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Representation varieties of once-punctured torus bundles

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Abstract

In this paper, we aim to compute the $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ character varieties of a certain family of once-punctured torus bundles. This family of 3-manifolds is defined by a family of automorphisms ϕ_n of $\pi_1 S$, where S denotes the once-punctured torus. We approach this problem via computing the topology of a fixed point set $X_{\phi_n}S$ since there is a generically 2:1 map between the $SL_2(\mathbb{C})$ characters of the bundle and the points in the fixed point set if we only consider irreducible characters in S. We study the defining equations of $X_{\phi_n}S$ and compute the genus for odd positive n. This enables us to gain the genera of the $PSL_2(\mathbb{C})$ character varieties of the 3-manifolds.

1 Introduction

The $SL_2(\mathbb{C})$ representation variety of a finitely-generated group Γ, denoted by $\mathcal{R}(\Gamma)$, is the set of representations from Γ into $SL_2(\mathbb{C})$. All characters of these representations form the $SL_2(\mathbb{C})$ character variety of Γ, denoted by $X(\Gamma)$. Both sets admit the structure of an affine algebraic set.

There are a number of connections found between the topology of 3-manifolds and the $SL_2(\mathbb{C})$-character varieties of their fundamental groups (Culler and Shalen, 1983; Boyer and Zhang, 1998). In this paper, we study an infinite family of once-punctured torus bundles M_n.

Let S denote the once-punctured torus and M_ϕ be an once-punctured torus bundle defined by the homeomorphism ϕ from S to itself. In this paper, in order to study the topology of $X(\pi_1 M_\phi)$, we consider the restriction map

$$r : X(\pi_1 M_\phi) \to X(\pi_1 S)$$

induced by the inclusion map $S \to M_\phi$.

A previous result (Horowitz, 1975) shows that we can identify $X(\pi_1 S)$ with $\mathbb{C}^3$. For any point $(x, y, z) \in \mathbb{C}$, there exists an $SL_2(\mathbb{C})$-representation ρ of $\pi_1 S$ such that $x = tr(\rho(a))$, $y = tr(\rho(b))$, $z = tr(\rho(ab))$. We can show that the image of r is a subset of the fixed point set $X_\phi(S)$ of the polynomial automorphism $\bar{\phi}$ defined by $\bar{\phi}(X_\rho) = X_{\rho\phi}$. Restricting to irreducible characters in $X(\pi_1 S)$, the map $r : X(\pi_1 M_\phi) \to X_\phi(S)$ is a 2:1 cover (with possible branch points).

In this paper, we compute the genera of the fixed point set $X_{\phi_n}S$ for an infinite family of monodromies ϕ_n for odd n. We prove the following theorem.

Theorem 6.1. *When n is a positive odd integer, the fixed point set $X_{\phi_n}S$ has one component and its genus is $\lfloor \frac{n}{2} \rfloor$.*

To extend this result further, we can compute the $PSL_2(\mathbb{C})$-character varieties of the infinite family of once-punctured torus bundles with monodromies ϕ_n.

Theorem 7.1. *For every positive odd integer n, the $PSL_2(\mathbb{C})$-character variety $\overline{X}(M_n)$ of M_n is birational equivalent to $\overline{X}_{\phi_n}S$ induced by the fixed point set $X_{\phi_n}S$. Its genus is zero for all positive odd n.*

1.1 Organisation

Section 2 gives the definition of once-punctured torus bundles and introduces our infinite family $\{M_n\}_{n>0}$ and their fundamental groups. We introduce the concepts of $SL_2(\mathbb{C})$ character varieties in section 3. In section 4, the restriction map r and the fixed point set $X_\phi S$ are introduced. Section 5 introduces the Newton polygon, which is used in the proof Theorem 6.1.

In section 6, we compute the defining equations for the fixed point set induced by our family of 3-manifolds M_n. We then compute the genus and the number of components of the fixed point set for each n. Section 7 further extends the result in section 6 to $PSL_2(\mathbb{C})$-character varieties of M_n.

1.2 Statement of authorship

The concepts about once-punctured torus bundle and character variety in section 2 and 3 source from many papers, mainly from (Boyer, Luft, and Zhang, 2002) and (Culler and Shalen, 1983).

The idea of the restriction map in section 4 comes from my supervisor. Under his direction, I figure out the result and proof in section 6 and 7.

2 Once-punctured torus bundle

Let S be an once-punctured torus in Figure 1. Its fundamental group is a free group $\pi_1 S = \langle a, b\rangle$ generated by the loops a and b as shown in Figure 1.

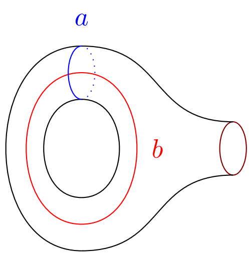

Figure 1: once-punctured torus S

We define the once-punctured torus bundle with corresponding framing ϕ as follows.

Definition 2.1. *Let $\Phi \in SL_2(\mathbb{Z})$ be a monodromy of S with the corresponding framing ϕ. Define the once-punctured torus bundles with monodromy Φ, denoted by M_ϕ, as*

$$M_\phi := {S \times [0,1]} \big/ \big((x,0),(\phi(x),1)\big)$$

The fundamental group of M_ϕ admits the presentation

$$\pi_1 M_\phi = \langle t, a, b | t^{-1}at = \phi(a), t^{-1}bt = \phi(b)\rangle$$

where a,b are the generators of $\pi_1 S$.

In previous results, Baker and Petersen studied the $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ character varieties of the infinite family of once-punctured torus bundles with monodromis AB^{n+2}. Here A and B are the right-handed Dehn-twists in terms of curve a and curve b respectively. And so A and B correspond to the following automorphisms α and β respectively.

$$\alpha = \begin{cases} a \to a \\ b \to ba \end{cases} \qquad \beta = \begin{cases} a \to ab^{-1} \\ b \to b \end{cases}$$

Baker and Petersen found a birational isomorphism between the $SL_2(\mathbb{C})$-character varieties of the 3-manifolds and a family of hyperelliptic curves to compute the genera of the $SL_2(\mathbb{C})$-character varieties. In this paper, we approach this problem via the restriction map and the fixed point set mentioned in the introduction. Our preliminary result shows that the corresponding fixed point sets for this family of 3-manifolds have genera 0 for all n, since the aforementioned restriction map r has branching points. Hence, we further study the family of monodromies $\Phi_n = AB^{n+2}A$, which gives us nonzero genus for the fixed point sets. This will be elaborated in section 6.

For the rest of this paper, we denote M_n as the infinite family of once-punctured torus bundles corresponding to the monodromy family $\Phi_n = AB^{n+2}A$. The corresponding family of framings ϕ_n admit the following form.

$$\phi_n = \begin{cases} a \to a(a^{-1}b^{-1})^{n+2} \\ b \to ba^2(a^{-1}b^{-1})^{n+2} \end{cases}$$

Then

$$\pi_1 M_\phi = \langle t, a, b | t^{-1}at = a(a^{-1}b^{-1})^{n+2}, t^{-1}bt = ba^2(a^{-1}b^{-1})^{n+2} \rangle$$

3 Character Variety

In this section, we introduce the $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ character varieties of finitely-generated group. We then adapt these concepts to the fundamental groups we introduced in section 1 and give some facts about their character varieties.

3.1 $SL_2(\mathbb{C})$ character variety

Given a finitely-generated group $\Gamma = \langle \gamma_1, ..., \gamma_n | r_j \rangle$. We define the $SL_2(\mathbb{C})$ representation variety of Γ as the following.

Definition 3.1. *An $SL_2(\mathbb{C})$ representation of Γ is a homomorphism $\rho : \Gamma \to SL_2(\mathbb{C})$. The set of all $SL_2(\mathbb{C})$ representation of Γ, denoted by $\mathcal{R}(\Gamma)$, is the $SL_2(\mathbb{C})$ representation variety of Γ.*

For a representation $\rho \in \mathcal{R}(\Gamma)$, we define the character X_ρ of ρ as a function $I_\rho : \Gamma \to \mathbb{C}$ defined by $I_\rho(p) = tr(\rho(p))$(Culler and Shalen, 1983).

Definition 3.2. *The $SL_2(\mathbb{C})$ character variety of Γ is the set of characters of all representations in $\mathcal{R}(\Gamma)$. We use $X(\Gamma)$ to denote the $SL_2(\mathbb{C})$ character variety of Γ.*

We say that a character X_ρ of Γ is an irreducible character if ρ is an irreducible representation of Γ. We denote the set of all reducible and irreducible characters of Γ as $X^{red}(\Gamma)$ and $X^{irr}(\Gamma)$ respectively.

Since Γ has n generators, we can identify $\mathcal{R}(\Gamma)$ with a subset of $SL_2(\mathbb{C})^n \subset \mathbb{C}^{4n}$ by identifying a representation ρ with the point $(\rho(\gamma_1), \rho(\gamma_2), ..., \rho(\gamma_n) \in \mathbb{C}^{4n})$(Culler and Shalen, 1983).

Similarly, according to (González-Acuña and Montesinos-Amilibia, 1993), the following lemma enables $X(\Gamma)$ to admit the structure of an affine algebraic set.

Lemma 3.3. *If we define the words $\{\gamma_i\gamma_j | 1 \leq i < j \leq n\} \cup \{\gamma_i\gamma_j\gamma_k | 1 \leq i < j < k \leq n\}$ as $\gamma_{n+1}, ..., \gamma_m$, a character X_ρ of Γ is uniquely determined by the words $\{\gamma_1, ..., \gamma_m\}$ by identifying X_ρ with the point $(tr\rho(\gamma_1), ..., tr\rho(\gamma_m)) \in \mathbb{C}^m$.*

This enables us to treat $X(\Gamma)$ as an affine algebraic set and study its topology in the affine coordinate.

3.2 $SL_2(\mathbb{C})$ character variety of free group of rank two

Throughout this paper, if we have a manifold M, we denote $\mathcal{R}(\pi_1 M)$ by $\mathcal{R}(M)$ and denote $X(\pi_1 M)$ by $X(M)$.

Recall from section 2, the fundamental group of once-punctured torus S is a free group of rank two. In symbols, $\pi_1 S = \langle a,b \rangle$. From Lemma 3.1, we can identify a character $X_\rho \in X(S)$ as a point $(x, y, z) \in \mathbb{C}^3$ by letting $x = tr\rho(a)$, $y = tr\rho(b)$, $z = tr\rho(ab)$. According to (Baumslag, 1993), the map $X^{irr}(S) \to \mathbb{C}^3$ is surjective. So we have the following theorem.

Theorem 3.4. $X^{irr}(S) = \mathbb{C}^3$.

3.3 $PSL_2(\mathbb{C})$ character variety

We can also define the $PSL_2(\mathbb{C})$ representation variety and $PSL_2(\mathbb{C})$ character variety of a finitely-generated group Γ. Let $\overline{\mathcal{R}}(\Gamma)$ be the set of all representations from Γ to $PSL_2(\mathbb{C})$. The representation variety $\overline{\mathcal{R}}(\Gamma)$ has an algebro-geometric quotient $\overline{X}(\Gamma)$, called the $PSL_2(\mathbb{C})$ character variety of Γ (Boyer and Zhang, 1998).

4 The restriction map and the fixed point set

Recall from section 2, the fundamental group of an once-punctured torus bundle M_ϕ with framing ϕ is

$$\pi_1 M_\phi = \langle t, a, b | t^{-1}at = \phi(a), t^{-1}bt = \phi(b) \rangle$$

From section 3.1, if we compute the topology of $X(M_\phi)$ directly, we are computing a subset of $\mathbb{C}^7$. Hence we consider the restriction map

$$r : X(M_\phi) \to X(S)$$
$$X \to X|_S$$

and try to gain information from the image of this map. If we choose a representation $\rho \in \mathcal{R}(M_\phi)$, then ρ needs to satisfy

$$\rho(t)^{-1}\rho(p)\rho(t) = \rho(\phi(p)), \forall p \in \pi_1 S$$

Take the traces on both sides, we get

$$tr(\rho(p)) = tr(\rho(\phi_*(p)), \forall p \in \pi_1 S$$

Combined with section 3.2, we have

$$\begin{aligned} Im(r) \subset \{(tr\rho(a), tr\rho(b), tr\rho(ab)) | tr\rho(a) &= tr\rho(\phi(a)), \\ tr\rho(b) &= tr\rho(\phi(b)), \\ tr(\rho(ab)) &= tr\rho(\phi(ab))\} \end{aligned}$$

We can rewrite this set as the fixed point set $X_\phi(S)$ of a polynomial automorphism $\bar{\phi}$ induced by ϕ. From the definition of $X_\phi(S)$, it is an affine variety (possibly reducible) in $\mathbb{C}^3$.

$$\begin{aligned} Im(r) \subset \{(tr\rho(a), tr\rho(b), tr\rho(ab)) | tr\rho(a) &= tr\rho(\phi_*(a)), \\ tr\rho(b) &= tr\rho(\phi_*(b)), \\ tr(\rho(ab)) &= tr\rho(\phi_*(ab))\} \\ = \{(x, y, z) | (x, y, z) = \bar{\phi}(x, y, z)\} = X_\phi S \end{aligned}$$

We then prove that the restriction map is a 2-to-1 (possible branched) cover and its image $Imr = X_\phi(S)$. Here we use the following extension lemma which extends from characters of S in $X_\phi(S)$ to characters of M_ϕ.

Lemma 4.1. *For any irreducible character X_ρ of S such that $X_\rho \in X_\phi(S)$, there exists a matrix $T = \rho(t)$ such that $\rho(t)^{-1}\rho(p)\rho(t) = \rho(\phi(p)), \forall p \in \pi_1 S$. Moreover, T is unique up to sign.*

Proof. Let $\rho \in \mathcal{R}(S)$ be an irreducible $SL_2(\mathbb{C})$ representation with $X_\rho \in X_\phi S$. Up to conjugation, we can assume that

$$\rho(a) = \begin{pmatrix} s & 0 \\ 1 & s^{-1} \end{pmatrix} \qquad \rho(b) = \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix}$$

where $u \neq 0$. Since $X_\rho \in X_\phi S$, we assume that

$$\rho(\phi(a)) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s + s^{-1} - s_1 \end{pmatrix} \qquad \rho(\phi(b)) = \begin{pmatrix} t_1 & t_2 \\ t_3 & t + t^{-1} - t_1 \end{pmatrix}$$

If there is an $SL_2(\mathbb{C})$ representation ρ' of $\pi_1 M_\phi$ extended from ρ, then $\rho(t) = T$ need to satisfy

$$T\rho(a)T^{-1} = \rho(\phi(a))$$

and

$$T\rho(b)T^{-1} = \rho(\phi(b))$$

If we set $T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$, then by direction calculation we obtain a linear homogeneous system of $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$ in terms of s_i, t_i, s, t. We can verify that this system has rank 3 and combine with the condition $t_1t_4 - t_3t_2 = 1$. Such T exists and is unique up to sign. □

According to Lemma 4.1 and Lemma 3.3, each irreducible character $X_\rho \in X_\phi(S)$ extends to two distinct characters in $X(M_\phi)$ unless $tr(\rho(t)) = tr(\rho(ta)) = tr(\rho(tb)) = tr(\rho(tab)) = 0$. This means that the restriction map r is generically two-to-one, though it has branch points if at least one of $tr(\rho(t))$, $tr(\rho(ta))$, $tr(\rho(tb))$, $tr(\rho(tab))$ is nonzero.

5 Newton Polygon of two variables

In this section, we introduce some facts about the Newton Polygon, which are used in the proof of our main result.

Definition 5.1. *Given a polynomial $f = \sum_{i,j} a_{i,j}x^iy^j$ of two variables x and y where $a_{i,j} \neq 0$, we define the Newton Polygon of f to be the convex hull of all points (i, j) in the plane.*

According to (Khovanskii, 1978), there is a nice connection between the Newton polygon of polynomial $f(x, y)$ and the genus of the variety $V(f)$ generated by f.

Lemma 5.2. *Suppose we have an irreducible polynomial $f \in \mathbb{C}[x, y]$. The genus of the variety $V(f)$ is equal to the number of integer points in the interior of the Newton polygon of f.*

6 Topology of the fixed point set

In section 2, we define the infinite family of once-punctured torus bundles M_n with monodromies $\Phi_n = AB^{n+2}A$. The corresponding automorphisms ϕ_n have the following explicit formula.

$$\phi_n = \begin{cases} a \to a(a^{-1}b^{-1})^{n+2} \\ b \to ba^2(a^{-1}b^{-1})^{n+2} \end{cases} \tag{6.1}$$

In section 4, we connect the $SL_2(\mathbb{C})$ character variety of an once-punctured torus bundle M_ϕ with the fixed point set $X_\phi(S)$ via a branching 2-to-1 cover.

In this section, we compute the topology of the fixed point sets corresponding to this infinite family of 3-manifolds M_n.

Our main result is summarised in the following theorem.

Theorem 6.1. *When n is a positive odd integer, the fixed point set $X_{\phi_n}S$ has one component and its genus is $\lfloor \frac{n}{2} \rfloor$.*

6.1 Computation of defining equations for $X_{\phi_n}S$

In this subsection, we compute the defining equations for the subvariety $X_{\phi_n}S$. We use the following trace identities in the calculation.

Lemma 6.2 (Trace Identities)**.** *Assume A,B,C are elements in $SL_2(\mathbb{C})$, then the following identities hold*

$$trA = trA^{-1} \tag{6.2}$$

$$trBA = trAB \tag{6.3}$$

$$trBAB^{-1} = trA \tag{6.4}$$

$$trA\,trB = trAB + trAB^{-1} \tag{6.5}$$

Proof. These trace identities can be proved by direct calculation. □

If we substitute the automorphism ϕ_n into the definition of $X_{\phi_n}S$, $X_{\phi_n}S$ is the variety of the ideal $I_n \subset \mathbb{C}[x, y, z]$ generated by the following equations

$$tr\rho(a) = tr\rho(a(a^{-1}b^{-1})^{n+2}) \tag{6.6}$$

$$tr\rho(b) = tr\rho(ba^2(a^{-1}b^{-1})^{n+2}) \tag{6.7}$$

$$tr\rho(ab) = tr\rho(a(a^{-1}b^{-1})^{n+2}ba^2(a^{-1}b^{-1})^{n+2}) \tag{6.8}$$

where $x = tr\rho(a)$, $y = tr\rho(b)$, $z = tr\rho(ab)$.

The first equation can be directly simplified as

$$tr\rho(a) = tr\rho(b^{-1}(a^{-1}b^{-1})^{n+1})$$

Using trace identity (6.4), the second equation (6.7) is

$$\begin{aligned}
tr\rho(b) &= tr\rho(ba^2(a^{-1}b^{-1})^{n+2}) \\
&= tr\rho((ba)\, b^{-1}(a^{-1}b^{-1})^n\, (a^{-1}b^{-1})) \\
&= tr\rho(b^{-1}(a^{-1}b^{-1})^n)
\end{aligned}$$

Using the first two equations and the trace identity (6.5) by letting $A = \rho(a(a^{-1}b^{-1})^{n+2})$ and $B = \rho(ba^2(a^{-1}b^{-1})^{n+2})$, the third equation (6.8) is

$$\begin{aligned}
tr\rho(ab) &= tr\rho(a(a^{-1}b^{-1})^{n+2}ba^2(a^{-1}b^{-1})^{n+2}) \\
&= tr\rho(a(a^{-1}b^{-1})^{n+2})\, tr\rho(ba^2(a^{-1}b^{-1})^{n+2}) - tr\rho(a(a^{-1}b^{-1})^{n+2}(ba)^{n+2}a^{-2}b^{-1}) \\
&= tr\rho(a)\, tr\rho(b) - tr\rho(a^{-1}b^{-1}) \\
&= tr\rho(a)\, tr\rho(b) - tr\rho(ab)
\end{aligned}$$

We define a sequence of polynomials in $\mathbb{C}[x, y, z]$ to help us explore the defining equations.

Definition 6.3. *Define* $P_h(x, y, z) = tr\rho(b^{-1}(a^{-1}b^{-1})^h)$, $h \in \mathbb{Z}$.

Proposition 6.4. *According to the above calculation, the fixed point set* $X_{\phi_n}S$ *is the affine variety* $V(I_n)$ *where* I_n *is the ideal* $I_n = \langle P_{n+1} - x, P_n - y, xy - 2z\rangle$.

6.2 The recursive polynomials

We collect facts about the sequence of polynomials P_n in this section.

Using the trace identity (6.5), we obtain a recurrence relation and the initial conditions for P_n in the following lemma.

Lemma 6.5. $\{P_n\}_{n\geq 0}$ *follows the linear recurrence relation*

$$P_n(x, y, z) = zP_{n-1}(x, y, z) - P_{n-2}(x, y, z)$$

with $P_0(x, y, z) = y$ *and* $P_1(x, y, z) = yz - x$.

Proof. Using trace identity (6.5),

$$\begin{aligned}
P_n(x, y, z) &= tr\rho(b^{-1}(a^{-1}b^{-1})^n) \\
&= tr\rho(b^{-1}(a^{-1}b^{-1})^{n-1})tr\rho(a^{-1}b^{-1}) - tr\rho(b^{-1}(a^{-1}b^{-1})^{n-2}) \\
&= zP_{n-1}(x, y, z) - P_{n-2}(x, y, z)
\end{aligned}$$

The initial conditions can be computed using the same trace identity.

$$P_0 = tr\rho(b^{-1}) = y$$

$$\begin{aligned} P_1(x,y,z) &= tr\rho(b^{-1}a^{-1}b^{-1}) \\ &= tr\rho(b^{-1})tr\rho(a^{-1}b^{-1}) - tr\rho(b^{-1}ba) \\ &= tr\rho(b)tr\rho(ab) - tr\rho(a) \\ &= yz - x \end{aligned}$$

□

Observing the recurrence relation, we find that P_n can be written as a linear combination of x and y.

Lemma 6.6.

$$P_n(x,y,z) = f_n(z)y - f_{n-1}(z)x$$

where $f_n(z)$ is a polynomial depending on z only. $f_n(z)$ is defined by the same recurrence relation $f_n(z) = zf_{n-1}(z) - f_{n-2}(z)$ and initial conditions $f_0(z) = 1,\ f_1(z) = z$.

Proof. This can be easily proved by induction. □

For an arbitrary fixed z, using the characteristic polynomial $\lambda^2 - 2z\lambda + 1 = 0$ of the linear recurrence relation, we get the following explicit formula for f_n.

Lemma 6.7.

$$f_n(z) = \begin{cases} n+1,\ \textit{if } z = 2 \\ n(-1)^{n+1},\ \textit{if } z = -2 \\ \dfrac{1}{\sqrt{z^2-4}}((\dfrac{z+\sqrt{z^2-4}}{2})^{n+1} - (\dfrac{z-\sqrt{z^2-4}}{2})^{n+1}),\ \textit{if } z \neq \pm 2 \end{cases}$$

We can also express $f_n(z)$ as a summation of monomials of some powers of z using the binomial theorem.

Lemma 6.8.

$$\begin{cases} \textit{When } n = 2k+1,\ f_n(z) = \sum_{m=0}^{k} p_{m,n} z^{2k-2m+1} \ \textit{where } p_{m,n} = \dfrac{(-1)^m}{2^{2k-2m+1}} \sum_{i=m}^{k} \binom{2k+2}{2i+1}\binom{i}{m} \\ \textit{When } n = 2k,\ f_n(z) = \sum_{m=0}^{k} q_{m,n} z^{2k-2m} \ \textit{where } q_{m,n} = \dfrac{(-1)^m}{2^{2k-2m}} \sum_{i=m}^{k} \binom{2k+1}{2i+1}\binom{i}{m} \end{cases}$$

The explicit formula for $p_{m,n}$ and $q_{m,n}$ are hard to simplify but we can retrieve some facts which are useful in our later proof.

Remark 6.9. *Since the constant coefficient is the summation of a product of binomial coefficients, we can see that $p_{m,n} \neq 0$ and $q_{m,n} \neq 0$ for all pairs $\{m,n\}$ well-defined.*

6.3 One simplified basis for I_n

In section 6.1, we compute the explicit defining equations generating the subvariety $X_{\phi_n}S$. To rigorously prove Theorem 6.1, we use the techniques from computational algebraic geometry.

To simplifying the 3 defining equations in Proposition 5.4, we use the idea of Buchberger's algorithm (Cox, Little, and O'Shea, 2007) which is used to compute the Groebner basis of an ideal of a polynomial ring.

Definition 6.10. *For the ideal* $I_n = \langle P_{n+1} - x, P_n - y, xy - 2z\rangle$, *we denote* $X_n = P_{n+1} - y$, $X_{n-1} = P_n - y$, $G = xy - 2z$.

We can rewrite X_n as $X_n = P_{n+1} - P_{-1}$ and $X_{n-1} = P_n - P_0$ where P_{-1} is defined in Definition 6.3.

Then in the following lemma, we compute a sequence of polynomials recursively to summarise X_n and X_{n-1} into one single polynomial.

Lemma 6.11. *Define* $X_{i+2} = zX_{i+1} - X_i$ *for* $i > 0$. *When* $n = 2k + 1$ *is a positive odd integer, the ideal* $I_n = \langle X_n, X_{n-1}, xy - 2z\rangle = \langle P_{k+1} - P_k, xy - 2z\rangle$.

Proof. Using induction, we can prove that $X_i = P_{i+1} - P_{n-i-1}$ for $0 < i \leq n$. Since X_{i+2} is a linear combination of X_{i+1} and X_i, $\langle X_{i+2}, X_{i+1}\rangle \subseteq \langle X_{i+1}, X_i\rangle$. By definition, $\langle X_{i+1}, X_i\rangle \subseteq \langle X_{i+2}, X_{i+1}\rangle$. Thus, $\langle X_{i+2}, X_{i+1}\rangle = \langle X_{i+1}, X_i\rangle$

Hence when $n = 2k + 1$,

$$\begin{aligned}\langle X_n, X_{n-1}\rangle &= \langle X_{n-1}, X_{n-2}\rangle \\ &= ... \\ &= \langle X_k, X_{k-1}\rangle \\ &= \langle P_{k+1} - P_k, P_k - P_{k+1}\rangle \\ &= \langle P_{k+1} - P_k\rangle\end{aligned}$$

□

6.4 Proof of Theorem 6.1

By Lemma 6.11,

$$X_{\phi_n} S = V\big(\langle P_{k+1} - P_k, xy - 2z\rangle\big), \text{ when } n = 2k + 1,\ k \geq 0$$

We prove Theorem 6.1 via constructing a birational isomorphism between $X_{\phi_n} S$ and its restriction on (x, z) coordinate. According to Lemma 6.6, we can express $P_{k+1} - P_k = 0$ as

$$(f_{k+1}(z) - f_k(z))y - (f_k(z) - f_{k-1}(z))x = 0 \tag{6.9}$$

When $x \neq 0$, we substitute $y = \frac{2z}{x}$ into Equation 6.9 and multiply x on both sides,

$$2(f_{k+1}(z) - f_k(z))z - (f_k(z) - f_{k-1}(z))x^2 = 0 \tag{6.10}$$

From Lemma 6.7, Equation (6.10) is of the following form,

$$g_n(x, z) := \sum_{i=1}^{k+2} p_i z^i + \sum_{i=0}^{k} q_i x^2 z^i = 0 \tag{6.11}$$

where p_i, q_i are nonzero constant coefficients.

Remark 6.12. *When* $x = 0$, *the only point in the variety* $X_{\phi_n} S$ *is* $(0, 0, 0)$.

Define the variety U_n as the variety in $\mathbb{C}^2$ generated by the Equation (6.11). Then there are rational maps between $X_{\phi_n}S$ and U_n,

$$r_1 : X_{\phi_n}S \dashrightarrow U_n$$
$$(x, y, z) \to (x, z)$$

and

$$r_2 : U_n \dashrightarrow X_{\phi_n}S$$
$$(x, z) \to (x, \frac{2z}{x}, z)$$

The varieties U_n and $X_{\phi_n}S$ are birational equivalent so we only need to compute the genus of U_n.

To use Lemma 5.2, we only need to explore the irreducibility of $g_n(x, z) \in \mathbb{C}[x, z]$ for every positive odd integer n.

Lemma 6.13. *$g_n(x, z)$ defined in Equation (6.11) is an irreducible polynomial in $\mathbb{C}[x, z]$ for every positive odd integer n.*

Proof. We prove the lemma by contradiction. Suppose $g_n(x, z)$ is reducible, then we can factorise it into $g_n(x, z) = s_n(x, z)t_n(x, z)$ where t_n, s_n are non-constant polynomials in $\mathbb{C}[x, z]$.

Then the Newton polygon of $g_n(x, z)$ is the Minkowski sum of the Newton polygons of $s_n(x, z)$ and $t_n(x, z)$. The Newton polygon of $g_n(x, z)$ is shown in Figure 2. If we travel through the boundary of the Newton polygon and consider the vectors from each lattice point to the next lattice point, we obtain the boundary vector sequence

$$v(g_n) = \{(1, -2), (1, 0), ..., (1, 0), (-1, 1), (-1, 1), (-1, 0), ..., (-1, 0)\}$$

where the numbers of $(1, 0)$ and $(-1, 0)$ in the sequence are $k + 1$ and k respectively. Since the Newton polygon of $g_n(x, z)$ is the Minkowski sum of the Newton polygons of $s_n(x, z)$ and $t_n(x, z)$, $v(g_n)$ can be partitioned into two disjoint nonempty subsequences $v(s_n)$ and $v(t_n)$, each of which sums to zero.

Given $(1, -2)$, $(-1, 1)$ and $(-1, 1)$ are the only three vectors with nonzero second components, they must be in the same sequence. Without loss of generality, we assume that $(1, -2)$ and $(-2, 2)$ are in $v(s_n)$. Then $v(s_n)$ is of the form $\{(1, -2), (1, 0), ..., (1, 0), (-1, 1), (-1, 1), (-1, 0), ..., (-1, 0)\}$, where there are $m+1$ of $(1, 0)$ and m of $(-1, 0)$ in the sequence and $0 \le m \le k - 1$, $m \in \mathbb{Z}$. $v(t_n)$ is of the form $\{(1, 0), ..., (1, 0), (-1, 0), ..., (-1, 0)\}$, where there are $k - m$ of $(1, 0)$ and $k - m$ of $(-1, 0)$ in the sequence.

According to the boundary vector sequences of $s_n(x, z)$ and $t_n(x, z)$, we can write $s_n(x, z) = x^2 s_n^{(1)}(z) + s_n^{(2)}(z)$ and $t_n(x, z) = t_n(z)$ where $s_n^{(i)}(z)$ and $t_n(z)$ are polynomial in $\mathbb{C}[z]$.

From Equation (6.10), $g_n(x, z) = 2(f_{k+1}(z) - f_k(z))z - (f_k(z) - f_{k-1}(z))x^2$. Hence

$$\begin{cases} 2(f_{k+1}(z) - f_k(z))z = s_n^{(2)}(z)t_n(z) \\ -(f_k(z) - f_{k-1}(z)) = s_n^{(1)}(z)t_n(z) \end{cases}$$

Since $t_n(z)$ is a non-constant polynomial in $\mathbb{C}[z]$, by the fundamental theorem of algebra, $t_n(z) = 0$ has a root $\alpha \in \mathbb{C}$. α is also the common roots of $2(f_{k+1}(z) - f_k(z))z = 0$ and $-(f_k(z) - f_{k-1}(z)) = 0$.

From Lemma 6.8, both $f_{k+1}(z) - f_k(z)$ and $f_k(z) - f_{k-1}(z)$ have a nonzero constant term, which means that $\alpha \neq 0$. Then we have

$$f_{k+1}(\alpha) - f_k(\alpha) = 0 \tag{6.12}$$

and

$$f_k(\alpha) - f_{k-1}(\alpha) = 0 \tag{6.13}$$

According to Lemma 6.7, if $\alpha = \pm 2$, we can easily check that the Equation (6.12) and (6.13) are not satisfied. If $\alpha \neq \pm 2$, then we can write Equation (6.12) as

$$\frac{1}{\sqrt{\alpha^2-4}}((\frac{\alpha+\sqrt{\alpha^2-4}}{2})^{k+2} - (\frac{\alpha-\sqrt{\alpha^2-4}}{2})^{k+2}) - \frac{1}{\sqrt{\alpha^2-4}}((\frac{\alpha+\sqrt{\alpha^2-4}}{2})^{k+1} - (\frac{\alpha-\sqrt{\alpha^2-4}}{2})^{k+1}) = 0$$

After direct calculation, we have

$$(\frac{\alpha+\sqrt{\alpha^2-4}}{2})^{k+1}(\frac{\alpha+\sqrt{\alpha^2-4}}{2} - 1) = (\frac{\alpha-\sqrt{\alpha^2-4}}{2})^{k+1}(\frac{\alpha-\sqrt{\alpha^2-4}}{2} - 1)$$

Similarly, from Equation (6.13), we have

$$(\frac{\alpha+\sqrt{\alpha^2-4}}{2})^{k}(\frac{\alpha+\sqrt{\alpha^2-4}}{2} - 1) = (\frac{\alpha-\sqrt{\alpha^2-4}}{2})^{k}(\frac{\alpha-\sqrt{\alpha^2-4}}{2} - 1)$$

Dividing the first equation by the second one, we have $\frac{\alpha+\sqrt{\alpha^2-4}}{2} = \frac{\alpha-\sqrt{\alpha^2-4}}{2}$, which contradicts that $\alpha \neq \pm 2$. □

Lemma 6.14. *The genus of U_n is k, where $n = 2k + 1$.*

Proof. The Newton polygon of $g_n(x, z)$ defined in Equation (6.11) is shown in Figure 2. There are k interior points in the Newton polygon, namely, (1,1),(2,1) until (k,1). By lemma 5.2, the genus of U_n is k. □

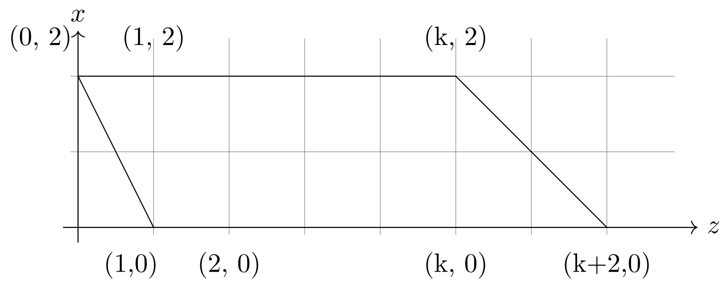

Figure 2: Newton polygon of Equation 6.11

7 $PSL_2(\mathbb{C})$ character varieties

In this section, we aim to prove the following theorem.

Theorem 7.1. *For every positive odd integer n, the $PSL_2(\mathbb{C})$-character variety $\overline{X}(M_n)$ of M_n is birational equivalent to $\overline{X}_{\phi_n}S$ induced by the fixed point set $X_{\phi_n}S$. Its genus is zero for all positive odd n.*

7.1 $PSL_2(\mathbb{C})$ character varieties for M_n

We consider the $PSL_2(\mathbb{C})$ character variety of the infinite family of once-punctured torus bundles M_n. In this section, we denote $\overline{X}(\pi_1 M_n)$ as $\overline{X}(M_n)$ and denote $\overline{X}(\pi_1 S)$ as $\overline{X}(S)$.

Given a character $X_\rho \in X(M_n)$, an element $\alpha \in Hom(\pi_1 M_n, \{\pm 1\})$ acts on X_ρ by $\alpha_{X_\rho}(p) = \alpha(p)X_\rho(p)$ for any $p \in \pi_1 M_n$. This is an action of $Hom(\pi_1 M_n, \{\pm 1\})$ on $X(M_n)$ and the quotient ${X(M_n)}/{Hom(\pi_1 M_n, \{\pm 1\})}$ is identified with a subset of $\overline{X}(M_n)$

Lemma 7.2. *Take a homomorphism $\alpha \in Hom(\pi_1 M_n, \{\pm 1\})$. When n is odd, then either $(\alpha(a), \alpha(b)) = (1, 1)$ or $(\alpha(a), \alpha(b)) = (-1, -1)$.*

Proof. Recall the fundamental group of M_n is

$$\pi_1 M_\phi = \langle t, a, b | t^{-1}at = a(a^{-1}b^{-1})^{n+2},\ t^{-1}bt = ba^2(a^{-1}b^{-1})^{n+2} \rangle$$

From the condition of the presentation, $\alpha(a)^{n+2}\alpha(b)^{n+2} = 1$. Since n is odd, $\alpha(a)\alpha(b) = 1$. □

Hence a representation $\bar{\rho} \in \overline{\mathcal{R}}(M_n)$ has either 0 lift or 4 lifts to $SL_2(\mathbb{C})$ representations. Let $\overline{X_0}(M_n)$ be the subvariety of $\overline{X}(M_n)$ consisting of characters of all $PSL_2(\mathbb{C})$ representations lift to representations of $SL_2(\mathbb{C})$ and its image is a subvariety $X_0(M_n)$ of $X(M_n)$. The natural map $X(M_n) \to \overline{X_0}(M_n)$ is a 4-to-1, branched cover.

7.2 Fixed point set for $PSL_2(\mathbb{C})$

As stated in section 7.1, the $PSL_2(\mathbb{C})$character variety $\overline{X}(M_n) \supseteq {X(M_n)}/{Hom(\pi_1 M_n, \{\pm 1\})}$ is determined by the involution $(\rho(a), \rho(b), \pm\rho(t)) \to (-\rho(a), -\rho(b), \pm\rho(t))$ for odd n.

If we also consider the restriction map in section 4

$$r : X(M_\phi) \to X_{\phi_n} S$$

We can construct a fixed point set for $PSL_2(\mathbb{C})$ by identifying the involution $(x, y, z) \to (-x, -y, z)$ in $X_{\phi_n} S$. In section 6, we construct a birational isomorphism between $X_{\phi_n} S$ with its restriction U_n. We can construct the corresponding $PSL_2(\mathbb{C})$ restriction $\overline{U_n}$ under identifying the involution $(x, z) \to (-x, z)$.

Recall the defining equation (Equation 6.11) for U_n is

$$\sum_{i=1}^{k+2} p_i z^i + \sum_{i=0}^{k} q_i x^2 z^i = 0$$

The corresponding defining equation for $\overline{U_n}$ can be constructed by letting $(X, Z) = (x^2, z)$. Then

$$\overline{U_n} = \{(X, Z) | \sum_{i=1}^{k+2} p_i Z^i + \sum_{i=0}^{k} q_i X Z^i = 0\}$$

Using Lemma 5.2, we can prove the genus of $\overline{U_n}$ is 0. Hence the genus of $\overline{X}_{\phi_n} S$ is also 0.

7.3 Proof of theorem 7.1

We can summarise the result from section 7.1, section 7.2 and section 4 in the Figure 3.

We can see that there is a 1-to-1 (possible branched) cover between $\overline{X}_{\phi_n}(S)$ and $\overline{X}(M_n)$. Hence $\overline{X}(M_n)$ is birational to $\overline{X}_{\phi_n}(S)$, which means that its genus is 0.

$$\begin{array}{ccc} X(M_n) & \xrightarrow{2:1} & X_{\phi_n}(S) \\ \downarrow{\scriptstyle 4:1} & & \downarrow{\scriptstyle 2:1} \\ \overline{X}(M_n) & \xrightarrow{1:1} & \overline{X}_{\phi_n}(S) \end{array}$$

Figure 3: The Commutative Diagram bewteen $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$

8 Further directions

Since we have a generically 2:1 map

$$r : X(M_n) \to X_{\phi_n}(S)$$

we may recover the topology of $X(\pi_1 M_{\phi_n})$ from the topology of $X_{\phi_n}(S)$. One possible way is to count the number of branching points in the map. Recall in Lemma 4.1, we extend a matrix $T = \rho(t)$ for irreducible character X_ρ. A character X_ρ is a branching point if and only if $tr\rho(t) = 0, tr\rho(ta) = 0, tr\rho(tb) = 0$ and $tr\rho(tab) = 0$ hold simultaneously.

Another direction is to find examples for once-punctured torus bundles such that the genera of their $PSL_2(\mathbb{C})$ character varieties are nonzero. Both the previous result from Baker and Petersen and our example have a genus zero $PSL_2(\mathbb{C})$ character variety.

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