which is isomorphic to $\mathbb{R}$. Thus $H^1(S^1) = \mathbb{R}$. For $k > 1$ we have $H^k(S^1) = 0$ since it is surrounded by zero maps.

A relatively straightforward induction argument yields the de Rham cohomology of the n-sphere as

$$H^k(S^n) = \begin{cases} \mathbb{R}, & k = 0,\, n \\ 0, & \text{otherwise.} \end{cases}$$

This tells us that, topologically speaking, the defining characteristic of an n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is that it is a connected manifold with a single n-dimensional hole: de Rham cohomology can indeed see holes!

Example 2.34. We now consider the 2-torus $T^2 = S^1 \times S^1$, decomposed as in Figure 3 into the union of two cylinders X_1, X_2.

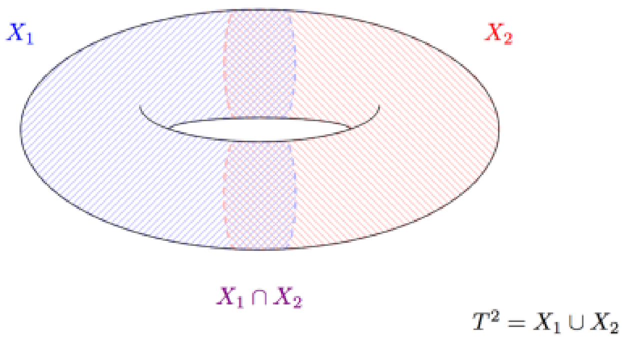

Figure 3: Two open subsets of T^2 whose union is T^2.

This example will be somewhat hand-wavy otherwise the details will get out of hand. Clearly $X_1 \cap X_2 = A_1 \sqcup A_2$ and each of X_1, X_2, A_1, A_2 and $S^1 \times \mathbb{R}$ are diffeomorphic. We use again that the cylinder is homotopically equal to S^1. So using that $H^k(U \cap V) = H^k(A_1 \sqcup A_2) = H^k(A_1) \oplus H^k(A_2)$ and that $H^k(S^1) = \mathbb{R}$ for $k = 0, 1$ our Mayer Vietoris sequence for T^2 is as shown in Figure 4.

$$\begin{array}{ccccc} H^0(T^2) = \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\ & \swarrow & & & \\ H^1(T^2) & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\ & \swarrow & & & \\ H^2(T^2) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Figure 4: The Mayer Vietoris sequence for T^2.

From the exactness of the diagram, and the definitions of the functions g^* and Δ^*, we find

$$H^k(T^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & k = 1 \\ 0 & k > 2. \end{cases}$$

3 Vector Bundles

Fibre bundles are topological spaces which give a generalisation of the Cartesian product in the sense that *locally* they resemble product spaces, but their global topological structure may differ. We make this idea precise in the following

Definition 3.1. A *fibre bundle* consists of topological spaces E, X and F (usually locally compact Hausdorff) together with a surjective and continuous map $\pi\colon E \to X$ such that $\pi^{-1}(\{x\}) \cong F$ and for all $x \in X$ there is an open neighbourhood U of x and a homeomorphism $\Phi_U : \pi^{-1}(U) \to U \times F$. The set of all $\{U, \Phi_U\}$ is called a *local trivialisation* of the fibre bundle.

Remark. We call E the total space, X the base space and F the fibre. The map π is called the *projection map*. Special conditions on these fibre bundles give rise to particular kinds of bundles, notably *vector bundles* (where F is a vector space, and E and X are both locally compact Hausdorff spaces) and *principal fibre bundles* (where F "looks like" a Lie group).

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \big\downarrow \pi \\ & & X \end{array}$$

Figure 5: General fibre bundle structure.

Examples 3.2.

1. Any Cartesian product space $E = X \times F$ together with the projection map $X \times F \to X$ is called the (globally) *trivial* fibre bundle with fibre F. The infinite cylinder $E = S^1 \times \mathbb{R}$ is a globally trivial fibre bundle.

2. Let $\exp\colon \mathbb{R} \to S^1$ be given by $\exp(t) = e^{2\pi i t} \in S^1$. Then exp is a fibre bundle with fibre the integers, since $e^{2\pi i t} = e^{2\pi i (t+n)}$ precisely when $n \in \mathbb{Z}$.

3. Consider the Möbius band $E = \{(e^{i\theta}, t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}) \subset \mathbb{C} \times \mathbb{R}^2 \colon t \in \mathbb{R},\, \theta \in [0, 2\pi]\}$. Then identifying $\mathbb{R}$ with $(-1, 1)$ (via the map $t \mapsto \tan^{-1} t$ for instance), this gives the familiar total space. The projection map $(e^{i\theta}, t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2}) \mapsto e^{i\theta}$ maps E onto the base space $X = S^1$. However this is a *nontrivial* fibre bundle because $E \neq S^1 \times \mathbb{R}$ (see Example 3.17). Locally the Möbius band *does* look like a cylinder in the sense that for all $U \subsetneq S^1$ we have $E|_U = U \times \mathbb{R}$. Alternatively, the Möbius

bundle may be described as the topological space $[0,1] \times \mathbb{R}$ equipped with the equivalence relation $(0,t) \sim (1,-t)$ for all $t \in \mathbb{R}$, where the projection map $\pi : E \to S^1$ is given by $\pi([x,t]) = e^{2\pi i x}$. See Figure 6.

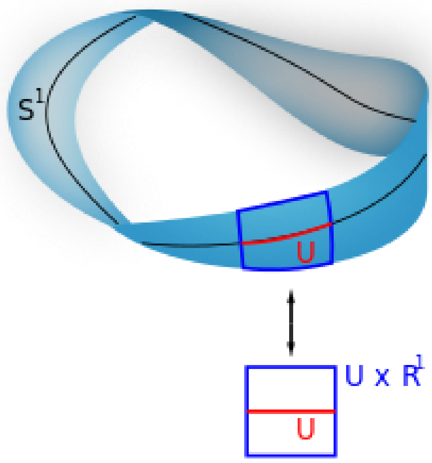

Figure 6: The Möbius band as a nontrivial fibre bundle. Image source: Wikipedia

The particular kind of fibre bundle in which we are interested is known as a *vector bundle*. The idea here is to make each of our fibres a vector space. In this way, the study of vector bundles may be considered *parametrised linear algebra*.

Definition 3.3. Let X be a locally compact Hausdorff space. A rank-k real vector bundle E over X is a fibre bundle satisfying $E_x := \pi^{-1}(\{x\}) \cong \mathbb{R}^k$ for all $x \in X$.

Remark. A rank-k *complex* vector bundle is defined analogously, replacing $\mathbb{R}$ with $\mathbb{C}$ where appropriate.

Examples 3.4.

1. The simplest example of a rank-k vector bundle is the trivial bundle $E = X \times \mathbb{R}^k$.
2. If we let $M \subseteq \mathbb{R}^n$ be a (sub)manifold, then recall the tangent space at $x \in U \subseteq M$ is

$$T_xM = \{\gamma'(0) : \gamma : (-\varepsilon, \varepsilon) \to M,\, \gamma(0) = x\}.$$

Then the disjoint union of all the tangent spaces

$$TM = \bigsqcup_{x \in M} T_xM \subseteq \mathbb{R}^{2n}$$

is called the *tangent bundle*. This is an example of a vector bundle, where at each point the associated vector space is the tangent space.

3. The cotangent bundle T^*M which consists of all cotangent spaces $T_x^*M = (T_xM)^*$ is also a vector bundle.

Bundle morphisms

We now define the structure-preserving maps between vector bundles.

Definition 3.5. Let $E, F \to X$ be vector bundles. A *homomorphism of vector bundles* $\phi\colon E \to F$ is a continuous map which is linear on the fibres, so that $\phi_x\colon E_x \to F_x$ is a homomorphism for all $x \in X$ and such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & F \\ {\scriptstyle \pi_E}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow[\text{id}]{} & X \end{array}$$

In Definition 3.5, we could replace id with a different homeomorphism $X \to X$ and this is sometimes useful for checking that two vector bundles are isomorphic.

Definition 3.6. Two vector bundles $E, F \to X$ are isomorphic if there exists a homomorphism $\phi : E \to F$ such that $\phi : E_x \overset{\cong}{\to} F_x$; that is, the homomorphism is a fibrewise isomorphism.

Example 3.7. We claim the tangent bundle TS^1 to the unit circle S^1 is isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. To see this, observe that the line tangent to $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ is perpendicular to the line through $0 \in \mathbb{R}^2$ and $(\cos\theta, \sin\theta)$, which is $t(\sin\theta, -\cos\theta)$ for $t \in \mathbb{R}$. Alternatively, identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we observe that the line through $e^{i\theta}$ and the origin is $te^{i\theta}$, $t \in \mathbb{R}$. Multiplication by i rotates this line by $\pi/2$ radians yielding $ite^{i\theta}$, $t \in \mathbb{R}$. See Figure 7.

With this description of tangent vectors to S^1, we may construct an isomorphism given by

$$(e^{i\theta}, ite^{i\theta}) \mapsto (e^{i\theta}, t), \quad \theta \in [0, 2\pi], \quad t \in \mathbb{R}.$$

The fact that $TS^1 \cong S^1 \times \mathbb{R}$ does not extend to arbitrary dimensions. Indeed, $TS^2 \neq S^2 \times \mathbb{R}^2$ follows from the famous Hairy Ball Theorem [3, Theorem 2.2.3]: there is *no* nowhere vanishing continuous vector field on $\mathbb{S}^2$. In fact, it's possible to show that the only spheres S^n whose tangent bundle is trivial are $n = 1, 3, 7$ [6, §2.3].

Definition 3.8. Let $E \overset{\pi}{\to} X$ be a rank-k vector bundle and, for $U, V \subset X$, let

$$\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$$
$$\Phi_V : \pi^{-1}(V) \to V \times \mathbb{R}^k$$

be local trivialisations, with $U \cap V \neq \emptyset$. Then the transition function of Φ_U, Φ_V is

$$g_{UV} : U \cap V \to GL_k(\mathbb{R})$$

defined by $\Phi_U \circ \Phi_V^{-1}(x, \xi) = (x, g_{UV}(x)\xi)$ for all $(x, \xi) \in U \cap V \times \mathbb{R}^k$.

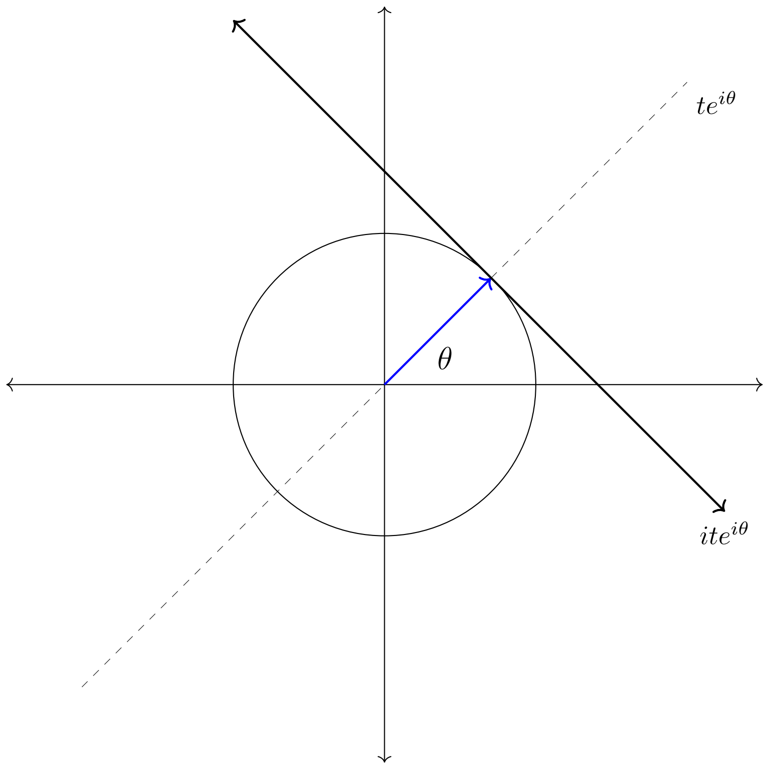

Figure 7: The tangent line to $e^{i\theta} = \cos\theta + i\sin\theta$ on S^1.

We can think of transition functions as a *continuous change of basis*. Indeed, given a local trivialisation $\Phi_U : E|_U \to U \times \mathbb{R}^k$, $(\Phi_U \circ v_1, \dots, \Phi_U \circ v_k)$ can be thought of as a continuous choice of basis in $\mathbb{R}^k$ which is parametrised by U.

Lemma 3.9. The transition maps in Definition 3.8 satisfy:

$$\begin{aligned} g_{UU}(x) &= \text{id} \quad \text{for all } x \in U \\ g_{UV}^{-1}(x) &= g_{VU}(x) \quad \text{for all } x \in U \cap V \\ g_{UV} \circ g_{VW}(x) &= g_{UW}(x) \quad \text{for all } x \in U \cap V \cap W. \end{aligned}$$

Proof. The first identity is clear from the definition. Now, for $(x, \xi) \in U \cap V \times \mathbb{R}^k$,

$$(x, g_{UV} \circ g_{VU}(x)\xi) = (\Phi_U \circ \Phi_V^{-1}) \circ (\Phi_V \circ \Phi_U^{-1})(x, \xi) = \Phi_U \circ \Phi_U^{-1}(x, \xi) = (x, \xi).$$

Relabelling also yields $g_{VU} \circ g_{UV} = \text{id}$ and so $g_{UV}^{-1}(x) = g_{VU}(x)$ for all $x \in U \cap V$. Similarly, for $x \in U \cap V \cap W, \xi \in \mathbb{R}^k$ we have

$$(x, g_{UV} \circ g_{VW}(x)\xi) = (\Phi_U \circ \Phi_V^{-1}) \circ (\Phi_V \circ \Phi_W^{-1})(x, \xi) = \Phi_U \circ \Phi_W^{-1}(x, \xi) = (x, g_{UW}(x)\xi). \qquad \square$$

Example 3.10. Let E be the Möbius bundle and consider the open cover of S^1 by the sets $U = S^1 \setminus \{(1, 0)\}$ and $V = S^1 \setminus \{(-1, 0)\}$. Then the intersection $U \cap V = A \sqcup B$ is the disjoint union of

$A = \{e^{i\theta} \in S^1 : \theta \in (0, \pi)\}$ and $B = \{e^{i\theta} \in S^1 : \theta \in (\pi, 2\pi)\}$. Let $v_t^\theta := (t\cos\frac{\theta}{2}, t\sin\frac{\theta}{2})$. Then

$$\pi^{-1}(U) = \{(e^{i\theta}, v_t^\theta) : \theta \in (0, 2\pi),\ t \in \mathbb{R}\}$$
$$\pi^{-1}(V) = \{(e^{i\theta}, v_t^\theta) : \theta \in (-\pi, \pi),\ t \in \mathbb{R}\}.$$

Define maps $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}$ and $\Phi_V : \pi^{-1}(V) \to V \times \mathbb{R}$ by

$$\Phi_U((e^{i\theta}, v_t^\theta)) = (e^{i\theta}, t)$$
$$\Phi_V((e^{i\theta}, v_t^\theta)) = (e^{i\theta}, t).$$

These are homeomorphisms because there is a continuous inverse $\Phi_V^{-1} : V \times \mathbb{R} \to \pi^{-1}(V)$ given by $(e^{i\theta}, t) \mapsto (e^{i\theta}, v_t^\theta)$ (similarly for Φ_U^{-1}). Hence they are local trivialisations. We now look at the function $\Phi_U \circ \Phi_V^{-1} : U \cap V$. We consider two cases.

(I) On $A \subset U \cap V$, we have $(0, \pi) \times \mathbb{R} \subset \mathrm{dom}\Phi_V^{-1}$ and $\Phi_V^{-1}((0, \pi) \times \mathbb{R}) \subseteq \pi^{-1}(U)$. So $\Phi_U \circ \Phi_V^{-1}$ simply maps

$$(e^{i\theta}, t) \mapsto (e^{i\theta}, v_t^\theta) \mapsto (e^{i\theta}, t).$$

Thus $g_{UV} : U \cap V \to \mathbb{R} \setminus \{0\}$ takes the value 1 on $A \subseteq U \cap V$.

(II) On $B \subset U \cap V$, we have $\theta \in (\pi, 2\pi)$ and $(\pi, 2\pi) \times \mathbb{R} \notin \mathrm{dom}(\Phi_V^{-1})$. So instead we set $\varphi = \theta + \pi$, then $\varphi \in (0, \pi)$ so that $\Phi_V^{-1}((0, \pi) \times \mathbb{R}) \subseteq \pi^{-1}(U)$. Importantly, we have $e^{i\varphi} = e^{i(\theta+\pi)} = -e^{i\theta}$ (so they belong to the same equivalence class). The transition function is no longer trivial, though. Since $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$, we have

$$(e^{i\varphi}, t) \mapsto (e^{i\varphi}, v_t^\varphi) = (e^{i\theta}, -v_t^\theta) \mapsto (e^{i\theta}, -t).$$

Thus $g_{UV} : U \cap V \to \mathbb{R} \setminus \{0\}$ takes the value -1 on B. In conclusion,

$$g_{UV}(x) = \begin{cases} 1 & x \in A \\ -1 & x \in B. \end{cases}$$

Example 3.11. Let $E = TM$ be the tangent bundle over M (a manifold with $\dim M = n$). Suppose (U, ϕ_U) is a chart of M with coordinate functions $x^1, \ldots, x^n$. Then for all $x \in U$, we have a basis

$$\Big\{\frac{\partial}{\partial x^1}(x), \ldots, \frac{\partial}{\partial x^n}(x)\Big\}$$

of $T_x M$. We define a trivialisation $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ of TM by

$$\Big(x, \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}(x)\Big) \mapsto (p, (a^1, \ldots, a^n)).$$

We can also calculate the transition functions. Suppose (V, ϕ_V) is another chart of M with coordinate functions $y^1, \ldots, y^n$ and $U \cap V \neq \emptyset$. For every $b = (b^1, \ldots, b^n) \in \mathbb{R}^n$ and $x \in U \cap V$ we have

$$\Phi_V^{-1}(x, b) = \Big(x, \sum_{i=1}^n b^i \frac{\partial}{\partial y^i}(x)\Big).$$

By the chain rule, on $U \cap V$,

$$\frac{\partial}{\partial y^i}(x) = \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}(x)$$

and thus

$$\Phi_U \circ \Phi_V^{-1}(x,b) = \Big(x, \sum_{i=1}^{n} b^i \frac{\partial x^1}{\partial y^i}, \ldots, \sum_{i=1}^{n} b^i \frac{\partial x^n}{\partial y^i}\Big).$$

This means the transition function g_{UV} is given by

$$g_{UV}(x) = d(\phi_U \circ \phi_V^{-1})$$

which is precisely the Jacobian matrix of the change of coordinates $x^1, \ldots, x^n \to y^1, \ldots, y^n$.

Properties of vector bundles depend on the gluing instructions encoded in the transition functions. It turns out that given local trivialisations and transition functions satisfying the relationships in Lemma 3.9, one can always construct a vector bundle. This is an example of the fibre bundle reconstruction theorem, which we will now prove.

Theorem 3.12 (Vector Bundle Reconstruction Theorem)**.** *Suppose $\bigcup_{\alpha \in I} U_\alpha$ is a locally finite open cover of X and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(\mathbb{R})$ are continuous functions satisfying*

$$g_{\alpha\alpha} = id$$
$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$
$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

Then there exists a vector bundle $E \xrightarrow{\pi} X$ such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^k$.

Proof. Let $E' = \bigcup_{\alpha \in I} \{\alpha\} \times U_\alpha \times \mathbb{R}^k$ and consider the equivalence relation on E' given by

$$(\alpha, x, v) \sim (\beta, y, w) \overset{\text{def}}{\iff} x = y, \quad w = g_{\alpha\beta}(x)v.$$

Then take $E := E'/\sim$ with the projection map $\pi : E \to X$ given by $\pi([\alpha, x, v]) = x$. Then $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ gives a local trivialisation for each $\alpha \in I$. □

3.1 The Hopf Bundle

Before defining the Hopf line bundle, we begin by considering its real analogue. It is much easier to visualise as it sits in three dimensions, not four. As a set, the real projective line $\mathbb{R}P_1$ consists of all lines through $0 \in \mathbb{R}^2$ as shown in Figure 8. We shall denote by $[x, y] \in \mathbb{R}P_1$ the line through 0 and $(x, y) \in \mathbb{R}^2$. The real projective line is the base space of the (real) Hopf line bundle. Observe that each line passes through the unit circle S^1 exactly twice, and so $\mathbb{R}P_1$ may be considered the unit circle modulo the equivalence relation identifying antipodal points. That is,

$$\mathbb{R}P_1 \cong S^1/\sim$$

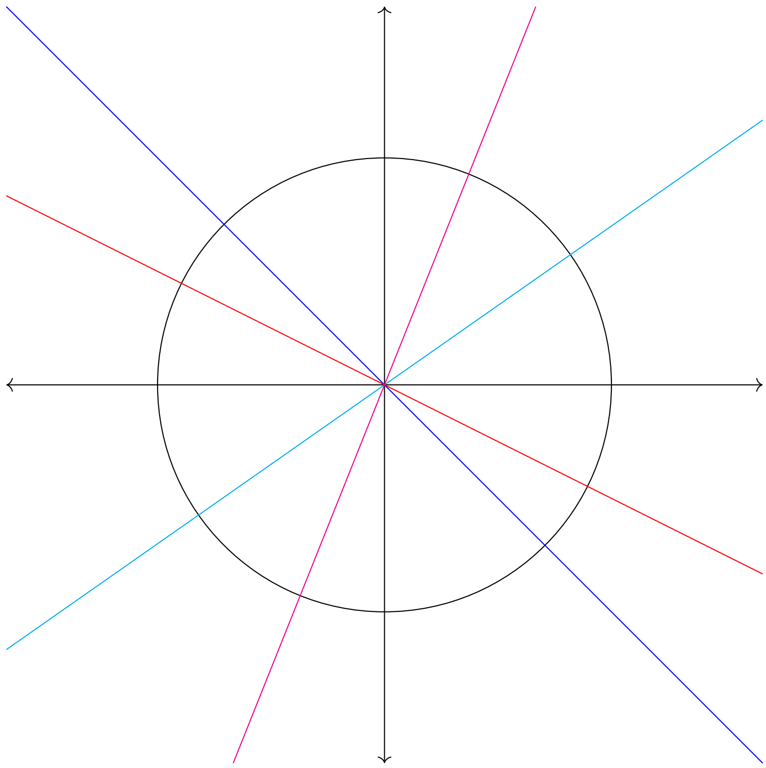

Figure 8: Elements of the real projective line $\mathbb{R}P_1$ are lines through the origin.

where two points $e^{i\theta}, e^{i\varphi} \in S^1$ are equivalent if and only $e^{i\theta} = \pm e^{i\varphi}$. Thus $\mathbb{R}P_1$ inherits the quotient topology from S^1 under this equivalence relation. Recalling $S^1 \cong \mathbb{R}\cup\{\infty\}$, the one-point compactification of the real line, we have another way of thinking about $\mathbb{R}P_1$: each real number $a \in \mathbb{R}$ uniquely defines a line $y = ax$ in the plane, and the point ∞ gives us the missing vertical line $x = 0$. This explains why $\mathbb{R}P_1$ is often called *the real projective line*, despite it consisting of many lines which live in $\mathbb{R}^2$. This framework also provides a homeomorphism between $\mathbb{R}P_1$ and $\mathbb{R} \cup \{\infty\} = S^1$ via the map $[x, y] \mapsto \frac{y}{x}$ where $\frac{y}{0} := \infty$. Thus $\mathbb{R}P_1$ is compact and Hausdorff.

Clearly, $[\lambda x, \lambda y] = [x, y]$ for all nonzero real numbers λ and so $[x, y]$ is really an equivalence class of points lying on a given line. To make a vector bundle over $\mathbb{R}P_1$ we simply attach to each point the line corresponding to that point. Explicitly, let $[x, y] \in \mathbb{R}P_1$ be any point in real projective space, and denote by $\ell_{[x,y]}$ the line in $\mathbb{R}^2$ corresponding to $[x, y]$. This notation helps us to distinguish between lines $[x, y]$ which belong to the base space $\mathbb{R}P_1$ and the fibres $\ell_{[x,y]}$ which lie over them. We define

$$H := \bigsqcup_{[x,y]\in\mathbb{R}P_1} \{[x, y]\} \times \ell_{[x,y]} = \Big\{([x, y], v) \in \mathbb{R}P_1 \times \mathbb{R}^2 : v \in \ell_{[x,y]}\Big\}.$$

We want to show that H constitutes a line bundle over $\mathbb{R}P_1$, called the *tautological line bundle*. It remains to show that H is a locally compact Hausdorff space, to find a projection map $\pi\colon H \to \mathbb{R}P_1$ such that $\pi^{-1}\{([x, y])\} \cong \mathbb{R}$ and find a local trivialisation for H. To see that H is a locally compact Hausdorff space, observe that it is equipped with the subspace topology from the product topology on

$\mathbb{R}P_1 \times \mathbb{R}^2$. As such, H readily inherits the properties of locally compact and Hausdorff. For further details see, for instance, [8, §2]. The projection map $\pi\colon H \to \mathbb{R}P_1$ given by $\pi(([x,y],v)) = [x,y]$ is a surjective and continuous map by definition of the product and subspace topology [8, §2]. Furthermore, for any $[x,y] \in \mathbb{R}P_1$, we have $\pi^{-1}(\{[x,y]\}) = \{([x,y],\lambda(x,y)) : \lambda \in \mathbb{R}\} \cong \mathbb{R}$, where we have naturally identified the fibre with $\ell_{[x,y]}$ which is homeomorphic to $\mathbb{R}$. This was pretty easy, because that's how we defined our bundle!

It just remains to show that $H \xrightarrow{\pi} \mathbb{R}P_1$ is locally trivial. For this, we define two open sets

$$U_1 := \{[x,y] \in \mathbb{R}P_1 : x \neq 0\} \subseteq \mathbb{R}P_1$$
$$U_2 := \{[x,y] \in \mathbb{R}P_1 : y \neq 0\} \subseteq \mathbb{R}P_1.$$

So U_1 consists of all lines except the vertical line $x = 0$ and U_2 consists of all lines except the horizontal line $y = 0$. Clearly $U_1 \cup U_2 = \mathbb{R}P_1$. Now we may define maps

$$\Phi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{R} \quad ([x,y],\lambda(x,y)) \mapsto \Big([x,y], \lambda\frac{y}{x}\Big)$$
$$\Phi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{R} \quad ([x,y],\lambda(x,y)) \mapsto \Big([x,y], \lambda\frac{x}{y}\Big).$$

The homeomorphism $\mathbb{R}P_1 \to S^1$ given by $[x,y] \mapsto \frac{y}{x}$ induces a natural isomorphism between H and the Möbius bundle. Indeed, each point of H can be expressed as $(\pm e^{i\theta}, te^{i\theta})$ for some $\theta \in [0,\pi]$, $t \in \mathbb{R}$. The point $e^{i\theta} \in S^1$ is the one identified with $\frac{y}{x} \in \mathbb{R} \cup \{\infty\}$ (concretely, $\theta = \tan^{-1}(y/x)$), and clearly $te^{i\theta}$ parameterises the line through this point and the origin. This representation is unique except that $(\pm e^{i0}, te^{i0}) = (\pm e^{i\pi}, -te^{i\pi})$ for each $t \in \mathbb{R}$. In other words, H is homeomorphic to $[0,\pi] \times \mathbb{R}$ by the identification $(0,t) \sim (\pi,-t)$, which is precisely one of our descriptions of the Möbius bundle. That is, $H \ni ([x,y],\lambda(x,y)) \mapsto (\tan^{-1}(y/x), \lambda) \in [0,\pi] \times \mathbb{R}$ is an isomorphism. This argument was inspired by [7, Theorem 2.1]. The transition functions for H are thus the same as those of Example 3.10.

We now turn our attention to the tautological line bundle over $\mathbb{C}P_1$, the complex projective line, which is often called the *Hopf line bundle*, which we shall denote $\mathcal{H}$ to distinguish it from its real cousin. The definitions are basically as expected, however the bundle we construct has a different structure. In particular, while the tautological real bundle H over $\mathbb{R}P_1$ is isomorphic to the Möbius bundle, the Hopf bundle is definitely not.

Complex projective space $\mathbb{C}P_1$ consists of the complex lines through the origin of $\mathbb{C}^2 \subseteq \mathbb{R}^4$. Once more we have a coordinate system on $\mathbb{C}P_1$ given by $[z,w] \mapsto \frac{w}{z} \in \mathbb{C} \cup \{\infty\} \cong S^2$, where $\frac{w}{0} := \infty$. As such the base space $\mathbb{C}P_1$ of the Hopf bundle is actually just (homeomorphic to) the 2-sphere. The Hopf bundle $\mathcal{H}$ is formed by attaching complex lines corresponding to each point of $\mathbb{C}P_1$, just as we did in the real case. The only significant difference between H and $\mathcal{H}$ is in the transition functions between local trivialisations. We cover $\mathbb{C}P_1$ with two open sets

$$U_1 := \{[1,z] \in \mathbb{C}P_1 : z \in \mathbb{C}\} \subseteq \mathbb{C}P_1$$
$$U_2 := \{[z,1] \in \mathbb{C}P_1 : z \in \mathbb{C}\} \subseteq \mathbb{C}P_1.$$

Realising once again that $\mathbb{C}P_1 \cong \mathbb{C} \cup \{\infty\}$, we have $U_1 \cong \mathbb{C}$ and $U_2 \cong (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$. Unlike the real case, the intersection $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ is connected. The homeomorphisms

$$\Phi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{C} \quad ([1, z], \lambda(1, z)) \mapsto \big([1, z], \lambda\big)$$
$$\Phi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{C} \quad ([z, 1], \lambda(z, 1)) \mapsto \big([z, 1], \lambda\big)$$

are local trivialisations. The inverse map $\Phi_2^{-1} : U_2 \times \mathbb{C} \to \pi^{-1}(U_2)$ is given by

$$\big([z, 1], \mu\big) \mapsto \big([z, 1], \mu(z, 1)\big) = \big([1, z^{-1}], \mu z(1, z^{-1})\big)$$

and as such

$$\Phi_1 \circ \Phi_2^{-1}\big([1, z^{-1}], \mu\big) = \Phi_1\big([1, z^{-1}], \mu z(1, z^{-1})\big) = \big([1, z^{-1}], \mu z\big).$$

Upon relabelling z^{-1} by z we find that $g_{12} : U_1 \cap U_2 \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ is given by

$$g_{12}(z) = z^{-1}.$$

3.2 Sections and Frames

We now wish to define an important concept called a *section* of a vector bundle, which generalises the notion of a vector-valued function.

Definition 3.13. Let $E \xrightarrow{\pi} X$ be a vector bundle. A *section* of E is a continuous map

$$\sigma : X \to E$$

such that $\sigma(x) \in \pi^{-1}(\{x\})$ for all $x \in X$ (or $\pi \circ \sigma = \mathrm{id}$). The set of all sections of E is denoted $\Gamma(E)$.

Examples 3.14.

1. If $E = X \times \mathbb{R}^k$ is the rank-k trivial real vector bundle then $\Gamma(E) = C(X; \mathbb{R})^k$ and so sections are just vector-valued functions.
2. If $E = X \times \mathbb{C}$ is the rank-1 trivial complex vector bundle then $\Gamma(E) = C(X; \mathbb{C})$ the set of continuous complex-valued functions.
3. Let M be a manifold. Then a section of the tangent bundle TM is called a *vector field*.
4. Let M be a manifold. Then a section of the cotangent bundle T^*M is called a *differentiable 1-form*. In general, sections of the kth exterior power $\Lambda^k(T^*M)$ of the cotangent bundle are differentiable k-forms.

Locally, *frames* give a generalised notion of a continuous choice of basis for the vector space over each point of our base space.

Definition 3.15. Let $E \xrightarrow{\pi} X$ be a rank-k vector bundle and suppose $U \subset X$ is an open set containing $x \in X$. Then a *local frame* over U, $(v_1, \ldots, v_k)$, is a collection of sections

$$v_j : U \to \pi^{-1}(U)$$

such that for all $x \in U$, $\{v_1(x), \ldots, v_k(x)\}$ is linearly independent and so a basis of E_x.

A local frame that can be defined over the entire base space X is called a *global frame*, which may not always exist. In fact, the existence of a global frame is equivalent to the vector bundle being trivial.

Theorem 3.16. *A vector bundle $E \xrightarrow{\pi} X$ of rank k is globally trivial if and only if there exists a global frame $\{v_1, \ldots, v_k\} : X \to E$.*

Proof. First suppose that $E = X \times \mathbb{R}^k$ is globally trivial. Then the frame where, for each $j = 1, \ldots, k$,

$$v_j(x) = e_j$$

for all $x \in X$ suffices (here $\{e_j\}$ is the standard basis on $\mathbb{R}^k$). Conversely, if $\{v_1, \ldots, v_k\}$ is a global frame we may define $\Phi : X \times \mathbb{R}^k \to E$ by $\Phi(e, (a_1, \ldots, a_k)) = \sum_{i=1}^k a_i v_i(x)$. This map is easily seen to be a linear isomorphism on each fibre. Also its composition with a trivialisation of E is continuous, and so it is continuous. Hence Φ is a vector bundle isomorphism. □

Remark. Since the zero vector does not span anything, the sections $\{v_1, \ldots, v_k\}$ of Theorem 3.16 are necessarily *non-zero* everywhere.

Example 3.17. We can now give a proof that the Möbius bundle E is not trivial. By Theorem 3.16, it suffices to show there is no global frame on E. For contradiction, suppose we have a global frame $\sigma \colon S^1 \to E$ (since E is rank-1, a global frame is the same as a global section). Then

$$\sigma(z) = (z, f(z))$$

for some $f : S^1 \to \mathbb{R}$. For σ to be continuous, f must satisfy $f(2\pi) = -f(0)$. Then by the Intermediate Value Theorem, there exists $z_0 \in [0, 2\pi]$ such that $f(z_0) = 0$ (see Figure 9). This contradicts our assumption σ is a global section, which cannot attain the value zero. So the Möbius bundle is nontrivial.

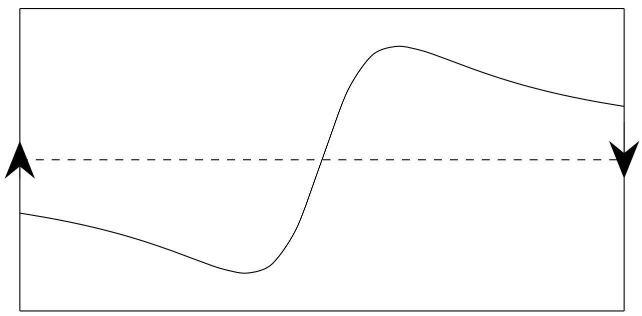

Figure 9: Every section of the Möbius bundle must intersect the zero section, so it is a nontrivial bundle.

4 Connections and Curvature

4.1 Connections

We now wish to generalise the notion of differentiation to sections of a vector bundle. The resulting *connections* are necessary to define the geometric notion of curvature.

Definition 4.1. Let M be a smooth manifold. A *smooth vector bundle* $E \xrightarrow{\pi} M$ is a vector bundle E over M such that E is also a smooth manifold, π is a *submersion*, that is,

$$(d\pi)_e : T_e E \to T_{\pi(e)} M$$

is a surjection for all $e \in E$ and for which the local trivialisation maps are smooth.

Remark. There is an analogous definition for a *smooth* principal fibre bundle which also requires that $P \times G \to G$ is a smooth map. From now on, all of our sections, vector bundles and principal bundles will be smooth.

Definition 4.2. Let $E \xrightarrow{\pi} M$ be a smooth vector bundle over a manifold M. A (left) *connection* on E is an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ which satisfies

$$\nabla(f\sigma) = f\nabla(\sigma) + df \otimes \sigma$$

for all $\sigma \in \Gamma(E)$, $f \in C^\infty(M)$. This is sometimes called a *Koszul connection*.

Remark. We may define a connection on a complex vector bundle in the obvious manner, replacing $\mathbb{R}$ by $\mathbb{C}$. There is also an analogous definition of a right connection on a vector bundle, which is a map from $\Gamma(E)$ to $\Gamma(E \otimes T^*M)$. This comes down to a matter of preference as the two definitions are equivalent. Our convention will be to always consider left connections.

An alternative but equivalent definition of a connection is to consider a map

$$\tilde{\nabla} : \Gamma(E) \otimes \Gamma(TM) \to \Gamma(E)$$

then we have $\tilde{\nabla}_Y X := \tilde{\nabla}(X, Y) = (\nabla X)(Y)$. The way to think about this is to observe that $T^*M \otimes E = \mathrm{Hom}(TM, E)$ represents linear maps from TM to E. So ∇ takes a section X of E and yields a map ∇X that sends a vector field $Y \in \Gamma(TM)$ to a section of E. The connection $\tilde{\nabla}$ acts on X and Y at the same time rather than first on X and then on Y.

We still need to prove that a connection actually exists as it is not enough to simply define such an object. First we will demonstrate that we really only need to find one connection, and from that we derive as many as we like.

Proposition 4.3. Assume there exists a connection ∇ on E, a smooth vector bundle. Then the space of all (left) connections on E is an affine space modelled on $\mathrm{Hom}(E, T^*M \otimes E)$. That is, given any connection ∇, we may take any $A \in \mathrm{Hom}(E, T^*M \otimes E)$ and then $\nabla + A$ is another connection. The map A is called the *connection 1-form*.

Proof. First suppose we have two connections ∇_1 and ∇_2 on a (smooth) vector bundle E. Then for $\sigma \in \Gamma(E), f \in C^\infty(M)$,

$$\nabla_1(f\sigma) - \nabla_2(f\sigma) = f\nabla_1(\sigma) + df \otimes \sigma - f\nabla_2(\sigma) - df \otimes \sigma = f(\nabla_1(\sigma) - \nabla_2(\sigma)).$$

Thus $\nabla_1 - \nabla_2$ is $C^\infty(M)$-linear (even though ∇_1 and ∇_2 individually are not!). Hence $\nabla_1 - \nabla_2$ comes from a vector bundle homomorphism

$$A : E \to T^*M \otimes E$$

where $A\sigma(x) := (\nabla_1 - \nabla_2)(\sigma)(x)$. Hence there is a unique $A \in \mathrm{Hom}(E, T^*M \otimes E)$ such that $\nabla_1 = \nabla_2 + A$. Furthermore, given a connection ∇ on E and $A \in \mathrm{Hom}(E, T^*M \otimes E)$, we can check $\nabla + A$ is also a connection. For $\sigma \in \Gamma(E), f \in C^\infty(M)$, we calculate

$$(\nabla + A)(f\sigma) = \nabla(f\sigma) + A(f\sigma) = f\nabla(\sigma) + df \otimes \sigma + fA(\sigma) = f(\nabla + A)(\sigma) + df \otimes \sigma.$$ □

It now suffices to find one connection. In order to do so, we need the following lemma.

Lemma 4.4. Let X be a compact Hausdorff space and $E \xrightarrow{\pi} X$ a (real) vector bundle. Then there exists an idempotent p and $N \in \mathbb{N}$ such that $\Gamma(E) \cong pC(X)^N$ (as modules).

Proof. Let $\bigcup_{\alpha=1}^n U_\alpha$ be an open cover of X such that $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ is a local trivialisation. Then define $p \in M_N(C(X))$ (where $N = nk$) by

$$p_{\alpha_j,\beta_l} = \sqrt{\varphi_\alpha\varphi_\beta}(g_{\alpha\beta})_{jl}$$

where φ_α is a partition of unity subordinate to the cover $\{U_\alpha\}$ (which exists since X is compact) and $g_{\alpha\beta}$ are the transition functions. Then

$$\begin{aligned}\sum_\beta p_{\alpha,\beta}p_{\beta,\gamma} &= \sum_\beta \sqrt{\varphi_\alpha\varphi_\gamma}\varphi_\beta g_{\alpha\beta}g_{\beta\gamma} \\ &= \sum_\beta \sqrt{\varphi_\alpha\varphi_\gamma}\varphi_\beta g_{\alpha\gamma} \quad \text{by Lemma 3.9} \\ &= \sqrt{\varphi_\alpha\varphi_\gamma}g_{\alpha\gamma} = p_{\alpha,\gamma}.\end{aligned}$$

Thus $p = p^2$ is a projection. Define $\Psi : \Gamma(E) \to p\bigoplus_{\alpha=1}^n C(U_\alpha)^k$ by

$$\sigma \mapsto p\begin{pmatrix}\Phi_1(\sigma|_{U_1}) \\ \vdots \\ \Phi_n(\sigma|_{U_n}).\end{pmatrix}$$

Moreover, this is a linear isomorphism (of modules) since each Φ_i is a homeomorphism. □

Example 4.5. Let $\mathcal{H} \xrightarrow{\pi} \mathbb{C}P_1$ be the Hopf line bundle. Then since $\mathbb{C}P_1 \cong S^2$ is compact and has a cover by two trivialisations, Theorem 4.4 guarantees the existence of an idempotent p such that $\Gamma(\mathcal{H}) \cong pC(S^2)^2$. In fact, we can explicitly construct a projection p_B such that

$$\Gamma(\mathcal{H}) \cong p_B C(S^2)^2.$$

Let

$$p_B(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix}.$$

Then we first check it is a projection. It is clearly self-adjoint because $\overline{\overline{z}} = z$. Furthermore,

$$p_B^2(z) = \frac{1}{(1+|z|^2)^2}\begin{pmatrix} 1+|z|^2 & \overline{z}+\overline{z}|z|^2 \\ z+z|z|^2 & |z|^2+|z|^4 \end{pmatrix} = \frac{1+|z|^2}{(1+|z|^2)^2}\begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix} = p_B(z).$$

Now for any $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in C(S^2)^2$, we have

$$p_B(z)\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \frac{1}{1+|z|^2}\begin{pmatrix} f(z)+\overline{z}g(z) \\ zf(z)+|z|^2g(z) \end{pmatrix} = \begin{pmatrix} h(z) \\ zh(z) \end{pmatrix} \in \Gamma(\mathcal{H})$$

where $h = \frac{1}{1+|z|^2}(f+\bar{z}g)$. Here we have implicitly used the fact that the section $\left(\begin{smallmatrix} f(z) \\ zf(z) \end{smallmatrix}\right)$, which at first glance may seem well-defined only on $\mathcal{H}\big|_{U_1}$, extends to a *globally defined* section of $\mathcal{H}$. This is because the limit as $z \to \infty$ exists, since f, g and so h are well-defined on the sphere.

Theorem 4.6 (Existence of a Connection)**.** *Let $E \xrightarrow{\pi} M$ be a smooth vector bundle and suppose $\Psi : \Gamma(E) \to pC^\infty(M)^N$ is a global module isomorphism with p idempotent. Then*

$$\nabla(\sigma) := (1 \otimes \Psi^{-1}) \circ (1 \otimes p) \circ (d \otimes 1)(\Psi(\sigma))$$

is a connection on E. The connection ∇ is called the Grassmann connection.

Proof. Let us first check Ψ is well-defined. Fix $\sigma \in \Gamma(E)$. Then $\Psi(\sigma) \in pC^\infty(M)^N$ is a column of N functions. Next, $(1 \otimes d)\Psi(\sigma)$ is a column of 1-forms, or a sum of (1-forms)$\otimes$(column of functions). Applying $1\otimes p$ to this yields a sum of (1-forms)$\otimes$(column of functions in $\mathrm{im}(p)$). Finally applying $1 \otimes \Psi^{-1}$ sends the second component back to a section of E, resulting in a sum of (1-forms)$\otimes$(sections of E), which is an element of $\Gamma(T^*M \otimes E)$. Now we check ∇ is a connection. For $f \in C^\infty(M)$,

$$\begin{aligned} \nabla(f\sigma) &= (1 \otimes \Psi^{-1}) \circ (1 \otimes p) \circ (d \otimes 1)(\Psi(f\sigma)) \\ &= (1 \otimes \Psi^{-1}) \circ (1 \otimes p) \circ (df \otimes \Psi(\sigma) + f(d \otimes 1)(\Psi(\sigma))) \\ &= (1 \otimes \Psi^{-1}) \circ (df \otimes \Psi(\sigma)) + f(1 \otimes \Psi^{-1}) \circ (1 \otimes p) \circ (d \otimes 1)(\Psi(\sigma)) \\ &= df \otimes \sigma + f\nabla(\sigma). \end{aligned}$$

Thus ∇ is a connection. □

Remark. In a shorthand abuse of notation, we write $\nabla = pd$ for the Grassmann connection we just constructed. The idea is that we first identify a section $\sigma \in \Gamma(E)$ of a vector bundle with an element of $pC(M)^N$ where it makes sense to 'differentiate' componentwise. We then project the result back onto $pC(M)^N$ and identify it once more to obtain a section $\nabla(\sigma) \in \Gamma(T^*M \otimes E)$. By Proposition 4.3 we obtain a global description of connections as operators of the form $\nabla = pdp + A, A \in pM_N(C(M))p$.

In general connections are difficult to write down explicitly, but we will look at two concrete examples.

Example 4.7 (Levi-Civita Connection)**.** Let $M \subseteq \mathbb{R}^n$ be a submanifold and $V \in \Gamma(TM)$ a section of its tangent bundle. Then we can write V in coordinates as $V = V^j \partial_j$ (where summation is implied). In general,

$$dV = V^j_{,k} dx^k \otimes \partial_j$$

belongs to $\Gamma(T^*\mathbb{R}^n \otimes T\mathbb{R}^n)$ but *not* $\Gamma(T^*M \otimes TM)$. That is, the exterior derivative itself fails to give a connection. However, taking p to be the orthogonal projection onto TM we have a connection $\nabla = pd$ which, in coordinates, is given by

$$\nabla_W V = V^j_{,k} W^k \partial_j + V^j \Gamma^k_{jm} W^m \partial_k,$$

where $\Gamma^k_{jm} = \nabla_{\partial_j} \partial_m$ are the *Christoffel symbols*. For our choice of p, this is called the *Levi-Civita connection* on the tangent bundle.

Example 4.8 (Hopf Bundle)**.** Let $\sigma \in \Gamma(\mathcal{H})$ be a section of the Hopf bundle. Then naively, we have

$$d\sigma = d \begin{pmatrix} f \\ zf \end{pmatrix} = \begin{pmatrix} df \\ dzf + zdf \end{pmatrix} = df \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} + dz \otimes \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

This does not belong to $\Gamma(T^*S^2 \otimes \mathcal{H})$ and so d once again does not constitute a connection. However we claim $\nabla = p_B d$, where p_B is the projection in Example 4.5, does constitute a connection. Let $\left(\begin{smallmatrix} f \\ zf \end{smallmatrix}\right) \in \Gamma(\mathcal{H})$. Then we calculate

$$\begin{aligned} \nabla(\sigma) &= (1 \otimes p) \begin{pmatrix} df \\ zdf + dzf \end{pmatrix} \\ &= df \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} + dz \otimes p \begin{pmatrix} 0 \\ f \end{pmatrix} \\ &= df \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} + \frac{dz}{1+|z|^2} \otimes \begin{pmatrix} \bar{z}f \\ |z|^2 f \end{pmatrix} \\ &= df \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} + \frac{f\bar{z}dz}{1+|z|^2} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} \in \Gamma(T^*S^2 \otimes \mathcal{H}). \end{aligned}$$

Moreover,

$$\begin{aligned} \nabla(f\sigma) &= (1 \otimes p)(df \otimes \sigma) + (1 \otimes p)(fd(\sigma)) \\ &= df \otimes \sigma + fpd(\sigma) \\ &= df \otimes \sigma + f\nabla(\sigma) \end{aligned}$$

and so $\nabla = p_B d$ is a connection.

Our picture of connections $\nabla = pd + A$ as projections of the usual exterior derivative on forms is a very global picture which is not commonly used in differential geometry and physics but is useful in proving existence. We now consider a local expression for a connection on a vector bundle.

Let $E \xrightarrow{\pi} M$ be a (smooth) vector bundle. Given a local trivialisation $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$, let $e = (e_1, \dots, e_k)$ be a local orthonormal frame (such that $e_i : U \to E|_U$ for all i). Any local section $\sigma : U \to E|_U$ may be expressed in the frame as a linear combination

$$\sigma = \sum_{i=1}^{k} \sigma^i e_i.$$

Now suppose ∇ is a connection on E. For any basis section $e_i : U \to E|_U$, there exist 1-forms $A_i^j \in \Omega^1(U)$ such that

$$\nabla(e_i) = \sum_{j=1}^{k} A_i^j \otimes e_j.$$

The matrix $A = (A_i^j)_{1 \leq i,j \leq k} \in \Omega^1(U, \mathrm{End}(E|_U))$ is called the *local connection 1-form* over U. Using the Leibniz rule for the connection, we calculate

$$\nabla(\sigma) = \nabla\Big(\sum_{i=1}^{k} \sigma^i e_i\Big) = \sum_{i=1}^{k} \Big(\sigma^i \nabla(e_i) + d\sigma^i \otimes e_i\Big) = \sum_{i,j=1}^{k} \Big(d\sigma^j + \sigma^i A_i^j\Big) \otimes e_j.$$

Using matrix shorthand we may write this as $\nabla(\sigma) = d\sigma + A\sigma$ and we see that $\nabla|_U = d + A$. The local connection 1-form A is precisely the restriction of the $\mathrm{End}(E)$-valued 1-form of Proposition 4.3 such that $\nabla|_U$ differs from the trivial connection d on $E|_U$. In other words, both our pictures look the same once we restrict to a local trivialisation.

A natural question to ask is how the connection changes under a change of local trivialisation (or sometimes *change of gauge*). Suppose $e' = (e'_1, \dots, e'_k)$ is another local frame over V, $U \cap V \neq \emptyset$. There is a matrix $g = (g_i^j)$ relating e and e' by $e' = eg$ and so we can calculate

$$\nabla(e') = \nabla(eg) = (\nabla e)g + e \otimes dg = e(Ag + dg) = e'(g^{-1}Ag + g^{-1}dg).$$

Hence the connection 1-form B of ∇ over V transforms under a change of trivialisation via

$$B = g^{-1}Ag + g^{-1}dg.$$

4.2 Curvature

Let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection on a vector bundle E compatible with an Hermitian (or nondegenerate) form $(\cdot|\cdot)$.

Definition 4.9. The *curvature* of ∇ is the map $R^E : \Gamma(E) \to \Gamma(\Lambda^2 T^*M \otimes E)$ given by

$$R^E\sigma = \wedge \otimes \mathrm{Id}_E \circ (d \otimes \mathrm{Id}_E - \mathrm{Id}_{T^*M} \otimes \nabla) \circ \nabla\sigma.$$

Remark. It should be noted there is a problem of left- and right-handedness in our definition. That is, we have chosen a convention $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ but the connection can be rephrased as $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ using the canonical isomorphism. However in this case, we have to change a minus sign in our definition.

$$R^E\sigma = \mathrm{Id}_E \otimes \nabla \circ (\mathrm{Id}_E \otimes d + \nabla \otimes \mathrm{Id}_{T^*M}) \circ \nabla(\sigma).$$

Remark. Despite its admittedly messy form when expressed concretely in Definition 4.9, the curvature is the obvious way of constructing a *second derivative* of a connection ∇ on E. To see this more clearly, observe that $\nabla : \Omega^0(E) \to \Omega^1(E)$ in the same manner in which $d : \Omega^k(M) \to \Omega^{k+1}(M)$. We extend this to a map $R^E : \Omega^0(E) \to \Omega^2(E)$ by composing ∇ with another map, which we might call $d^\nabla : \Omega^1(E) \to \Omega^2(E)$ which satisfies the Leibniz rule $d^\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma - \omega \wedge \nabla\sigma$. Indeed, if you squint your eyes, then $R^E\sigma$ looks exactly the same as $(d^\nabla \circ \nabla)\sigma$. In the case of the exterior derivative, $d^{k+1} \circ d^k = 0$ for all k, but this is not true for d^∇. The *curvature of the connection* measures precisely the obstruction to the second derivative vanishing.

Lemma 4.10. The curvature R^E is an $\mathrm{End}(E)$-valued 2-form. That is,

$$R^E \in \Gamma(\Lambda^2 T^*M \otimes E \otimes E^*) = \mathrm{Hom}(E, \Lambda^2 T^*M \otimes E).$$

Proof. It suffices to prove that R^E is $C^\infty(M)$-linear. Let $f \in C^\infty(M), \sigma \in \Gamma(E)$. Then we calculate

$$\begin{aligned} R^E(f\sigma) &= \wedge \otimes \mathrm{Id}_E(d \otimes \mathrm{Id}_E - \mathrm{Id}_{T^*M} \otimes \nabla) \circ (f\nabla\sigma + df \otimes \sigma) \\ &= df \wedge \nabla\sigma + f \wedge \otimes \mathrm{Id}_E \circ d \otimes \mathrm{Id}_E \nabla(\sigma) - f \wedge \otimes \mathrm{Id}_E \circ \mathrm{Id}_{T^*M} \otimes \nabla \circ \nabla(\sigma) - df \wedge \nabla\sigma \\ &= fR^E\sigma. \end{aligned}$$

□

Just as we thought of the connection locally as being a matrix A of 1-forms, we may think of the curvature (locally) as a matrix of 2-forms.

Before continuing, we will explicitly define some operations on $\mathrm{End}(E)$-valued forms. This will make our discussion of curvature under local trivialisations make a lot more sense. For $A \in \Omega^k(\mathrm{End}(E))$, we may write $A = \eta^i \otimes N_i$ for some $\eta^i \in \Omega^k(M)$ and $N_i \in \Omega^0(\mathrm{End}(E))$. Similarly, an $\mathrm{End}(E)$-valued l-form $B \in \Omega^l(\mathrm{End}(E))$ may be written $B = \mu^j \otimes M_j$. We define their wedge product as

$$A \wedge B := (\eta^i \wedge \mu^j) \otimes (N_i M_j).$$

The product $N_i M_j$ of endomorphisms is just composition. Importantly, there are two different wedges here; the usual wedge product $\eta^i \wedge \mu^j$ of differential forms, and the wedge we just *defined*. Because we are lazy, $\wedge$ will adopt whichever definition makes sense in context. Crucially, some properties of the wedge product of differential forms do not extend to the wedge product of $\mathrm{End}(E)$-valued forms. In the first instance, $\omega \wedge \omega$ always vanishes however the latter expression is not trivial as

$$A \wedge A = (\eta^i \wedge \eta^j) \otimes (N_i N_j)$$

and each $\eta^i \wedge \eta^j$ may not vanish if $i \neq j$. This is not the only way of defining a product between $\mathrm{End}(E)$-valued forms. We may define the commutator $[A, B]$ of A and B by

$$[A, B] := (\eta^i \wedge \mu^j) \otimes [N_i, M_j]$$

where $[N_i, M_j] = N_i M_j - M_j N_i$. The two products are connected via the relation

$$[A, B] = A \wedge B - (-1)^{\deg(A)\deg(B)} B \wedge A$$

which in particular implies $A \wedge A = \frac{1}{2}[A, A]$. This follows from the anticommutativity of the standard wedge product of differential forms. In local descriptions of curvature, both notations are used but $[\cdot, \cdot]$ is preferred in more general setting where the commutator may be taken to be the Lie bracket. We will use the wedge product.

Proposition 4.11 (Cartan's Structure Equation)**.** Suppose ∇ is a connection on a vector bundle E with local connection form A on a trivialising set U. Then the curvature R^E locally has the form $\Omega_A = dA + A \wedge A$, where $\Omega_A \in \Omega^2(\mathrm{End}(E)|_U)$.

Proof. Let $e = (e_1, \dots, e_k)$ be a local frame on U. Then (suppressing our summation notation)

$$\begin{aligned} R^E(e_i) &= \wedge \otimes Id_E \circ (d \otimes Id_E - Id_{T^*M} \otimes \nabla) \circ \nabla(e_i) \\ &= \wedge \otimes Id_E \circ (d \otimes Id_E - Id_{T^*M} \otimes \nabla) \circ (A_i^j \otimes e_j) \\ &= \wedge \otimes Id_E \circ (dA_i^j \otimes e_j - A_i^j \otimes \nabla(e_j)) \\ &= dA_i^p \otimes e_p - A_i^j \wedge A_j^p \otimes e_p \\ &= (dA_i^p + A_j^p \wedge A_i^j) \otimes e_p. \end{aligned}$$

That is, $(\Omega_A)_i^p = dA_i^p + A_j^p \wedge A_i^j$ which is precisely the wedge product between $\mathrm{End}(E)$-valued forms we just defined. In particular, we have shown

$$\Omega_A = dA + A \wedge A. \qquad \square$$

In Lemma 4.10, we showed that R^E is globally well-defined. As such, the local curvature forms Ω_A should be compatible on overlapping trivialisations. This is in contrast to the local connection forms A which don't transform quite as nicely under a change of local trivialisations.

Theorem 4.12. *Let $E \xrightarrow{\pi} M$ be a vector bundle with connection ∇. Suppose Φ_U, Φ_V are local trivialisations with $U \cap V \neq \emptyset$ and A and B are the local connection 1-forms associated to U and V, respectively. Then*

$$\Omega_A = g_{UV} \Omega_B g_{UV}^{-1}.$$

Proof. We will write $g := g_{VU}$ for notational simplicity. Then recall $A = g^{-1}dg + g^{-1}Bg$, so we calculate

$$\begin{aligned}
\Omega_A &= d(g^{-1}dg + g^{-1}Bg) + (g^{-1}dg + d^{-1}Bg) \wedge (g^{-1}dg + d^{-1}Bg) \\
&= dg^{-1} \wedge dg + dg^{-1} \wedge Bg + g^{-1}dBg - g^{-1}B \wedge dg + g^{-1}dg \wedge g^{-1}dg \\
&\quad + g^{-1}dg \wedge g^{-1}Bg + g^{-1}B \wedge dg + g^{-1}B \wedge Bg \\
&= g^{-1}(dB + B \wedge B)g + dg^{-1} \wedge dg + dg^{-1} \wedge Bg + g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Bg.
\end{aligned}$$

Now note that $0 = d(gg^{-1}) = dgg^{-1} + gdg^{-1}$, we have $dg = -gdg^{-1}g$. So

$$\begin{aligned}
\Omega_A &= g^{-1}\Omega_B g - dg^{-1} \wedge gdg^{-1}g + dg^{-1}g \wedge dg^{-1}g + dg^{-1} \wedge Bg - dg^{-1} \wedge Bg \\
&= g^{-1}\Omega_B g \\
&= g_{UV}\Omega_B g_{UV}^{-1}
\end{aligned}$$

□

We will now look at some examples. Our first example will be given as a lemma, which shows us that the definition of curvature often given in the setting of differential geometry for the tangent bundle coincides with Definition 4.9.

Lemma 4.13. Let M be a smooth manifold and consider its tangent bundle TM equipped with a connection ∇. Then for $X, Y \in \Gamma(TM)$,

$$R^{TM}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

where $[X,Y] = XY - YX$ is the commutator bracket.

Proof. It suffices to check the identity holds on a basis element of TM. So let $e = (e_1, \ldots, e_k)$ be a local section of TM. Then for all $X \in \Gamma(TM)$,

$$\nabla_X(e_i) = \nabla(e_i)(X) = \sum_{j=1}^{k} A_i^j(X)e_j.$$

Therefore, for $X, Y \in \Gamma(TM)$ (with summation implied)

$$\begin{aligned}
\nabla_X \nabla_Y(e_i) &= \nabla_X(A_i^j(Y)e_j) \\
&= XA_i^j(Y)e_j + A_i^j(Y)\nabla_X(e_j) \\
&= XA_i^j(Y)e_j + A_i^l(Y)A_l^j(X)e_j.
\end{aligned}$$

Similarly,

$$\nabla_Y \nabla_X(e_i) = YA_i^j(X)e_j + A_i^l(X)A_l^j(Y)e_j.$$

Furthermore,

$$\nabla_{[X,Y]}(e_i) = \nabla(e_i)([X,Y]) = A_i^j([X,Y])e_j.$$

Putting these all together, we calculate

$$\begin{aligned}(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(e_i) &= XA_i^j(Y)e_j + A_i^l(Y)A_l^j(X)e_j - YA_i^j(X)e_j \\ &\quad - A_i^l(X)A_l^j(Y)e_j - A_i^j([X,Y])e_j \\ &= \Big(XA_i^j(Y) - YA_i^j(X) - A_i^j([X,Y])\Big)e_j \\ &\quad + \Big(A_i^l(Y)A_l^j(X) - A_i^l(X)A_l^j(Y)\Big)e_j \\ &= dA_i^j(X,Y)e_j + A_i^l \wedge A_l^j(X,Y)e_j \\ &= (dA_i^j + A_i^l \wedge A_l^j)(X,Y)e_j \\ &= (\Omega_A)_i^j(X,Y)e_j.\end{aligned}$$

By Proposition 4.11, this proves the claim. □

For the Levi-Civita connection on TM, we can write the curvature tensor R^{TM} associated to this connection, called the *Riemann curvature tensor*, in coordinates using Lemma 4.13. Let $\{\partial_i\}$ be the coordinate basis of the tangent bundle TM for some choice of local coordinates $x_\alpha : U_\alpha \to \mathbb{R}^k$ at $x \in M$. Then since $[\partial_i, \partial_j] = 0$ for all i, j,

$$\begin{aligned}R^{TM}(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\ &= \Gamma^m_{jk,i}\partial_m + \Gamma^m_{jk}\Gamma^l_{im}\partial_l - \Gamma^m_{ik,j}\partial_m - \Gamma^m_{ik}\Gamma^l_{jm}\partial_l \\ &= (\Gamma^m_{jk,i} - \Gamma^m_{ik,j} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl})\partial_m \\ &=: R^m_{ijk}\partial_m\end{aligned}$$

where the $\Gamma^k_{ij} := \nabla_{\partial_i}\partial_j$ are the Christoffel symbols and R^m_{ijk} are called the *components of the Riemann curvature tensor*. This example illustrates that the study of Riemannian geometry is really just a very special case of the study of vector bundles, where we restrict ourselves to a single vector bundle (the tangent bundle to a smooth manifold) and a certain connection (the Levi-Civita connection).

Example 4.14 (Hopf Bundle)**.** Recall we defined a connection $\nabla = p_B d p_B$ on $\mathcal{H}$ in Example 4.8. We wish to calculate the curvature of this connection. Let $\sigma = \begin{pmatrix} f \\ zf \end{pmatrix} \in \Gamma(\mathcal{H})$. Then we calculate

$$\begin{aligned}R^{\mathcal{H}}\begin{pmatrix} f \\ zf \end{pmatrix} &= (\wedge \otimes \mathrm{Id}_E) \circ (d \otimes \mathrm{Id}_E - \mathrm{Id}_{T^*M} \otimes \nabla) \circ \nabla \begin{pmatrix} f \\ zf \end{pmatrix} \\ &= (\wedge \otimes \mathrm{Id}_E) \circ (d \otimes \mathrm{Id}_E - \mathrm{Id}_{T^*M} \otimes \nabla) \circ \left\{ df \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} + \frac{f\bar{z}dz}{1+|z|^2} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} \right\} \\ &= d(df) \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} - df \wedge \nabla \begin{pmatrix} 1 \\ z \end{pmatrix} + d\Big(\frac{f\bar{z}dz}{1+|z|^2}\Big) \otimes \begin{pmatrix} 1 \\ z \end{pmatrix} - \frac{f\bar{z}dz}{1+|z|^2} \wedge \nabla \begin{pmatrix} 1 \\ z \end{pmatrix}.\end{aligned}$$

Now $d^2 f = 0$ so the first term vanishes and moreover since the derivative of a constant is zero,

$$\nabla \begin{pmatrix} 1 \\ z \end{pmatrix} = \frac{\bar{z}dz}{1+|z|^2} \otimes \begin{pmatrix} 1 \\ z \end{pmatrix}.$$

We also calculate, applying the Leibniz rule repeatedly,

$$
\begin{aligned}
d\Big(\frac{f\bar{z}dz}{1+|z|^2}\Big) &= \frac{df\wedge\bar{z}dz}{1+|z|^2} + \frac{fd\bar{z}\wedge dz}{1+|z|^2} - f\bar{z}dz\wedge\Big(-\frac{\bar{z}dz+zd\bar{z}}{(1+|z|^2)^2}\Big)\\
&= \frac{df\wedge\bar{z}dz}{1+|z|^2} + \frac{fd\bar{z}\wedge dz(1+|z|^2)+f|z|^2d\bar{z}\wedge dz + f\bar{z}^2dz\wedge dz}{(1+|z|^2)^2}\\
&= \frac{df\wedge\bar{z}dz}{1+|z|^2} + \frac{fd\bar{z}\wedge dz}{(1+|z|^2)^2}.
\end{aligned}
$$

Continuing our calculation above, we find

$$
\begin{aligned}
R^{\mathcal{H}}\begin{pmatrix} f \\ zf \end{pmatrix} &= \Big\{-\frac{df\wedge\bar{z}dz}{1+|z|^2} + \frac{df\wedge\bar{z}dz}{1+|z|^2} + \frac{fd\bar{z}\wedge dz}{(1+|z|^2)^2} + \frac{f\bar{z}dz}{1+|z|^2}\wedge\frac{\bar{z}dz}{1+|z|^2}\Big\}\otimes\begin{pmatrix} 1 \\ z \end{pmatrix}\\
&= \frac{d\bar{z}\wedge dz}{(1+|z|^2)^2}\otimes\begin{pmatrix} f \\ zf \end{pmatrix},
\end{aligned}
$$

where we have used that $\omega\wedge\omega = 0$ for any k-form ω.

Remark. In the previous example, we calculated the curvature for just one connection, but recall every other connection looks like some perturbation $\nabla = \nabla^G + A$ of the Grassmann connection ∇^G and so if one wants to compute the curvature for a general connection, you simply include this A term in the calculation above. Note that regardless of your choice of A, the term we just calculated will not vanish, as it does not depend on A.

5 Chern-Weil Theory

Roughly speaking, Chern-Weil theory provides a method of constructing cohomology classes from geometric information encoded in connections, thus marrying the two concepts we have explored previously; algebraic topology and geometry. This discussion mostly follows [10].

5.1 Characteristic Classes

We begin by studying how the curvature of a connection can be used to construct de Rham classes known as *characteristic classes.* Any connection on a vector bundle E may be expressed locally by a matrix ω of 1-forms, and similarly the curvature can be represented locally by a matrix Ω of 2-forms. Under a change of trivialisation, curvature transforms by conjugation $\tilde{\Omega} = g^{-1}\Omega g$ and thus if P is some polynomial which is invariant under conjugation, the differential form $P(\Omega)$ will be independent of the frame and so define a global form on M. We will show that in fact, it is a closed form which is *independent of the connection.* This gives rise to the aforementioned characteristic classes.

Definition 5.1. Let $X = (x^i_j)$ be a $k\times k$ matrix. A polynomial $P(X)$ on $\mathfrak{gl}_k(\mathbb{R})$ is called *invariant* if $P(A^{-1}XA) = P(X)$ for all $A\in \mathrm{GL}_k(\mathbb{R})$.

Example 5.2. Let $X = (x^i_j)$ be a $k \times k$ matrix, $\lambda \in \mathbb{C}$. Then

$$\det(\lambda 1 + X) = \lambda^k + f_1(X)\lambda^{k-1} + \ldots + f_{k-1}(X)\lambda + f_k(X).$$

The functions f_j are called *characteristic polynomials*, and they are all invariant since

$$\det(\lambda 1 + A^{-1}XA) = \det(A^{-1}(\lambda 1 + X)A) = \det(\lambda 1 + X)$$

using the fact det is a homomorphism.

Example 5.3. For all $j \in \mathbb{N}$ the map $T_j(X) := \mathrm{tr}(X^j)$ is an invariant polynomial. This simply follows from the fact $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. So in particular,

$$\mathrm{tr}(A^{-1}X^jA) = \mathrm{tr}(A^{-1}AX^j) = \mathrm{tr}(X^j).$$

Each T_j is called the j^{th} *trace polynomial.*

The characteristic polynomials and trace polynomials are related by the identity [10, Theorem B.14]

$$T_j - f_1T_{j-1} + f_2T_{j-2} - \cdots + (-1)^{j-1}f_{j-1}T_1 + (-1)^j jf_j = 0.$$

These are especially important examples of invariant polynomials because of the following algebraic result.

Theorem 5.4. *The ring* $\mathrm{Inv}(\mathfrak{gl}_k(\mathbb{R})$ *of invariant polynomials is generated by the characteristic polynomials* $\{f_j\}$ *or the trace polynomials* $\{T_j\}$ *for* $j = 0, 1, \ldots, k$.

Proof. [10, Theorem 23.4] □

This means it generally suffices to prove results about characteristic classes for the trace polynomials. We will employ this technique in proving the main result of this section.

Before continuing, we check that it makes sense to 'take powers of Ω'. In fact, we have already seen this. Recall that our connection $\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$. When we defined the curvature, we got an operator $\Omega : \Gamma(E) \to \Gamma(\Lambda^2T^*M \otimes E)$. Thus $\Omega = d^\nabla \circ \nabla$ was really just a second derivative of the connection. In general, for any $k \in \mathbb{N}$ we have a map

$$\nabla : \Gamma(\Lambda^kT^*M \otimes E) \to \Gamma(\Lambda^{k+1}T^*M \otimes E)$$

and thus we can just repeatedly apply ∇ to obtain higher powers of Ω.

Lemma 5.5. Let E be a smooth rank-k vector bundle over M and ∇ a connection on E. Then for all invariant polynomials $P \in \mathrm{Inv}(\mathfrak{gl}_k(\mathbb{R}))$, $P(\Omega)$ is a global $2r$-form on M where r is the degree of P.

Proof. Let Ω be the curvature matrix relative to a frame $e = (e_1, \ldots, e_k)$ on $U \subset M$. For $p \in U$, Ω_p is a $k \times k$ matrix of 2-forms and if $e' = eg$ is another local frame on U we have

$$\Omega'_p = g(p)^{-1}\Omega_p g(p)$$

for some $g(p) \in \mathrm{GL}_k(\mathbb{R})$. Thus since P is invariant, as p varies over U we have $P(\Omega) = P(\Omega')$ is independent of a choice of frame. Now given a local trivialisation $\{U_\alpha\}$, let e^α be any frame on U_α, and Ω_α the curvature matrix relative to this frame. Then $P(\Omega_\alpha)$ is a $2r$-form on U_α. On the overlap $U_\alpha \cap U_\beta$ we have two $2r$-forms $P(\Omega_\alpha)$ and $P(\Omega_\beta)$ which must equal since $P(\Omega_\alpha)$ is independent of frame. Therefore, the collection $\{P(\Omega_\alpha)\}$ gives rise to a global $2r$-form on M. □

Theorem 5.6. *Let $E \xrightarrow{\pi} M$ be a smooth vector bundle and ∇ and connection on E with local curvature Ω. Then for any $r \in \mathbb{N}$, $T_k(\Omega)$ is a closed $2r$-form on M. If $\tilde{\nabla}$ is another connection on E and $\alpha = \nabla - \tilde{\nabla} \in \Omega^r(M, \mathrm{End}E)$ then*

$$T_r(\Omega) - T_r(\tilde{\Omega}) = d\Big(r\int_0^1 \mathrm{tr}\big(\alpha \wedge \Omega_t^{r-1}\big)\,dt\Big)$$

where Ω_t is the curvature of the connection $\nabla_t = (1-t)\tilde{\nabla} + t\nabla$. In particular, $[T_r(\Omega)] = [T_r(\tilde{\Omega})]$ so the cohomology class of $T_r(\Omega)$ is independent of the connection.

Lemma 5.7. Given two connections ∇ and $\tilde{\nabla}$ on a vector bundle E over M,

$$\nabla_t = t\nabla + (1-t)\tilde{\nabla}$$

is another connection for all $t \in (0,1)$.

Proof. It suffices to check $\nabla_t(f\sigma) = f\nabla_t(\sigma) + df \otimes \sigma$ for all $\sigma \in \Gamma(E)$, $f \in C^\infty(M)$. We calculate,

$$\begin{aligned}
\nabla_t(f\sigma) &= t\nabla(f\sigma) + (1-t)\tilde{\nabla}(f\sigma) \\
&= t(f\nabla(\sigma) + df \otimes \sigma) + (1-t)(f\tilde{\nabla}(\sigma) + df \otimes \sigma) \\
&= f(t\nabla(\sigma) + (1-t)\tilde{\nabla}(\sigma)) + df \otimes \sigma \\
&= f\nabla_t(\sigma) + df \otimes \sigma.
\end{aligned}$$

□

Lemma 5.8. Let ∇ be a connection on a vector bundle E. Then for all $A \in \Omega^r(M, \mathrm{End}E)$,

$$d(\mathrm{tr}(A)) = \mathrm{tr}([\nabla, A]).$$

Proof. Locally, the connection may be written as $\nabla = d + \omega$, and A is a matrix of r-forms relative to a frame e on U. So,

$$\begin{aligned}
[\nabla, A]e &= [d + \omega, A]e \\
&= [\omega, A]e + d(A_e) - Ad(e) \\
&= [\omega, A]e + d(A)e + (-1)^{2k}Ad(e) - Ad(e) \\
&= [\omega, A]e + d(A)e.
\end{aligned}$$

Taking the trace of both sides, we have

$$\operatorname{tr}([\nabla, A]) = \operatorname{tr}([\omega, A]) + \operatorname{tr}(d(A)) = 0 + \operatorname{tr}(d(A)) = d\operatorname{tr}(A). \qquad \square$$

Proof of Theorem 5.6. By Lemma 5.8, we have

$$\begin{aligned} dT_r(\Omega) &= \operatorname{tr}([\nabla, \Omega^r]) \\ &= \operatorname{tr}([\nabla, (\nabla^2)^r]) \\ &= \operatorname{tr}(\nabla^{2r+1} - \nabla^{2r+1}) = 0. \end{aligned}$$

So $T_k(\Omega)$ is a closed form. Now,

$$\begin{aligned} \frac{d}{dt}\Big(T_r(\Omega_t)\Big) &= \operatorname{tr}\Big(\sum_{j=0}^{r-1} \Omega_t^j \frac{d\Omega_t}{dt} \Omega_t^{r-j-1}\Big) \\ &= r\operatorname{tr}\Big(\frac{d}{dt}(\nabla_t^2)\Omega_t^{r-1}\Big) \\ &= r\operatorname{tr}\Big(\big(\frac{d}{dt}(\nabla_t)\nabla_t + \nabla_t \frac{d}{dt}(\nabla_t)\big)\Omega_t^{r-1}\Big) \\ &= r\operatorname{tr}\Big(\big(\alpha\nabla_t + \nabla_t\alpha\big)\Omega_t^{r-1}\Big) \\ &= r\operatorname{tr}\big(d(\alpha \wedge \Omega^{r-1})\big) \\ &= r\, d\Big(\operatorname{tr}(\alpha \wedge \Omega^{r-1})\Big). \end{aligned}$$

Hence,

$$\begin{aligned} T_r(\Omega) - T_r(\tilde{\Omega}) &= \int_0^1 \frac{d}{dt}\operatorname{tr}(\Omega_t^r)\, dt \\ &= d\Big(r\int_0^1 \operatorname{tr}(\alpha \wedge \Omega^{r-1})\, dt\Big). \end{aligned}$$

Thus we have shown $T_r(\Omega) - T_r(\tilde{\Omega})$ is exact, and so $[T_r(\Omega)]$ is independent of the choice of connection on E. $\square$

By Theorem 5.4, it immediately follows that Theorem 5.6 can be extended to any invariant polynomial $P(\Omega)$.

Corollary 5.9 (Chern-Weil Homomorphism)**.** Let E be a vector bundle of rank k on a manifold M, ∇ a connection on E, and P an invariant polynomial of degree r on $\mathfrak{gl}_k(\mathbb{R})$. Then the global $2r$-form $P(\Omega)$ on M is closed and $[P(\Omega)] \in H^{2r}(M)$ is independent of the connection.

The map $c : \operatorname{Inv}(\mathfrak{gl}_k(\mathbb{R})) \to H^\bullet(M)$ given by $P(\Omega) \mapsto [P(\Omega)]$ is called the *Chern-Weil homomorphism.* We call each equivalence class $[P(\Omega)]$ a *characteristic class.*

Remark. One might ask, what about odd degree cohomology classes? It turns out that if k is odd, then the cohomology class $[T_k(\Omega)] = 0$ for any connection on any vector bundle. For a proof of this, see [10, Theorem 24.3]. Another legitimate question is to ask if we can obtain every de Rham cohomology class via the Chern-Weil homomorphism. The answer is yes, but proving this requires some much more complicated topological machinery (see [7]).

Save for a few technical details, we now have a way to perfectly translate information about curvature to information about de Rham cohomology in a manner which is independent of the choice of connection. This is really only the very beginnings of the powerful Chern-Weil theory, but we will finish by looking at the notion of Chern classes.

Definition 5.10. Let E be a complex vector bundle over M and define the *total Chern class of* E as

$$\mathrm{Ch}(E) = \det\left(1 + \frac{i}{2\pi}\Omega\right) = 1 + c_1(E) + \cdots + c_k(E).$$

The $c_i(E)$ are called the *Chern classes* of E.

Remark. There is an analogue of the Chern class for real vector bundles, called the *Pontrjagin class* but they differ in some important ways. Most notably, one must contend with questions of orientability. For simplicity, we just consider the example of the Chern class as it is the one relevant to our complex line bundle $\mathcal{H}$.

Example 5.11. Recall that for the connection $\nabla = p_B dp_B$ on the Hopf bundle, we computed its curvature as

$$R^{\mathcal{H}} = \frac{d\bar{z} \wedge dz}{(1+|z|^2)^2}.$$

Now the coordinate z is not globally defined on S^2, however it is defined almost everywhere, so we may integrate over S^2. Alternatively, we may identify S^2 with $\mathbb{R}^2$ using stereographic projection. Then the curvature term becomes

$$R^{\mathcal{H}} = \frac{2i dx \wedge dy}{(1+|x|^2+|y|^2)^2}.$$

Using polar coordinates, we calculate

$$\begin{aligned}\int_{S^2} R^{\mathcal{H}} &= 2i \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1+|x|^2+|y|^2)^2}\, dxdy \\ &= 4\pi i \int_0^\infty \frac{r}{(1+r^2)^2}\, dr \\ &= 2\pi i.\end{aligned}$$

Therefore, since $\mathcal{H}$ is a line bundle, its only Chern class is

$$\mathrm{Ch}(\mathcal{H}) = \frac{2\pi i}{2\pi i} = 1.$$

This finally gives us a proof the $\mathcal{H}$ is nontrivial, since the trivial bundle has a Chern class of 0, and thus $\mathcal{H} \ncong S^2 \times \mathbb{C}$.

For more complicated examples, much more sophisticated tools exist for calculating characteristic classes, but that will be the topic for further research. Both [7] and [10] delve further into the theory of characteristic classes for the interested reader.

6 Discussion and Conclusion

In this report, we have explored in considerable detail two key areas of research in modern mathematics, namely differential geometry and algebraic topology. We have demonstrated how calculus may be used to compute topological invariants of some underlying space, which has powerful ramifications since calculus generally allows for more explicit computation. In particular, we used the methods we developed to characterise the de Rham cohomology of the n-sphere and the torus. Cohomology theory, and algebraic topology more generally, extend far beyond what we had time to study in this summer project but many of the motivations remain the same. We want to find ways of assigning algebraic invariants to topological spaces. Rephrasing problems in algebraic terms can often make problems of topology more tractable and so this provides a rich area of future research.

We have also introduced and studied vector bundles as something of a generalisation of Riemannian geometry. This intrinsic approach to studying geometry required us to carefully define notions of sections, connections and curvature. Our research was aided once again by studying concrete examples, most notably the Hopf bundle and tangent bundles. Particular emphasis was placed on the dependence of curvature on the choice of connection, which at first glance seemed to present a major obstacle to extracting any kind of global topological data from the geometry of vector bundles.

Nevertheless, we showed that Chern-Weil theory provides us with exactly the right tool to calculate topological invariants from curvature, which is independent of the choice of connection. The theory of vector bundles, connections and characteristic classes is another field of very active research both within mathematics, to things like noncommutative geometry and K-theory, and beyond. Notably, these ideas find many applications in physics, with the language of vector bundles and connections being central to gauge theories such as the celebrated standard model of particle physics.

References

[1] Bleecker, D. (1981). *Gauge Theory and Variational Principles.* Addison-Wesley Publishing Company. p. 1-50.

[2] Bott, R; Tu, L. (1982). *Differential Forms in Algebraic Topology.* Springer-Verlag New York Inc. p. 13-50.

[3] Burns, K; Gidea, M (2005). *Differential Geometry and Topology: With a View to Dynamical Systems.* CRC Press. p. 77.

[4] Darling, R. (1994) *Differential Forms and Connections.* Cambridge University Press.

[5] Hatcher, A. *Algebraic Topology.* Available at `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`.

[6] Hatcher, A. *Vector Bundles and K-Theory*. Available at `https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html`.

[7] Milnor, J; Stasheff, J. (1974). *Characteristic Classes*. Princeton.

[8] Munkres, J. (2014). *Topology*, 2nd Edition. Pearson Education Limited. p. 73-145, 259.

[9] Richeson, D. (2008). *Euler's Gem: The Polyhedron Formula and the Birth of Topology.* Princeton University Press.

[10] Tu, L. (2017). *Differential Geometry: Connections, Curvature, and Characteristic Classes.* Springer International Publishing.