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Entanglement Harvesting In Flat Spacetime

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Contents

1 Abstract

Entanglement harvesting is the process wherein two faraway particle detectors can become entangled simply by interacting with the quantum vacuum in their separate spatial locations. In this work we investigate how the entanglement harvesting process would appear to an observer moving at a constant velocity with respect to the detectors. Our analysis shows that the speed of the observer does not affect whether or not an entangled state is assigned to the detectors.

2 Introduction

The behaviour of quantum states under the influence of gravity is not well understood. Relativistic quantum information (RQI) is a fairly new field of physics that uses information theory as a framework for investigating phenomena that involve both quantum mechanics (QM) and general relativity (GR). For example, it has been shown that spatially separated particle detectors can become entangled through local interactions with the quantum vacuum - a phenomenon known as entanglement harvesting (Kerstjens and Martinez 2018). The degree to which the detectors become entangled is dependent on the properties of the spacetime they exist in, such as its curvature and expansion, hence entanglement harvesting provides a way of exploring the overlap of QM and GR (Ver Steeg and Menicucci 2009).

Given the connection that entanglement harvesting provides between these two fields, one may wonder what influence the effects of special relativity have on the process. Specifically, in this work we calculated the amount of entanglement harvested by two point-like detectors as a function of their separation, interaction time, and energy as would be observed in an inertial reference frame moving with respect to the detectors. What we found is that the degree to which the detectors are entangled does not depend on the motion of the observer.

In achieving this result we explore the mathematical techniques that are used in current work on entanglement harvesting, to which this report may serve as an introduction for an interested reader. We also calculate the effect of using detectors that do not interact with the field in synchronicity.

3 Statement of Authorship

This report was wholly written by Angus Walsh under the supervision of A/Prof Nicolas Menicucci. The background section on quantum mechanics covers standard knowledge in the field and in particular we have drawn upon (Nielsen and Chuang 2010; Sakurai and Napolitano 2011) for reference. That entanglement could be extracted from the quantum vacuum was first shown in (Reznik 2003). It was then shown that this process could be used as a probe of spacetime curvature and expansion in (Ver Steeg and Menicucci 2009). The contour integration methods used in this project are based on those used in (Nambu 2013). To our best knowledge this is the first time that the calculation of entanglement harvesting from a moving reference frame and the consideration of asynchronous switching leading to the Meijer-G function in equation (85) have been presented.

4 Background: Quantum Mechanics

This section of the report is concerned with establishing the conceptual and mathematical prerequisites for understanding the work done in this project. It should hopefully also serve as a brief introduction to quantum mechanics for a reader who is unfamiliar with the field.

4.1 Representing Quantum States

In quantum mechanics a *pure state* is a state of maximal information about a physical system and is denoted by a *ket*. Kets are written as $|\psi\rangle$ where ψ is a label that identifies the particular state. A pure state is a unit vector in *Hilbert space* - a complex vector space with inner product. The Hilbert space of a system has a number of dimensions equal to the number of possible outcomes in a particular measurement of the system. At this point it is worthwhile introducing a simple but relevant example. A classical bit is one of two possible numbers $\{0, 1\}$, in contrast a quantum bit or 'qubit' is a complex vector of dimension two and can be written as a column vector

$$|\psi\rangle \doteq \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{1}$$

where α and β are complex numbers and the dot over the equals sign indicates that this representation is in a particular basis. The real numbers $|\alpha|^2$ and $|\beta|^2$ are the probabilities that when measured in this basis the system will be found in the corresponding basis state. Because the measurement must have an outcome we have $|\alpha|^2 + |\beta|^2 = 1$. Every ket has an associated *bra* $\langle\psi|$ that is the Hermitian conjugate of the ket

$$\langle\psi| = |\psi\rangle^\dagger \doteq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^\dagger = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}. \tag{2}$$

The inner product of two states $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by

$$\langle\psi_1|\psi_2\rangle \doteq \begin{bmatrix} \alpha_1^* & \beta_1^* \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \alpha_1^*\alpha_2 + \beta_1^*\beta_2, \tag{3}$$

and the outer product is

$$|\psi_2\rangle\langle\psi_1| \doteq \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \alpha_1^* & \beta_1^* \end{bmatrix} = \begin{bmatrix} \alpha_1^*\alpha_2 & \beta_1^*\alpha_2 \\ \alpha_1^*\beta_2 & \beta_1^*\beta_2 \end{bmatrix}. \tag{4}$$

Note that in general these operations are not commutative. A more general description of a quantum state is given by the *density matrix*

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{5}$$

where p_i are the probabilities of the system being in each pure state $|\psi_i\rangle$, the hat is used to indicate that this is an operator on the Hilbert space of the system. The density matrix is more general because it allows for the representation of *mixed states* - states that contain less than maximal information. An example of a mixed state is

$$\hat{\rho} \doteq \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \neq |\psi\rangle \langle\psi| \,. \tag{6}$$

As indicated by the inequality this density matrix cannot correspond to a pure state.[1] The significance of a mixed state is that it can only be used to make probabilistic predictions about measurements, whereas a pure state predicts with certainty the outcome of a single measurement.

4.2 Entanglement and Negativity

What happens if we have multiple quantum systems that we want to describe? In the classical case there are four possible arrangements of two bits, so in the quantum case our state vector should be four dimensional. This relation is satisfied if the composite system is formed by taking the tensor product of the two sub-systems. If we have two qubits in pure states then their combined state is

$$|\psi_1\rangle \otimes |\psi_2\rangle \doteq \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}. \tag{7}$$

Composite quantum systems have a peculiar feature. Consider the following pure state of a two qubit system:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \neq |\psi_1\rangle \otimes |\psi_2\rangle \tag{8}$$

This state cannot be separated into pure states of each subsystem,[2] so we say that this is an *entangled state.* Entangled pure states contain maximal information not about the individual subsystems but about the relationship between them. In the density matrix representation there is a general procedure for separating composite systems using the partial trace

$$\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho}_A \otimes \hat{\rho}_B). \tag{9}$$

Where

$$\mathrm{Tr}_B(|A\rangle\langle A| \otimes |B\rangle\langle B|) = |A\rangle\langle A| \,\mathrm{Tr}(|B\rangle\langle B|) = |A\rangle\langle A| \langle B|B\rangle \,. \tag{10}$$

However if the composite system is entangled then the density matrices of the subsystems will be mixed states. It is not always obvious by inspection that a particular state is entangled, but we can quantify the entanglement of

[1] For a pure state the off-diagonal terms imply that $\alpha\beta^* = 0$, but this requires that either α or β is zero - a contradiction because the on-diagonal terms are both non-zero.

[2] Similar reasoning to the mixed state example shows that this state doesn't fit the form of equation (7).

a state is by calculating the *negativity*. The negativity of a density matrix is defined as the sum of the negative eigenvalues of the partial transpose of the matrix. For the two qubit system that we will be considering, a non-zero negativity is a sufficient condition for the qubits to be entangled.

4.3 Time Evolution and Pictures of Quantum Mechanics

In the Schrodinger picture of quantum mechanics state vectors evolve with time while the operators corresponding to observable properties are constant. Given an initial state $|\psi_S(t)\rangle$ the time dependent state is found by applying the unitary time evolution operator $\hat{U}(t,t_0)$

$$|\psi_S(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle = e^{-i(t-t_0)\hat{H}}\,|\psi(t_0)\rangle\,. \tag{11}$$

In the above equation $\hat{H}$ is the time-independent Hamiltonian operator that defines the energy levels of the system, the 'S' subscript indicates that we are in the Schrodinger picture, and we have assumed units in which the reduced Planck's constant is equal to one. These dynamics obey the Schrodinger equation

$$i\frac{\partial}{\partial t}\,|\psi_S(t)\rangle = \hat{H}_S\,|\psi_S(t)\rangle\,. \tag{12}$$

In this work we are interested in the effect of a time-dependent interaction Hamiltonian of the form

$$\hat{H}_S(t) = \hat{H}_0 + \hat{V}_S(t). \tag{13}$$

In order to handle this time-dependent observable we will work in what is known as the interaction picture. In this picture both observables and states evolve with time. We define the interaction picture state to be

$$|\psi_I(t)\rangle = e^{i\hat{H}_0 t}\,|\psi_S(t)\rangle\,. \tag{14}$$

This obeys an equation of the same form as (12)

$$i\frac{\partial}{\partial t}\,|\psi_I(t)\rangle = \hat{V}_I(t)\,|\psi_I(t)\rangle\,, \tag{15}$$

where

$$\hat{V}_I(t) = e^{i\hat{H}_0 t}\hat{V}_S(t)e^{-i\hat{H}_0 t}. \tag{16}$$

In the interaction picture the time evolution operator is found using the Dyson series

$$\hat{U}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\hat{V}_I(t_1) - \frac{1}{2}\int_{\tau_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 T[\hat{V}_I(t_2),\hat{V}_I(t_1)] + \cdots = \sum_{j=0}^{\infty}\hat{U}^{(j)}(t,t_0), \tag{17}$$

where $T[,]$ is the time-ordering operator that enforces $t_1 > t_2$.

5 Entanglement Harvesting in Flat Spacetime

5.1 Unruh-deWitt Detectors

We use a highly simplistic model of a particle detector known as an Unruh-deWitt detector that consists of a qubit interacting with a scalar field (Kerstjens and Martinez 2018). We consider two of these qubits that both have the Schrodinger picture Hamiltonian

$$\hat{H}_{S,qubit} = \frac{\Omega}{2}(|e\rangle\langle e| - |g\rangle\langle g|) = \frac{\Omega}{2}\hat{\sigma}_z, \tag{18}$$

where $\hat{\sigma}_Z$ is the Pauli-Z operator and Ω is the difference in energy between the excited state $|e\rangle$ and the ground state $|g\rangle$. These qubits are separated by a distance L and are stationary with respect to each other. The interaction between the qubits and the field is modelled with the interaction picture Hamiltonian

$$\hat{V}_I(\tau) = \sum_{k\in\{A,B\}} \eta(\tau)_k \hat{\phi}_k(\mathbf{x}_k(\tau))(e^{i\Omega\tau}\hat{\sigma}_k^+ + e^{-i\Omega\tau}\hat{\sigma}_k^-), \tag{19}$$

where τ is the proper time as measured in the reference frame stationary with respect to the detectors, $\hat{\phi}(\mathbf{x}(\tau))$ is the field operator as a function of the detector's trajectory $\mathbf{x}(\tau)$, and $\hat{\sigma}^+ = |e\rangle\langle g|$ and $\hat{\sigma}^- = |g\rangle\langle e|$ are the raising and lowering operators for the qubit. $\eta(\tau)$ is called the *switching function* and determines the strength of the interaction as a function of time. The switching functions for the two detectors are both assumed to be Gaussian with variance σ so that the detectors can be considered 'on' for $\sigma > |\tau|$, and 'off' otherwise. However we do not require that the detectors turn on synchronously, so we allow for a delay, τ_0, in the switching function of one of the detectors giving

$$\eta_A(\tau) = \eta_0 e^{-\tau^2/2\sigma^2}, \quad \eta_B(\tau) = \eta_0 e^{-(\tau-\tau_0)^2/2\sigma^2}, \tag{20}$$

where η_0 is called the coupling constant.

5.2 Perturbative Solution

In the initial state of the system both of the qubits are in their ground states and the field is in its vacuum, or minimum energy state, denoted $|0\rangle$. The density matrix prior to the interaction is therefore

$$\hat{\rho}_0 = |g_A\rangle\langle g_A| \otimes |g_B\rangle\langle g_B| \otimes |0\rangle\langle 0|. \tag{21}$$

After the interaction the system will have density matrix

$$\hat{\rho} = \hat{U}(t,t_0)\hat{\rho}_0\hat{U}(t,t_0)^\dagger, \tag{22}$$

where the time evolution operator is given by the Dyson series (17). Because this is an infinite power series the problem appears intractable, however if the coupling constant is very small, $\eta_0 << 1$, then we can approximate the time evolution by only considering the first few terms in the expansion and obtain a perturbative solution. We will consider up to second order in η_0 and take the limits $t_0 \to -\infty$ and $t \to \infty$ so that

$$\hat{U} = 1 - i\int_{-\infty}^{\infty} d\tau_1 \hat{V}_I(\tau_1) - \frac{1}{2}\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 T[\hat{V}_I(\tau_2), \hat{V}_I(\tau_1)] + \mathcal{O}(\eta_0^3), \tag{23}$$

where we have written the time evolution in terms of the proper time of the detector. Using this approximation we can then calculate the density matrix of the system after the interaction. (See the appendix for details.)

Using the partial trace to remove the state of the field, the state of the detectors after the interaction is

$$\hat{\rho}_{det} \doteq \begin{pmatrix} 1-2A & 0 & 0 & X^* \\ 0 & A & E_{AB} & 0 \\ 0 & E_{BA} & A & 0 \\ X & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\eta_0^3), \tag{24}$$

where we have used basis vectors $\{|g_A\rangle |g_B\rangle, |e_A\rangle |g_B\rangle, |g_A\rangle |e_B\rangle, |e_A\rangle |e_B\rangle\}$. Up to second order in η_0 the only eigenvalue of the partial transpose of this matrix that can be negative is $A - |X|$ so we define the negativity of the state to be

$$\mathcal{N} = |X| - A, \tag{25}$$

where

$$A = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \eta_A(\tau_1)\eta_A(\tau_2) e^{-i\Omega(\tau_1-\tau_2)} \langle 0| \hat{\phi}_A(\mathbf{x}_A(\tau_1))\hat{\phi}_A(\mathbf{x}_A(\tau_2)) |0\rangle, \tag{26}$$

and

$$X = -2\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \eta_A(\tau_1)\eta_B(\tau_2) e^{i\Omega(\tau_1+\tau_2)} \langle 0| \hat{\phi}_A(\mathbf{x}_A(\tau_1))\hat{\phi}_B(\mathbf{x}_B(\tau_2)) |0\rangle. \tag{27}$$

Our goal now becomes the evaluation of these integrals. To do so we introduce the Wightman function

$$D^+(t_A(\tau_1), \mathbf{x}_A(\tau_1); t_B(\tau_2), \mathbf{x}_B(\tau_2)) = \langle 0| \hat{\phi}_A(\mathbf{x}_A(\tau_1))\hat{\phi}_B(\mathbf{x}_B(\tau_2)) |0\rangle. \tag{28}$$

For Minkowski spacetime the Wightman function is

$$D^+(t_A(\tau_1), \mathbf{x}_A(\tau_1); t_B(\tau_2), \mathbf{x}_B(\tau_2)) = \frac{-1}{4\pi^2[(t_A(\tau_1) - t_B(\tau_2) - i\epsilon)^2 - |\mathbf{x}_A(\tau_1) - \mathbf{x}_B(\tau_2)|^2]}, \tag{29}$$

where the variables t_A, t_B, $\mathbf{x}_A$ and $\mathbf{x}_B$ are functions of the detectors' proper time that depend on which reference frame is assumed in the calculation (Peskin and Schroeder 1995). Of particular importance in this equation is the ϵ term. This is a positive constant that is arbitrarily small but non-zero. It has the effect of removing the pole of the Wightman function away from the real axis and in doing so makes the above integrals computable. After solving the integrals we take the limit $\epsilon \to 0^+$.

5.3 Special Relativity

Lets begin this section by reminding ourselves of the hypothetical setup and how it was analysed in prior work. The two detectors are separated by a distance L and are stationary with respect to each other. In a reference frame co-moving with the detectors the spacetime diagram showing the detector trajectories is shown in Figure 1. In the co-moving frame, indicated with primed variables, the detector trajectories are simply

$$t'_A = \tau, \qquad x'_A = 0, \tag{30}$$

$$t'_B = \tau, \qquad x'_B = L. \tag{31}$$

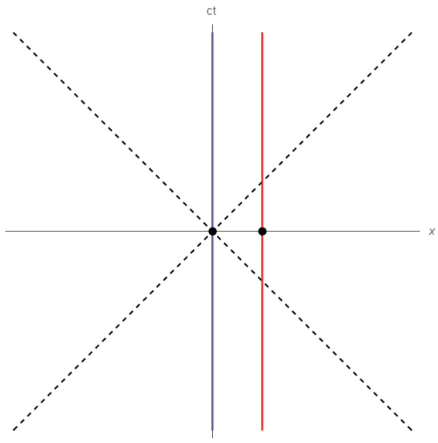

Figure 1: Spacetime diagram in a reference frame co-moving with the detectors. The vertical axis is time scaled by the speed of light, and the horizontal axis is distance. Dashed black lines form the light cones. The detector trajectories are shown in blue and red and are spaced a distance L apart. The black dots indicate the point in time around which the switching function is centered.

Using these variables the Wightman function is

$$D^{+}(t'_A(\tau_1), \mathbf{x}'_A(\tau_1); t'_B(\tau_2), \mathbf{x}'_B(\tau_2)) = \frac{-1}{4\pi^2[(\tau_1 - \tau_2 - i\epsilon)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2]}, \tag{32}$$

and this is the form used in prior work on entanglement harvesting (Ver Steeg and Menicucci 2009). However in this work we want to describe the process from a reference frame moving at an arbitrary (sub-luminal) velocity v in the positive x direction, that we will refer to as the lab frame. To reformulate the problem in the lab frame we need to apply the *Lorentz transformations*. These are the appropriate coordinate transformations according to special relativity, and using them to represent the detector frame in terms of lab frame variables we have

$$ct' = \gamma(ct - \beta x), \tag{33}$$

$$x' = \gamma(x - \beta ct), \tag{34}$$

where c is the speed of light, $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. The inverse transformation is then

$$ct = \gamma(ct' + \beta x'), \tag{35}$$

$$x = \gamma(x' + \beta ct'). \tag{36}$$

To simplify this we assume units in which $c = 1$ giving

$$t = \gamma(t' + \beta x'), \tag{37}$$

$$x = \gamma(x' + \beta t'). \tag{38}$$

Substituting the detector frame trajectories into these equations we obtain

$$t_A(\tau) = \gamma\tau, \qquad x_A(\tau) = \beta\gamma\tau, \tag{39}$$

$$t_B(\tau) = \gamma(\tau + \beta L), \qquad x_B(\tau) = \gamma(L + \beta\tau). \tag{40}$$

Note that at $\tau = 0$, $t_A = 0$ but $t_B \neq 0$. This means that even if the detectors were switched on simultaneously in the detector frame, this will not be the case in the lab frame. This is because Lorentz transformations do not preserve simultaneity. This can be seen in the lab frame spacetime diagram in Figure 2.

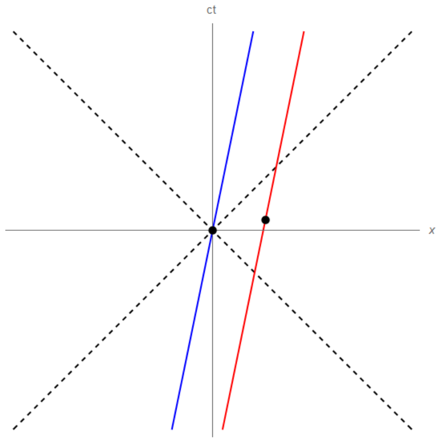

Figure 2: Spacetime diagram in a reference frame moving at constant velocity with respect to the detectors. The vertical axis is time scaled by the speed of light, and the horizontal axis is distance. Dashed black lines form the light cones. The detector trajectories are shown in blue and red. The black dots indicate the point in time around which the switching function is centered.

The Wightman function in the lab frame is therefore

$$D^+(t_A(\tau_1), \mathbf{x}_A(\tau_1); t_B(\tau_2), \mathbf{x}_B(\tau_2)) = \frac{-1}{4\pi^2[(\gamma\tau_1 - \gamma\tau_2 - i\epsilon)^2 - |\beta\gamma\tau_1 - \beta\gamma\tau_2|^2]}. \tag{41}$$

5.4 Contour Integration

In the calculation of the integrals (26) and (27) we will use the variable substitutions

$$x = (\tau_1 + \tau_2)/2\sigma, \qquad y = (\tau_1 - \tau_2)/2\sigma. \tag{42}$$

We will also make use of the Fourier transform identity

$$\mathcal{F}\{e^{-x^2}\} = \int_{-\infty}^{\infty} dx e^{-x^2} e^{-i\omega x} = \sqrt{\pi} e^{-\omega^2/4}. \tag{43}$$

An important note: because ϵ is arbitrarily small and we will take the limit $\epsilon \to 0^+$ after integration we will use terms such as ϵ and ϵ/σ interchangeably. Starting with the A integral we have

$$A = -\eta_0^2 \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 e^{-(\tau_1^2+\tau_2^2)/2\sigma^2} \frac{e^{-i\Omega(\tau_1-\tau_2)}}{4\pi^2[(\gamma\tau_1 - \gamma\tau_2 - i\epsilon)^2 - |\beta\gamma\tau_1 - \beta\gamma\tau_2|^2]}, \tag{44}$$

we then apply the change of variables (42) to obtain

$$\begin{aligned} A &= -\frac{\eta_0^2}{8\pi^2\gamma^2} \int_{-\infty}^{\infty} dx e^{-x^2} \int_{-\infty}^{\infty} dy e^{-y^2} \frac{e^{-i\Omega\sigma 2y}}{(y - i\epsilon)^2 - \beta^2 y^2} \\ &= -\frac{\eta_0^2\sqrt{\pi}}{8\pi^2\gamma^2} \int_{-\infty}^{\infty} dy e^{-y^2} \frac{e^{-i\Omega\sigma 2y}}{(y - i\epsilon)^2 - \beta^2 y^2}, \end{aligned} \tag{45}$$

where we have immediately evaluated the x integral. We then recognise $e^{-y^2}e^{-i\Omega\sigma 2y}$ as the Fourier transform of a time-shifted Gaussian

$$e^{-y^2}e^{-i\Omega\sigma 2y} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dk e^{-(k+\sigma\Omega)^2}e^{2iky}, \tag{46}$$

giving us

$$A = -\frac{\eta_0^2}{8\pi^2\gamma^2}\int_{-\infty}^{\infty} dk e^{-(k+\sigma\Omega)^2}\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2}. \tag{47}$$

The y-integrand has poles at

$$y = \frac{i\epsilon}{1\pm\beta}. \tag{48}$$

Because $1 > \beta \geq 0$ both of these poles are on the positive half of the imaginary axis, they are also both simple poles. To make the process of finding residues easier we use the partial fraction decomposition

$$\frac{1}{(y-i\epsilon)^2-\beta^2y^2} = \frac{1}{2y\beta[(1-\beta)y-i\epsilon]} - \frac{1}{2y\beta[(1+\beta)y-i\epsilon]}, \tag{49}$$

we then split the k integral into positive and negative regions

$$\begin{aligned} A &= -\frac{\eta_0^2}{8\pi^2\gamma^2}\Big[\int_0^{\infty} dk e^{-(k+\sigma\Omega)^2}\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2} + \int_{-\infty}^{0} dk e^{-(k+\sigma\Omega)^2}\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2}\Big] \\ &= -\frac{\eta_0^2}{8\pi^2\gamma^2}\Big[\int_0^{\infty} dk e^{-(k+\sigma\Omega)^2}\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2} + \int_{0}^{\infty} dk e^{-(k-\sigma\Omega)^2}\int_{-\infty}^{\infty} dy \frac{e^{-2iky}}{(y-i\epsilon)^2-\beta^2y^2}\Big]. \end{aligned} \tag{50}$$

Breaking this down into workable pieces we first look at the integral

$$\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2}. \tag{51}$$

We can evaluate this integral by considering a contour consisting of two curves, the first being a straight line along the real line from $-R$ to R, and the second being a semi-circular arc from R to $-R$ in the upper half of the complex plane so that the contour encloses the two poles as shown in Figure (3). In the limit $R \to \infty$ the straight segment is equivalent to the above integral while by Jordan's lemma the arc segment will go to zero because of the exponential in the integrand. Then applying the residue theorem

$$\int_{-\infty}^{\infty} dy \frac{e^{2iky}}{(y-i\epsilon)^2-\beta^2y^2} = 2\pi i \sum_k \mathrm{Res}_k, \tag{52}$$

where the residues are

$$\begin{aligned} \mathrm{Res}_1 &= \lim_{y\to\frac{i\epsilon}{1+\beta}} (y - \frac{i\epsilon}{1+\beta})[\frac{e^{2iky}}{2y\beta[(1-\beta)y-i\epsilon]} - \frac{e^{2iky}}{2y\beta[(1+\beta)y-i\epsilon]}] = \frac{i}{2\beta\epsilon}e^{-2k\epsilon/(1+\beta)}, \\ \mathrm{Res}_2 &= \lim_{y\to\frac{i\epsilon}{1-\beta}} (y - \frac{i\epsilon}{1-\beta})[\frac{e^{2iky}}{2y\beta[(1-\beta)y-i\epsilon]} - \frac{e^{2iky}}{2y\beta[(1+\beta)y-i\epsilon]}] = \frac{-i}{2\beta\epsilon}e^{2k\epsilon/(1-\beta)}. \end{aligned} \tag{53}$$

We then apply the same process to the integral

$$\int_{-\infty}^{\infty} dy \frac{e^{-2iky}}{(y-i\epsilon)^2-\beta^2y^2}, \tag{54}$$

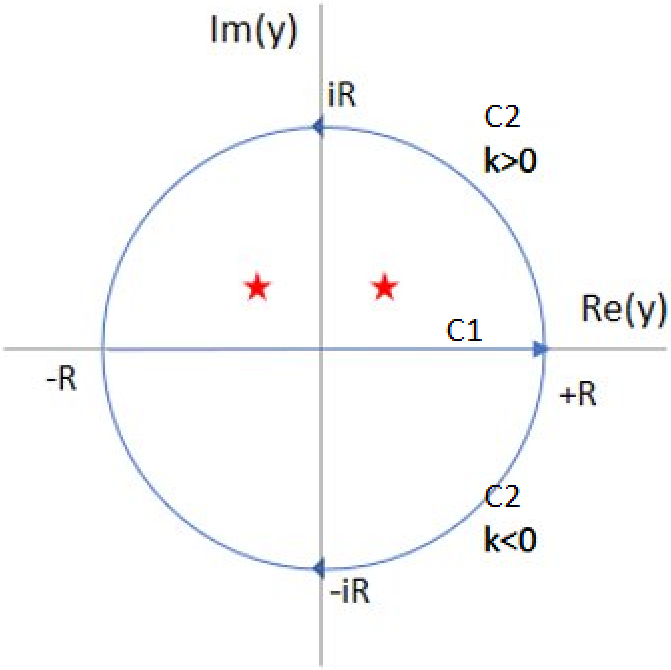

Figure 3: Contours used in the calculation of the local term A. The upper path is taken when $k > 0$ and the lower when $k < 0$. Poles are indicated by the red stars.

however because in this case the argument of the exponential is negative, for the function to be bounded the contour must have a semi-circular arc in the lower half of the complex plane. This means that the contour will enclose no poles, so the above integral is zero. Putting this together we now have

$$A = -\frac{\eta_0^2}{8\pi^2\gamma^2}\Big[\int_0^\infty dke^{-(k+\sigma\Omega)^2}2\pi i[\frac{i}{2\beta\epsilon}e^{-2k\epsilon/(1+\beta)} - \frac{i}{2\beta\epsilon}e^{2k\epsilon/(1-\beta)}]\Big]. \tag{55}$$

Taking the limit $\epsilon \to 0^+$ and evaluating the integral

$$A = \frac{\eta_0^2}{8\pi^2\gamma^2}[\frac{2\pi(e^{-\sigma^2\Omega^2} - \sqrt{\pi}\sigma\Omega\text{Erfc}[\sigma\Omega])}{1-\beta^2}], \tag{56}$$

where Erfc is the complementary error function defined as

$$\text{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt, \tag{57}$$

but by definition

$$\gamma^2(1-\beta^2) = 1, \tag{58}$$

so

$$A = \frac{\eta_0^2}{4\pi}(e^{-\sigma^2\Omega^2} - \sqrt{\pi}\sigma\Omega\text{Erfc}[\sigma\Omega]). \tag{59}$$

In the X integral the time delay in the switching of one detector is relevant, we have

$$X = 2\eta_0^2\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\frac{e^{-\tau_1^2/2\sigma^2}e^{-\tau_2^2/2\sigma^2}e^{-\tau_0^2/2\sigma^2}e^{\tau_0\tau_1/\sigma^2}e^{i\Omega(\tau_1+\tau_2)}}{4\pi^2[(\gamma\tau_1-\gamma\tau_2-\gamma\beta L-i\epsilon)^2-(\gamma\beta\tau_1-\gamma\beta\tau_2-\gamma L)^2]}. \tag{60}$$

Applying the change of variables (42)

$$X = 4\eta_0^2 e^{-\tau_0^2/2\sigma^2} \int_{-\infty}^{\infty} dx e^{-x^2} e^{\tau_0 x/\sigma} e^{i2\sigma\Omega x} \int_0^{\infty} dy e^{-y^2} e^{\tau_0 y/\sigma} \frac{1}{16\pi^2\gamma^2[(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2]}, \quad (61)$$

note that the requirement that $\tau_2 < \tau_1$ translates to $y > 0$ in the new variables, so the second integral is over the positive real line. We then complete the squares in the exponentials using

$$e^{-x^2+x\tau_0/\sigma} = e^{-x^2+x\tau_0/\sigma} e^{-(\tau_0/2\sigma)^2} e^{(\tau_0/2\sigma)^2} = e^{-(x-\tau_0/2\sigma)^2} e^{(\tau_0/2\sigma)^2}. \quad (62)$$

This gives

$$X = 4\eta_0^2 \int_{-\infty}^{\infty} dx e^{-(x-\tau_0/2\sigma)^2} e^{i2\sigma\Omega x} \int_0^{\infty} dy e^{-(y-\tau_0/2\sigma)^2} \frac{1}{16\pi^2\gamma^2[(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2]} \quad (63)$$

The x integral is now the Fourier transform of a time-shifted Gaussian so we can evaluate it immediately

$$X = 4\eta_0^2 e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} \sqrt{\pi} \int_0^{\infty} dy e^{-(y-\tau_0/2\sigma)^2} \frac{1}{16\pi^2\gamma^2[(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2]}. \quad (64)$$

We then substitute the y integrand as the Fourier transform of a frequency-shifted Gaussian

$$e^{-(y-\tau_0/2\sigma)^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk e^{-k^2} e^{-ik\tau_0/\sigma} e^{2iky}, \quad (65)$$

in doing so we end up with

$$X = \frac{\eta_0^2}{4\pi^2\gamma^2} e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} \int_{-\infty}^{\infty} dk e^{-k^2} e^{-ik\tau_0/\sigma} \int_0^{\infty} dy \frac{e^{2iky}}{[(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2]}. \quad (66)$$

Note that the integrand only has one pole for $\Re(y) > 0$ at $y = \frac{L}{2\sigma} + i\frac{\epsilon}{1+\beta}$. The residue of this simple pole can be calculated as follows

$$\text{Res} = \lim_{y \to \frac{L}{2\sigma} + i\frac{\epsilon}{1+\beta}} [y - (\frac{L}{2\sigma} + i\frac{\epsilon}{1+\beta})][\frac{e^{2iky}}{(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2}] = \frac{e^{\frac{ik(L+L\beta+2i\epsilon\sigma)}{(1+\beta)\sigma}}\sigma}{L - L\beta^2 - 2i\beta\epsilon\sigma}, \quad (67)$$

which for $\epsilon \to 0^+$ is equal to

$$\frac{\sigma e^{ikL/\sigma}}{L(1-\beta^2)}. \quad (68)$$

We then split the k integral for positive and negative values

$$X = \frac{\eta^2}{4\pi^2\gamma^2} e^{-(\Omega\sigma)^2} [\int_0^{\infty} dk e^{-k^2} e^{-ik\tau_0/\sigma} \int_0^{\infty} dy \frac{e^{2iky}}{(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2}$$
$$+ \int_{-\infty}^{0} dk e^{-k^2} e^{-ik\tau_0/\sigma} \int_0^{\infty} dy \frac{e^{2iky}}{(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2}], \quad (69)$$

and we will evaluate the y integrals using a different contour in each case. When k is positive we consider a contour consisting of three segments: a straight line along the positive real axis from 0 to R, a quarter circle arc from R to iR, and a straight line along the positive imaginary axis from iR to 0 as shown in Figure (4). This contour encloses the pole considered above so by the residue theorem we have

$$\oint_{k>0} f(y)dy = \int_{C1|k>0} f(y)dy + \int_{C2|k>0} f(y)dy + \int_{C3|k>0} f(y)dy = 2\pi i \frac{\sigma e^{ikL/\sigma}}{L(1-\beta^2)}, \quad (70)$$

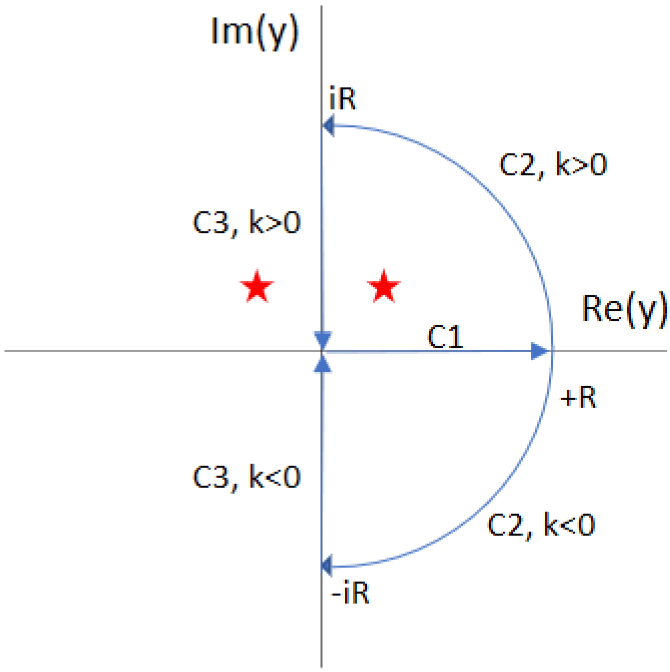

Figure 4: Contours used in the calculation of the non-local term X. The upper path is taken when $k > 0$ and the lower when $k < 0$. Poles are indicated by the red stars.

where C1 is the integral along the positive real y axis, C2 is the quarter circle arc, and C3 is along the the positive imaginary y axis. By Jordan's lemma the arc integral goes to zero in the limit, so $R \to \infty$

$$\int_0^\infty dy \frac{e^{2iky}}{(y - \beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2} = 2\pi i \frac{\sigma e^{ikL/\sigma}}{L(1-\beta^2)} - \int_{C3|k>0} f(y)dy. \tag{71}$$

We simplify the C3 integral by substituting $y = iu$

$$\begin{aligned}\int_{C3|k>0} f(y)dy &= \int_{i\infty}^0 dy \frac{e^{2iky}}{(y - \beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2} \\ &= -i\int_0^\infty du \frac{e^{-2ku}}{(iu - \beta L/2\sigma - i\epsilon)^2 - (\beta iu - L/2\sigma)^2}.\end{aligned} \tag{72}$$

Because the C3 integral does not pass over the pole even in the limit $\epsilon \to 0^+$ we can immediately apply that limit to obtain

$$\begin{aligned}\int_{C3|k>0} f(y)dy &= -i\int_0^\infty du \frac{e^{-2ku}}{(iu - \beta L/2\sigma)^2 - (\beta iu - L/2\sigma)^2} \\ &= i\int_0^\infty du \frac{e^{-2ku}}{(1-\beta^2)(u^2 + (L/2\sigma)^2)}.\end{aligned} \tag{73}$$

When k is negative we follow a similar procedure, however for the integrand to be bounded the contour must reside in the lower half of the complex plane. In this case the contour encloses no poles, so we have

$$\oint_{k<0} f(y)dy = \int_{C1|k<0} f(y)dy + \int_{C2|k<0} f(y)dy + \int_{C3|k<0} f(y)dy = 0. \tag{74}$$

This time substituting $y = -iu$

$$\int_{C3|k<0} f(y)dy = \int_{-i\infty}^{0} dy \frac{e^{2iky}}{(y-\beta L/2\sigma - i\epsilon)^2 - (\beta y - L/2\sigma)^2}$$
$$= i\int_{0}^{\infty} du \frac{e^{2ku}}{(-iu-\beta L/2\sigma - i\epsilon)^2 - (-\beta iu - L/2\sigma)^2}, \tag{75}$$

and again applying the ϵ limit

$$\int_{C3|k<0} f(y)dy = i\int_{0}^{\infty} du \frac{e^{2ku}}{(-iu-\beta L/2\sigma)^2 - (-\beta iu - L/2\sigma)^2}$$
$$= -i\int_{0}^{\infty} du \frac{e^{2ku}}{(1-\beta^2)(u^2+(L/2\sigma)^2)}. \tag{76}$$

Putting this all together gives

$$X = \frac{\eta^2}{4\pi^2\gamma^2} e^{-(\Omega\sigma)^2} [\int_0^\infty dk e^{-k^2} e^{-ik\tau_0/\sigma} (2\pi i \frac{\sigma e^{ikL/\sigma}}{L(1-\beta^2)} - i\int_0^\infty du \frac{e^{-2ku}}{(1-\beta^2)(u^2+(L/2\sigma)^2)})$$
$$+ \int_{-\infty}^{0} dk e^{-k^2} e^{-ik\tau_0/\sigma} (0 + i\int_0^\infty du \frac{e^{2ku}}{(1-\beta^2)(u^2+(L/2\sigma)^2)})]$$
$$= \frac{\eta^2}{4\pi^2\gamma^2} e^{-(\Omega\sigma)^2} [\int_0^\infty dk e^{-k^2} e^{-ik\tau_0/\sigma} 2\pi i \frac{\sigma e^{ikL/\sigma}}{L(1-\beta^2)}$$
$$- i\int_{-\infty}^{\infty} dk e^{-k^2} e^{-ik\tau_0/\sigma} sign(k) \int_0^\infty du \frac{e^{-2|k|u}}{(1-\beta^2)(u^2+(L/2\sigma)^2)}]. \tag{77}$$

We can then factor out the $(1-\beta^2)$ term

$$X = \frac{\eta^2}{4\pi^2} e^{-(\Omega\sigma)^2} [\int_0^\infty dk e^{-k^2} e^{-ik\tau_0/\sigma} 2\pi i \frac{\sigma e^{ikL/\sigma}}{L} - i\int_{-\infty}^{\infty} dk e^{-k^2} e^{-ik\tau_0/\sigma} sign(k) \int_0^\infty du \frac{e^{-2|k|u}}{(u^2+(L/2\sigma)^2)}]. \tag{78}$$

Expanding the complex exponential with Euler's formula

$$X = \frac{\eta_0^2}{4\pi^2} e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} [\int_0^\infty dk e^{-k^2} e^{-ik\tau_0/\sigma} 2\pi i \frac{\sigma e^{ikL/\sigma}}{L}$$
$$- i\int_{-\infty}^{\infty} dk e^{-k^2} [\cos(k\tau_0/\sigma) - i\sin(k\tau_0/\sigma)] sign(k) \int_0^\infty du \frac{e^{-2|k|u}}{(u^2+(L/2\sigma)^2)}]$$
$$= \frac{\eta_0^2}{4\pi^2} e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} [\int_0^\infty dk e^{-k^2} e^{-ik\tau_0/\sigma} 2\pi i \frac{\sigma e^{ikL/\sigma}}{L}$$
$$- i\int_{-\infty}^{\infty} dk e^{-k^2} \cos(k\tau_0/\sigma) sign(k) \int_0^\infty du \frac{e^{-2|k|u}}{(u^2+(L/2\sigma)^2)}$$
$$- \int_{-\infty}^{\infty} dk e^{-k^2} \sin(k\tau_0/\sigma) sign(k) \int_0^\infty du \frac{e^{-2|k|u}}{(u^2+(L/2\sigma)^2)}]. \tag{79}$$

The integrand with the cosine is an odd function of k, so that integral is zero. The sine integrand however is even so we can simplify as follows

$$X = \frac{\eta_0^2}{4\pi^2} e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} [\frac{2\pi i\sigma}{L} \int_0^\infty dk e^{-k^2} e^{ik(L-\tau_0)/\sigma} - 2\int_0^\infty dk e^{-k^2} \sin(k\tau_0/\sigma) \int_0^\infty du \frac{e^{-2ku}}{(u^2+(L/2\sigma)^2)}]. \tag{80}$$

The first integral can be easily calculated, however the second is more difficult. We first replace the sine function with its Taylor series

$$\begin{aligned} I_2 &= \int_0^\infty du \frac{1}{(u^2+(L/2\sigma)^2)} \int_0^\infty dk e^{-k^2} e^{-2ku} \sin(k\tau_0/\sigma) \\ &= \int_0^\infty du \frac{1}{(u^2+(L/2\sigma)^2)} \int_0^\infty dk e^{-k^2} e^{-2ku} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}(k\tau_0/\sigma)^{2n+1} \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \int_0^\infty du \frac{1}{(u^2+(L/2\sigma)^2)} \int_0^\infty dk e^{-k^2} e^{-2ku} (k\tau_0/\sigma)^{2n+1}, \end{aligned} \tag{81}$$

where the interchange of the integral and summation is justified by the fact that each function in the integrand is absolutely convergent. We then consider a more general function

$$f(\tau/\sigma, L/2\sigma, n) = \int_0^\infty du \frac{1}{(u^2+(L/2\sigma)^2)} \int_0^\infty dk e^{-k^2} e^{-2ku} (k\tau_0/\sigma)^n, \tag{82}$$

where n is a natural number. For positive L/σ these integrals can be evaluated as follows

$$\begin{aligned} f(\tau/\sigma, L/2\sigma, n) &= \int_0^\infty du \frac{1}{(u^2+(L/2\sigma)^2)} [2^{-1-n} n! (\tau_0/\sigma)^n U(\frac{1+n}{2}, \frac{1}{2}, u^2)] \\ &= \frac{(\tau_0/\sigma)^n (L/2\sigma)^{-2}}{4\sqrt{\pi}} G^{3,2}_{2,3}\left(\begin{matrix} 1, 1-n/2 \\ 1/2, 1, 1 \end{matrix} \middle| \left(\frac{L}{2\sigma}\right)^2 \right), \end{aligned} \tag{83}$$

where $U(,,)$ is the confluent hypergeometric function of the second kind, and $G^{m,n}_{p,q}$ is the Meijer G function. Substituting this back into the integral equation

$$\begin{aligned} I_2 &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} f(\tau/\sigma, L/2\sigma, 2n+1) \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \frac{(\tau_0/\sigma)^{2n+1}(L/2\sigma)^{-2}}{4\sqrt{\pi}} G^{3,2}_{2,3}\left(\begin{matrix} 1, 1-(2n+1)/2 \\ 1/2, 1, 1 \end{matrix} \middle| \left(\frac{L}{2\sigma}\right)^2 \right) \\ &= \frac{(L/2\sigma)^{-2}}{4\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} (\tau_0/\sigma)^{2n+1} G^{3,2}_{2,3}\left(\begin{matrix} 1, 1/2-n \\ 1/2, 1, 1 \end{matrix} \middle| \left(\frac{L}{2\sigma}\right)^2 \right). \end{aligned} \tag{84}$$

This is a power series in terms of τ_0/σ. For slight deviations in the switching time $\tau_0 << \sigma$ an approximate numerical solution could be obtained by truncating the sum. Our result for X is therefore

$$\begin{aligned} X = \frac{\eta_0^2}{4\pi^2} e^{i\tau_0\Omega} e^{-(\sigma\Omega)^2} [\pi^{3/2} \frac{\sigma}{L} e^{-(L-\tau_0)^2/4\sigma^2} [i - \text{Erfi}[\frac{(L-\tau_0)}{2\sigma}]] \\ - 2\frac{(L/2\sigma)^{-2}}{4\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} (\tau_0/\sigma)^{2n+1} G^{3,2}_{2,3}\left(\begin{matrix} 1, 1/2-n \\ 1/2, 1, 1 \end{matrix} \middle| \left(\frac{L}{2\sigma}\right)^2 \right)], \end{aligned} \tag{85}$$

where Erfi is the imaginary error function

$$\text{Erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt. \tag{86}$$

6 Results

In this section we will assume that the detector switching is synchronous in order to make the analysis simpler. Setting $\tau_0 = 0$ gives

$$A = \frac{\eta_0^2}{4\pi}(e^{-\sigma^2\Omega^2} - \sqrt{\pi}\sigma\Omega\text{Erfc}[\sigma\Omega]), \tag{87}$$

$$X = -\frac{\eta_0^2\sigma}{4\sqrt{\pi}L}e^{-(\Omega\sigma)^2}e^{-L^2/4\sigma^2}[\text{Erfi}(\frac{L}{2\sigma}) - i]. \tag{88}$$

The negativity of the state of the detectors is therefore

$$\mathcal{N} = |X| - A = \frac{\eta^2}{4\pi}\Big[\sqrt{\pi}\frac{\sigma}{L}e^{-(\Omega\sigma)^2}e^{-L^2/4\sigma^2}|i - \text{Erfi}(\frac{L}{2\sigma})| - (e^{-(\sigma\Omega)^2} - \sqrt{\pi}\sigma\Omega\text{Erfc}[\sigma\Omega])\Big]. \tag{89}$$

Note that nowhere in this equation do either β or γ appear; in fact this equation is identical to what we would find if we calculated the negativity in a reference frame co-moving with the detectors. This is our key result: observers in any of the reference frames that we have considered will agree on whether the detectors have become entangled or not. We can then ask 'what are the conditions in which entanglement actually occurs?'. For the detectors to be entangled we require that $\mathcal{N} > 0$. We have three parameters of interest, the detector energy gap, Ω, the switching time of the detectors, σ, and the distance of the two detectors, L. Because we have been working in Planck units ($\hbar = c = 1$) we have Length = Time = Energy^{-1}. Therefore we can consider only two of our three parameters at a time, and use the third parameter to normalise the others into unitless variables. Using the energy gap of the detectors to normalise the other parameters we can write the negativity in terms of $\sigma' = \Omega\sigma$ and $L' = L\Omega$

$$\mathcal{N} = \frac{\eta^2}{4\pi}\Big[\sqrt{\pi}\frac{\sigma'}{L'}e^{-(\sigma')^2}e^{-L'^2/4\sigma'^2}|i - \text{Erfi}(\frac{L'}{2\sigma'})| - (e^{-(\sigma')^2} - \sqrt{\pi}\sigma'\text{Erfc}[\sigma'])\Big]. \tag{90}$$

A contour plot of this function can be used to determine when entanglement harvesting is possible, this is shown in Figure (5).

Looking at this plot we can see that the detectors can indeed become entangled, and that the interaction time required for this to happen increases when the detectors are further apart. However it's important to note that we have an upper bound on how long the detectors are switched on in order for the entanglement to have actually been harvested from the vacuum. This is because when $\sigma \geq L/2$ we cannot rule out the possibility that some unknown mechanism has led to the state of one detector having a causal influence on the other. Such a causal influence could at most propagate at the speed of light so by applying the restriction that $\sigma < L/2$ we can be sure that any entanglement in the detectors has come from the underlying correlations in the vacuum state of the field (Nambu 2013). Entanglement harvesting is made possible by the fact that even though the field is in a minimal energy state there are still vacuum fluctuations in global modes of the field.

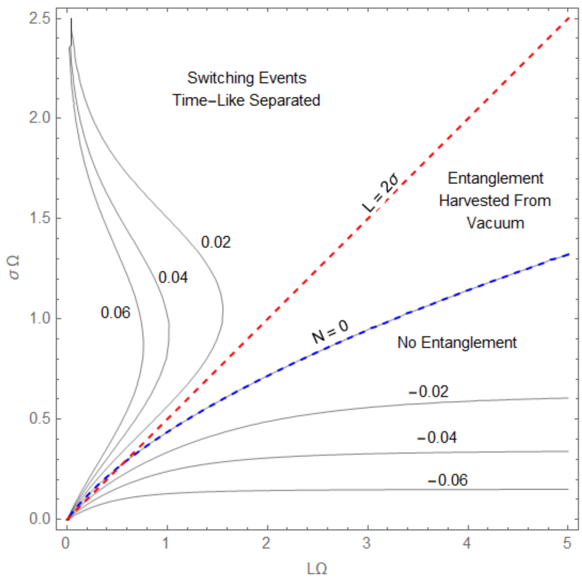

Figure 5: Contour plot of the negativity of the state of the two detectors as a function of σ the switching time of their interaction with the field and L the distance between them. Both of these parameters are scaled by the energy gap Ω of the detectors. The dashed blue line shows the cutoff for entanglement when $\mathcal{N} = 0$, and the dashed red line shows the cutoff for signalling $L = 2\sigma$.

7 Discussion and Conclusion

We have modeled the interaction of a pair of spatially separated particle detectors with the vacuum state of a quantum field. From a reference frame that is moving with respect to the detectors we determined under what conditions do the detectors become entangled. We found that for certain system parameters the detectors will become entangled but not because of any possible communication between the detectors. Instead the entanglement is harvested from the correlations in the vacuum fluctuations of the field.

Our work demonstrates that the class of observers that we considered will all assign the same entangled state to the detectors, and that they would agree with a stationary observer. Even though in the end the result we obtained was the same as prior work (Ver Steeg and Menicucci 2009), we have shown how the change in reference frame is handled mathematically, and this may prove useful to future studies of entanglement harvesting. We have also shown that a far more complex result is obtained when the detectors do not switch on and off simultaneously (in their own reference frame).

Directions for further research can be seen in the assumptions that we have made in the present work. In our model we assumed point-like detectors and smooth, perfectly-Gaussian switching, but these are both idealisations. Entanglement harvesting by detectors with different spatial profiles and switching functions is an active research topic (Kerstjens and Martinez 2018). Additionally, we have only looked at motion parallel to to displacement vector between the detectors and further work could generalise our calculation to motion in an arbitrary direction.

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8 Appendix

8.1 Calculation of the Density Matrix

This method of calculating the density matrix was developed by (Nambu 2013). Our initial state consists of the qubits in their ground states and the field in its vacuum state $|0\rangle = \prod_k |0_k\rangle$

$$\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0| = |g_A g_B\rangle\langle g_A g_B| \otimes |0\rangle\langle 0|. \tag{91}$$

We then introduce the following operators:

$$\hat{\Phi}^{\pm}_k = \int_{\tau_0}^{\tau} d\tau_1 \eta(\tau_1)_k \hat{\phi}_k(\mathbf{x}_k(\tau_1)) e^{\pm i\Omega\tau_1} \tag{92}$$

$$\hat{S}(\tau) = -i\int_{\tau_0}^{\tau} d\tau_1 \hat{V}(\tau_1)_I = -i\sum_{j=\pm}\sum_{k=A,B} \hat{\Phi}^j_k \hat{\sigma}^j_k. \tag{93}$$

This will allow us to separate the action on the detectors and the field and simplify the calculation of $\hat{\rho}(\tau)$. We can approximate $\hat{U}(\tau)$ as

$$\hat{U}(\tau) = 1 + \hat{S} + \frac{1}{2}T[\hat{S}\hat{S}]. \tag{94}$$

Therefore

$$\hat{\rho}(\tau) = [1 + \hat{S} + \frac{1}{2}T[\hat{S}\hat{S}]](|g_A g_B\rangle|0\rangle\langle 0|\langle g_A g_B|)[1 + \hat{S} + \frac{1}{2}T[\hat{S}\hat{S}]] = |\psi(\tau)_I\rangle\langle\psi(\tau)_I|. \tag{95}$$

Looking at just the ket vector:

$$|\psi(\tau)_I\rangle = |g_A g_B\rangle|0\rangle + \hat{S}|g_A g_B\rangle|0\rangle + \frac{1}{2}T[\hat{S}\hat{S}]|g_A g_B\rangle|0\rangle. \tag{96}$$

The first order term gives us

$$\hat{S}|g_A g_B\rangle|0\rangle = -i\sum_{j=\pm}\sum_{k=A,B}\hat{\Phi}^j_k\hat{\sigma}^j_k|g_A g_B\rangle|0\rangle = -i[(\hat{\Phi}^+_A\hat{\sigma}^+_A + \hat{\Phi}^-_A\hat{\sigma}^-_A) + (\hat{\Phi}^+_B\hat{\sigma}^+_B + \hat{\Phi}^-_B\hat{\sigma}^-_B)]|g_A g_B\rangle|0\rangle. \tag{97}$$

Applying the lowering operator to the ground state of a qubit will annihilate the state so we can simplify:

$$\hat{S}\,|g_A g_B\rangle\,|0\rangle = -i[\hat{\Phi}_A^+\hat{\sigma}_A^+ + \hat{\Phi}_B^+\hat{\sigma}_B^+]\,|g_A g_B\rangle\,|0\rangle = -i[|e_A g_B\rangle\,\hat{\Phi}_A^+\,|0\rangle + |g_A e_B\rangle\,\hat{\Phi}_B^+\,|0\rangle]. \tag{98}$$

Because we can't go any lower than the vacuum state we have $\hat{\Phi}_k^j\,|0\rangle \propto |1\rangle$, so we can write $\hat{\Phi}_k^j\,|0\rangle = |1\rangle\,\langle 1|\,\hat{\Phi}_k^j\,|0\rangle$. Therefore

$$\hat{S}\,|g_A g_B\rangle\,|0\rangle = -i[\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|e_A g_B\rangle\,|1\rangle + \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|g_A e_B\rangle\,|1\rangle]. \tag{99}$$

For the second order term

$$\begin{aligned}
\hat{S}\hat{S}\,|g_A g_B\rangle\,|0\rangle &= -i\hat{S}[\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|e_A g_B\rangle\,|1\rangle + \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|g_A e_B\rangle\,|1\rangle] \\
&= -[(\hat{\Phi}_A^+\hat{\sigma}_A^+ + \hat{\Phi}_A^-\hat{\sigma}_A^-) + (\hat{\Phi}_B^+\hat{\sigma}_B^+ + \hat{\Phi}_B^-\hat{\sigma}_B^-)][\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|e_A g_B\rangle\,|1\rangle + \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|g_A e_B\rangle\,|1\rangle] \\
&= -\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,[\hat{\Phi}_A^-\hat{\sigma}_A^- + \hat{\Phi}_B^+\hat{\sigma}_B^+]\,|e_A g_B\rangle\,|1\rangle - \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,[\hat{\Phi}_A^+\hat{\sigma}_A^+ + \hat{\Phi}_B^-\hat{\sigma}_B^-]\,|g_A e_B\rangle\,|1\rangle \\
&= -\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,[\hat{\Phi}_A^-\,|1\rangle\,|g_A g_B\rangle + \hat{\Phi}_B^+\,|1\rangle\,|e_A e_B\rangle] - \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,[\hat{\Phi}_A^+\,|1\rangle\,|e_A e_B\rangle + \hat{\Phi}_B^-\,|1\rangle\,|g_A g_B\rangle] \\
&= -\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,[|2\rangle\,\langle 2|\,\hat{\Phi}_A^-\,|1\rangle\,|g_A g_B\rangle + |0\rangle\,\langle 0|\,\hat{\Phi}_A^-\,|1\rangle\,|g_A g_B\rangle \\
&\qquad + |2\rangle\,\langle 2|\,\hat{\Phi}_B^+\,|1\rangle\,|e_A e_B\rangle + |0\rangle\,\langle 0|\,\hat{\Phi}_B^+\,|1\rangle\,|e_A e_B\rangle] \\
&\quad - \langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,[|0\rangle\,\langle 0|\,\hat{\Phi}_A^+\,|1\rangle\,|e_A e_B\rangle + |2\rangle\,\langle 2|\,\hat{\Phi}_A^+\,|1\rangle\,|e_A e_B\rangle \\
&\qquad + |0\rangle\,\langle 0|\,\hat{\Phi}_B^-\,|1\rangle\,|g_A g_B\rangle + |2\rangle\,\langle 2|\,\hat{\Phi}_B^-\,|1\rangle\,|g_A g_B\rangle] \\
&= -[\langle 2|\,\hat{\Phi}_A^-\,|1\rangle\,\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|g_A g_B\rangle\,|2\rangle + \langle 0|\,\hat{\Phi}_A^-\,|1\rangle\,\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|g_A g_B\rangle\,|0\rangle \\
&\qquad + \langle 2|\,\hat{\Phi}_B^+\,|1\rangle\,\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|e_A e_B\rangle\,|2\rangle + \langle 0|\,\hat{\Phi}_B^+\,|1\rangle\,\langle 1|\,\hat{\Phi}_A^+\,|0\rangle\,|e_A e_B\rangle\,|0\rangle] \\
&\quad - [\langle 0|\,\hat{\Phi}_A^+\,|1\rangle\,\langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|e_A e_B\rangle\,|0\rangle + \langle 2|\,\hat{\Phi}_A^+\,|1\rangle\,\langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|e_A e_B\rangle\,|2\rangle \\
&\qquad + \langle 0|\,\hat{\Phi}_B^-\,|1\rangle\,\langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|g_A g_B\rangle\,|0\rangle + \langle 2|\,\hat{\Phi}_B^-\,|1\rangle\,\langle 1|\,\hat{\Phi}_B^+\,|0\rangle\,|g_A g_B\rangle\,|2\rangle].
\end{aligned} \tag{100}$$

We obtain the density operator for the detectors by tracing out the field:

$$\hat{\rho}_{det} = \sum_n \langle n|\psi(t)_I\rangle\,\langle\psi(t)_I|n\rangle\,. \tag{101}$$

We simplify the notation by introducing coefficients:

$$\begin{aligned}
d_1 &= -i\,\langle 1|\,\hat{\Phi}_A^+\,|0\rangle = \mathcal{O}(\eta_0) & d_2 &= -i\,\langle 1|\,\hat{\Phi}_B^+\,|0\rangle = \mathcal{O}(\eta_0) \\
d_{3A} &= -\langle 0|\,\hat{\Phi}_B^+\hat{\Phi}_A^+\,|0\rangle = \mathcal{O}(\eta_0^2) & d_{3B} &= -\langle 0|\,\hat{\Phi}_A^+\hat{\Phi}_B^+\,|0\rangle = \mathcal{O}(\eta_0^2) \\
d_{4A} &= -\langle 0|\,\hat{\Phi}_A^-\hat{\Phi}_A^+\,|0\rangle = \mathcal{O}(\eta_0^2) & d_{4B} &= -\langle 0|\,\hat{\Phi}_B^-\hat{\Phi}_B^+\,|0\rangle = \mathcal{O}(\eta_0^2) \\
d_{5A} &= -\langle 2|\,\hat{\Phi}_B^+\hat{\Phi}_A^+\,|0\rangle = \mathcal{O}(\eta_0^2) & d_{5B} &= -\langle 2|\,\hat{\Phi}_A^+\hat{\Phi}_B^+\,|0\rangle = \mathcal{O}(\eta_0^2) \\
d_{6A} &= -\langle 2|\,\hat{\Phi}_B^-\hat{\Phi}_B^+\,|0\rangle = \mathcal{O}(\eta_0^2) & d_{6B} &= -\langle 2|\,\hat{\Phi}_A^-\hat{\Phi}_A^+\,|0\rangle = \mathcal{O}(\eta_0^2)
\end{aligned}$$

For the Vacuum state:

$$\langle 0|\psi(t)_I\rangle = (1 + d_{4A} + d_{4B})\,|g_A g_B\rangle + (d_{3A} + d_{3B})\,|e_A e_B\rangle \tag{102}$$

Therefore

$$\langle 0|\psi(t)_I\rangle \langle\psi(t)_I|0\rangle = (1 + d_{4A} + d_{4B} + d_{4A}^* + d_{4B}^*) |g_A g_B\rangle \langle g_A g_B| + (d_{3A} + d_{3B}) |e_A e_B\rangle \langle g_A g_B| + (d_{3A}^* + d_{3B}^*) |g_A g_B\rangle \langle e_A e_B| + \mathcal{O}(\eta_0^4) \tag{103}$$

For the 1-particle state

$$\langle 1|\psi(t)_I\rangle = d_1 |e_A g_B\rangle + d_2 |g_A e_B\rangle \tag{104}$$

Therefore

$$\langle 1|\psi(t)_I\rangle \langle\psi(t)_I|1\rangle = d_1 d_1^* |e_A g_B\rangle \langle e_A g_B| + d_1 d_2^* |e_A g_B\rangle \langle g_A e_B| + d_2 d_1^* |g_A e_B\rangle \langle e_A g_B| + d_2 d_2^* |g_A e_B\rangle \langle g_A e_B| \tag{105}$$

For the 2-particle state

$$\langle 2|\psi(t)_I\rangle = (d_{6A} + d_{6B}) |g_A g_B\rangle + (d_{5A} + d_{5B}) |e_A e_B\rangle \\ \Rightarrow \langle 2|\psi(t)_I\rangle \langle\psi(t)_I|2\rangle = \mathcal{O}(\eta_0^4) \tag{106}$$

So the density matrix for the detectors using basis vectors $\{|gg\rangle, |eg\rangle, |ge\rangle, |ee\rangle\}$ is

$$\hat{\rho}_{det} = \begin{pmatrix} 1-2A & 0 & 0 & X^* \\ 0 & A & E_{AB} & 0 \\ 0 & E_{BA} & A & 0 \\ X & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\eta^4) \tag{107}$$

Where

$$A = d_1 d_1^* = d_2 d_2^* = \langle 0| \hat{\Phi}_A^- \hat{\Phi}_A^+ |0\rangle = \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \eta_A(\tau_1)\eta_A(\tau_2) e^{-i\Omega(\tau_1-\tau_2)} \langle 0| \hat{\phi}_A(\mathbf{x}_A(\tau_1))\hat{\phi}_A(\mathbf{x}_A(\tau_2)) |0\rangle \tag{108}$$

$$E_{AB} = E_{BA}^* = d_1 d_2^* = \langle 0| \hat{\Phi}_A^- \hat{\Phi}_B^+ |0\rangle \tag{109}$$

$$X = d_{3A} + d_{3B} = -\langle 0| \hat{\Phi}_B^+ \hat{\Phi}_A^+ |0\rangle - \langle 0| \hat{\Phi}_A^+ \hat{\Phi}_B^+ |0\rangle = -2\int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \eta_A(\tau_1)\eta_B(\tau_2) e^{i\Omega(\tau_1+\tau_2)} \langle 0| \hat{\phi}_A(\mathbf{x}_A(\tau_1))\hat{\phi}_B(\mathbf{x}_B(\tau_2)) |0\rangle \tag{110}$$