

# The Super Box-Ball System

# Benjamin Solomon

Supervised by Dr Travis Scrimshaw The University of Queensland

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#### Abstract

We generalise the supersymmetric box-ball system devised by Hikami–Inoue for the affine general linear Lie superalgebra using the Kirillov–Reshetikhin (KR) crystals constructed by Kwon–Okado. We prove solitonic behaviour for a certain class of solitons in this generalised system.

#### 0.1 Statement of Authorship

The theoretical background in this report stems from many texts, all of which are listed in the references. Where information for others is presented, it has been varied in notation and structure from the original sources. Benjamin Solomon performed research and wrote this report. Benjamin Solomon and Mitchell Ryan shared in the proving of Theorems 4.1 and 4.2, and in developing SAGEMATH code used for analysis. Travis Scrimshaw gave direction and advice on research, and proofread and edited this report.

#### 0.2 Acknowledgments

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## 1 Introduction

The Kortweg–de Vries (KdV) equation is an non-linear partial differential equation used to model shallow water waves moving through a narrow channel. In 1965, Zubusky and Kruskal found that the solutions to this equation decompose into solitons, which are solitary waves within the channel. The solitons are known to propagate with speed proportional to their size and retain their shapes after collisions. There exists a ultradiscrete analogue to the KdV equation called the Takahashi–Satsuma box-ball system (BBS) (Takahashi and Satsuma, 1990). The BBS is a discrete integrable dynamical system that is composed of finitely many balls in



an infinite line of boxes with an algorithm describing its time evolution. The ultradiscrete analogue of the soliton solutions to the KdV equation in the BBS are packets of balls that move together with speed corresponding to their number and are stable under propagation and collision. The box-ball system has deep links with mathematical physics and representation theory through Kashiwara's crystal theory of quantum groups (Kashiwara, 1990).

It was realised that the BBS could be described using tensor products of Kirillov–Reshetikhin (KR) crystals for  $U_q(\widehat{\mathfrak{sl}}_2)$  (Fukuda et al., 2000; Hatayama et al., 2001) with the time evolution is described by using the combinatorial *R*-matrix, the unique isomorphism that interchanges tensor product of KR crystals. Utilising this crystal theory, it was natural to extend the system to  $U_q(\widehat{\mathfrak{sl}}_{n+1})$ , producing a system often called the coloured BBS (Hatayama et al., 2001). This was then further generalised by Yamada (Yamada, 2004) using the KR crystals  $B^{r,s}$  rather than  $B^{1,s}$ .

A generalisation to the BBS was later devised by Hikami–Inoue (Hikami and Inoue, 2000) using ideas from supersymmetry called the supersymmetric box-ball system (SBBS). Subsequently, Kwon–Okado (Kwon and Okado, 2020) constructed the analogue of KR crystals for the generalised quantum group of type A, providing the structure in order generalise the SBBS further. We define the structure of a single soliton within this generalised system (Theorem 4.1). We then prove a certain class of solitons interact such that their shapes are preserved after collisions (Theorem 4.2).

### 2 Background

#### 2.1 Partitions and hook shape

A partition  $\lambda$  of  $N \in \mathbb{N}$ , is a set of positive integers  $\{\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k\}$  such that  $N = \sum_{i=1}^k \lambda_i$ . Young diagrams are used to represent these partitions as boxes pushed into the upper-left corner.



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**Example 2.1.** The partition  $\lambda = (4, 3, 2, 2)$  has a Young diagram of



An (m|n)-hook shape it is a Young diagram such that  $\lambda_{m+1} \leq n$ . Pictorially, it means the Young diagram fits inside the shaded area:



#### 2.2 The Box-ball System

The Takahashi–Satsuma *box-ball system (BBS)* (Takahashi and Satsuma, 1990) is an ultradiscrete analogue of Kortweg–de Vries (KdV) equation. The BBS is composed of a finite number of balls in an infinite line of boxes, with the following algorithm to describe how these balls propagate. Moving from left to right, every time a ball is encountered, unless it had already been moved, place the leftmost ball in the strictly nearest right empty box.

The KdV equation admits soliton solutions, which are solitory waves that maintain their shape under collision and exhibit speed proportional to their amplitude. Solitons in the context of the BBS are the discrete analogue of this phenomena. A BBS *soliton* is a group of s balls that exhibit the following behaviour:

- Move with speed s.
- Maintain shape under propagation and collision.



#### 2.3 The general linear Lie superalgebra

The generalised BBS we aim to construct obtains its structure in part from the general linear Lie superalgebra  $\mathfrak{gl}(m|n)$  and its corresponding (Drinfeld–Jimbo) quantum group  $U_q(\mathfrak{gl}(m|n))$ . Let  $I = I_{\text{even}} \sqcup I_{\text{odd}}$  be the indexing set of simple roots, where  $I_{\text{even}} = \{\overline{m-1}, \ldots, \overline{1}, 1, \ldots, n-1\}$ and  $I_{\text{odd}} = 0$ . Let P be the weight lattice

$$P = \bigoplus_{b \in B} \mathbb{Z} \epsilon_b,$$

where  $B = B_+ \sqcup B_-$ , with  $B_+ = \{\overline{m}, \ldots, \overline{1}\}$  and  $B_- = \{1, \ldots, n\}$ . We define an inner product

$$(\epsilon_a, \epsilon_{a'}) = \begin{cases} 1 & \text{if } a = a' \in B_+, \\ -1 & \text{if } a = a' \in B_-, \\ 0 & \text{otherwise.} \end{cases}$$

The values in  $B_+$  are referred to as *bosonic* and values in  $B_-$  as *fermionic*. The simple roots indexed by I are then given by,

$$\alpha_i = \begin{cases} \epsilon_{\overline{a+1}} - \epsilon_{\overline{a}} & \text{if } i = a \in m-1, \dots, 1\\ \epsilon_{\overline{1}} - \epsilon_1 & \text{if } i = 0,\\ \epsilon_i - \epsilon_{i+1} & \text{if } i \in 1, \dots, n-1. \end{cases}$$

Let  $\{h_i\}_{i \in I}$  denote the simple coroots with the canonical pairing denoted by  $\langle h_i, \alpha_j \rangle$ , which is given by the Cartan matrix.

The fundamental representation V of  $U_q(\mathfrak{gl}(m|n))$  is an (m+n)-dimensional representation. Let  $V^{\bigotimes k}$  be the k-th tensor power of representation of V. It can be shown that all tensor powers of representations for  $k \ge 1$  are completely reducible, of which the irreducible summands  $V(\lambda)$ correspond to partitions  $\lambda$  of (m|n)-hook shape.





Figure 1: Dynkin Diagram for  $\mathfrak{gl}(m|n)$  using the standard Borel.

#### 2.4 Crystals

A  $U_q(\mathfrak{gl}(m|n))$ -crystal B is a set with mappings called the *Kashiwara operators*  $e_i, f_i: B \to B \sqcup \{0\}$ , for all  $i \in I$ , and *weight function* wt:  $B \to P$  that satisfy the following axioms:

- 1. For all  $b \in B$  and  $i \in I$ , there exists k > 0 such that  $e_i^k = f_i^k = 0$ .
- 2.  $e_i 0 = f_i 0 = 0.$
- 3. For  $b_1, b_2 \in B$ , then  $f_2b_2 = b_1$  and  $e_1b_1 = b_2$ .
- 4.  $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \operatorname{wt}(b) \rangle$  for all  $b \in B$  and  $i \in I_{\text{even}}$ , where  $\varepsilon_i, \varphi_i \colon B \to \mathbb{Z}_{\geq 0}$ , are the statistics

$$\varepsilon_i(b) = \max\{k | e_i^k(b) \neq 0\}, \qquad \varphi_i(b) = \max\{k | f_i^k(b) \neq 0\}.$$

5. For all  $b \in B$  and  $i \in I_{\text{odd}}$ :

$$\epsilon_i(b) + \varphi_i(b) = \begin{cases} 0 & \text{if } \langle h_i, \operatorname{wt}(b) \rangle = 0, \\ 1 & \text{otherwise.} \end{cases}$$

We say element  $b \in B$  is *highest weight* if  $e_i(b) = 0$  for all  $i \in I$ .

For crystals  $B_1, B_2, \ldots, B_L$  we can define their tensor product  $B = B_L \otimes \cdots \otimes B_2 \otimes B_1$ as the set  $B_L \times \cdots \times B_2 \times B_1$  with the crystal structure given as follows. Fix an element  $b = b_L \otimes \cdots \otimes b_1 \in B$  and  $i \in I_{\text{even}}$ . We define  $e_i b$  and  $f_i b$  using the *signature rule* (Benkart et al., 2000). Reading left to right, we construct the signature as the sequence

$$\operatorname{sgn}_i(b) = \underbrace{-\cdots}_{\varphi_i(b_L)} \underbrace{+\cdots}_{\varepsilon_i(b_L)} \underbrace{\cdots}_{\varphi_i(b_1)} \underbrace{+\cdots}_{\varphi_i(b_1)} \underbrace{+\cdots}_{\varepsilon_i(b_1)}$$

Then, successively removing +- pairs, we obtain the *reduced signature* 

$$\operatorname{rsg}_i(b) = \underbrace{-\cdots}_{\varphi_i(b)} \underbrace{+\cdots}_{\varepsilon_i(b)} \cdot$$



The operator  $e_i$  (resp.  $f_i$ ) acts on the factor containing the rightmost – (resp. leftmost +). If there is no such – (resp. +), we define  $e_i b = 0$  (resp.  $f_i b = 0$ ). The operators  $e_0$  and  $f_0$  have a different algorithm (Kwon and Okado, 2020):

- If the first occurrence of  $\overline{1}$  in word(x) is before the first occurrence of 1, then  $f_0(x)$  replaces the corresponding  $\overline{\overline{1}}$  in x with  $\overline{1}$  and  $e_0(x) = 0$ .
- If the first occurrence of 1 in word(x) is before the first occurrence of  $\overline{1}$ , then  $e_0(x)$  replaces the corresponding  $\overline{\overline{1}}$  in x with  $\overline{1}$  and  $f_0(x) = 0$

For more information on crystals, we refer the reader to (Benkart et al., 2000; Bump and Schilling, 2017).

#### 2.5 Semistandard tableaux and crystals

A Young tableau is a filling of the boxes of a Young diagram with an element of  $B_+ \sqcup B_i$  such that rows and columns are weakly increasing with respect to the ordering  $\overline{m} < \cdots < \overline{1} < 1 < \cdots < n$ . A semistandard Young tableau (SSYT) is a Young tableau such that the bosonic (resp. fermionic) values strictly increase column-wise (resp. row-wise).

We define the reading word of a tableau by the Japanese reading word, reading right to left and top to bottom. For example, consider the tableau

|     | $t_{11}$ | $t_{12}$ |   | $t_{1s}$ |
|-----|----------|----------|---|----------|
| h — | $t_{21}$ | $t_{22}$ |   | $t_{2s}$ |
| 0 — | ÷        | :        | · | :        |
|     | $t_{r1}$ | $t_{r2}$ |   | $t_{rs}$ |

Then the reading word of b is then given by

word(b) = 
$$t_{1s} \cdots t_{rs} \cdots t_{12} \cdots t_{r2} t_{11} \cdots t_{r1}$$
,

where tensors are omitted for convenience.

As previously mentioned, the tensor powers of the fundamental representation of  $U_q(\mathfrak{gl}(m|n))$ are completely reducible, where their irreducible summands  $V(\lambda)$  are indexed by partitions  $\lambda$ 





Figure 2: The crystal  $B(\Box)$  for  $U_q(\mathfrak{gl}(m|n))$ .

that have (m|n)-hook shape. The irreducible module  $V(\lambda)$  admits a crystal base denoted by  $B(\lambda)$ , which is described by semistandard tableaux of shape  $\lambda$  with the crystal structure given by the reading word and signature rule (Benkart et al., 2000).

#### 2.6 Affine crystals

Let  $\mathcal{U}(\varepsilon)$  be the generalised quantum group for type A (Kuniba et al., 2015). It can be interpreted as the affine analogue to the quantum group for the general linear Lie superalgebra. There exists a irreducible  $\mathcal{U}(\varepsilon)$ -module  $W^{r,s}$  with crystal base, where we denote the corresponding crystal  $B^{r,s}$  called a *Kirillov–Reshetikhin (KR) crystal* (Kwon and Okado, 2020). As a  $U_q(\mathfrak{gl}(m|n))$ -crystal,  $B^{r,s}$  corresponds to (m|n)-hook semistandard Young tableaux of shape  $s^r$ , a partition of an  $r \times s$  rectangle.

#### 2.7 The combinatorial *R*-matrix

Consider two  $U_q(\widehat{\mathfrak{gl}}(m|n))$ -crystals  $B^{r_1,s_1}$  and  $B^{r_2,s_2}$ . Then, there exists a unique isomorphism called the *combinatorial R-matrix* (Kwon and Okado, 2020)

$$R\colon B^{r_1,s_1}\otimes B^{r_2,s_2}\to B^{r_2,s_2}\otimes B^{r_1,s_1},$$

which means it commutes with the Kashiwara operators  $e_i$  and  $f_i$  for all  $i \in I$ . We will focus on the *R*-matrix when  $B^{r_1,s_1} = B^{r,s}$  and  $B^{r_2,s_2} = B^{r,1}$ .

In order to describe how the *R*-matrix acts, we introduce *Robinson–Schensted–Knuth (RSK)* insertion. We denote column insertion of a tableau  $T_2$  into another tableau  $T_1$  as  $col(T_2) \to T_1$ . We will often write this simply as  $T_2 \to T_1$ . We do this by inserting  $word(T_2)$  element-wise. Let  $i \in word(T_1)$  be the element being inserted. Beginning with column 1 of  $T_1$ . The process



is as follows:

- 1. For  $i \in B_-$  (resp.  $i \in B_+$ ). If there does not exist any element j > i (resp.  $j \ge i$ ), then place i in a box at the end of the column.
- 2. If there exists a topmost element j > i (resp.  $j \ge i$ ) then replace j with i and repeat process (1) and (2) beginning with the next column.
- 3. Repeat process for each  $i \in word(T_2)$  each time beginning at column 1 until all elements have been inserted.

From Kwon–Okado (Kwon and Okado, 2020, Theorem 7.9), we have that  $R: B^{r_1,s_1} \otimes B^{r_2,s_2}$ sends  $T_1 \otimes T_2 \to \widetilde{T}_2 \otimes \widetilde{T}_1$  if and only if  $T_2 \to T_1 = \widetilde{T}_2 \to \widetilde{T}_1$ . This means that we need not know explicitly how the *R*-matrix acts if we know the result. We use this theorem extensively in our proofs to avoid this explicit calculation.

**Example 2.2.** We perform column insertion on the following pair of tableaux  $T_2 \rightarrow T_1$ :



The image of the R-matrix is

|                     | 4 | 3 | $\overline{3}$ |                       | $\overline{4}$ |   |
|---------------------|---|---|----------------|-----------------------|----------------|---|
| $\widetilde{T}_2 =$ | 3 | 1 | 2              | , $\widetilde{T}_1 =$ | 1              | , |
|                     | 1 | 2 | 3              |                       | 3              |   |

and a direct computation shows that  $\widetilde{T}_2 \to \widetilde{T}_1 = T_2 \to T_1$ .

# 3 The Super Box-ball System

The *super box-ball system (SBBS)* is a generalisation of the Hikame–Inoue (Hikami and Inoue, 2000) BBS using the KR modules constructed by Kwon–Okado (Kwon and Okado, 2020).



#### 3.1 SBBS and Crystals

The SBBS is constructed using the  $U_q(\widehat{\mathfrak{gl}}(m|n))$ -crystal  $B^{r,s}$ . The balls in this system are now column tableaux of height r.

$$b = [x_1, x_2, \cdots, x_r]^T \in B^{r,1},$$

where  $C^T$  denotes a column and  $x_i \in B_+ \sqcup B_-$ . The empty box, also denoted the *vacuum* element v, is chosen to be the highest weight element with restricted indexing choice i for the Kashiwara operators by removing  $i = \overline{m-r}$  so that the vacuum remains unchanged. The vacuum v is defined as follows:

$$v = [\overline{m}, \overline{m-1}, \dots, \overline{m-r+1}]^T \in B^{r,1}.$$

A state of the SBBS is constructed with  $v, b_{\alpha} \in B^{r,1}$ , as follows,

$$b_0 \otimes b_1 \otimes \cdots \otimes b_L \otimes (v)^{\otimes \infty} \in (B^{r,1})^{\otimes \infty}.$$

#### 3.2 Carrier

We now introduce the *carrier* in order to describe time evolution in the SBBS. This is analogous to the time evolution used in the generalised BBS for  $\widehat{\mathfrak{sl}}_{n+1}$  given by Yamada (Yamada, 2004). The empty carrier, denoted  $v_{\ell}$ , is given as follows:

|              | $\overline{m}$     | $\overline{m}$     | ••• | $\overline{m}$     |
|--------------|--------------------|--------------------|-----|--------------------|
| $v_{\ell} =$ | :                  | :                  | ·   | ÷                  |
|              | $\overline{m-r+1}$ | $\overline{m-r+1}$ |     | $\overline{m-r+1}$ |

where  $\ell$  is sufficiently large.

For a state  $p = v^{\otimes \infty} \otimes b_1 \otimes \cdots \otimes b_N \otimes v^{\otimes \infty}$ , we define the *time evolution operator*  $T_{\ell}(p)$  by

$$T_{\ell}(p) \otimes v_{\ell} = R(v_{\ell} \otimes p)$$

given by the appropriate composition of R-matrices. This is well-defined because we have



 $R(v_{\ell} \otimes v) = v \otimes v_{\ell}$  and eventually the carrier returns back to  $v_{\ell}$ . More precisely, we can represent this pictorially as



where each crossing is an application of the R-matrix.

**Example 3.1.** Fix some r < m. Consider some  $b \in B^{r,1}$  such that  $b \neq v$ , and consider the (truncated) state  $v^{\otimes N} \otimes b \otimes v^{\otimes L}$ . The *R*-matrix acts as the identity on the vacuum,  $R(v_{\ell} \otimes v) = v \otimes v_{l}$ . Thus, the *R*-matrix acts on each vacuum, leaving it and the carrier unchanged, until we reach the non-vacuum element *b*. Here, we have

$$R(v_{\ell} \otimes b) = v \otimes v v \cdots v b$$
.

We see b has been replaced by a vacuum and picked up by the carrier in the rightmost column. Now the carrier moves to the next vacuum

We see the carrier unloads b and returns to its initial state. We then obtain the state

$$v^{\otimes N+1} \otimes b \otimes v^{\otimes L-1}.$$

Thus, we see that b has propagated to the right with speed 1.

#### 4 Solitons

We say a state in the SBBS has *solitonic behaviour* if the following conditions are satisfied.

1. Groups of size s of adjacent column tableaux not equal to the vacuum v, move with speed s when far apart. Such a group is called a *soliton*.



2. Solitons maintain their sizes after collisions.

Note that the values in the columns are free to be exchanged between the interacting bodies.

#### 4.1 Satisfying the speed condition

We now give the structural requirements for satisfying the speed condition.

Theorem 4.1. Consider



Then,

$$T_{\ell}^{t}(u_{1}^{\otimes c} \otimes x \otimes u_{1}^{\otimes \infty}) = u_{1}^{\otimes (c+t\min\{d,\ell\})} \otimes x \otimes u_{1}^{\otimes \infty}.$$

In order to prove this theorem, first construct the highest weight semistandard group of column tableaux. Using the insertion rule involving the *R*-matrix, then show how the carrier and *R*-matrix operate on the system in a series of loading and unloading steps. Since crystal  $(B^{r,1})^{\otimes s}$  is connected (Kwon and Okado, 2020) and the *R*-matrix commutes with the Kashiwara operators. This proves the result for all semistandard groups of column tableaux.

#### 4.2 Collision Stability

In order to describe the structural requirements for collision stability, we must specify structure of every group of elements in the system. This is because no group of elements acts solitonically universally. From extensive examples generated using SAGEMATH (Developers, 2020), we observe a requirement of separation between values greater than or equal to  $\overline{m-r}$  and less than  $\overline{m-r}$  among the interacting groups. Our current results are given in the following theorem.



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#### **Theorem 4.2.** Let r < m. Consider solitons of the form

$$x = \boxed{\begin{array}{c}x_{11}\\x_{21}\\\vdots\\x_{r1}\end{array}} \otimes \boxed{\begin{array}{c}x_{12}\\x_{22}\\\vdots\\x_{r2}\end{array}} \otimes \cdots \otimes \boxed{\begin{array}{c}x_{1s}\\x_{2s}\\\vdots\\x_{rs}\end{array}} \in (B^{r,1})^{\otimes s}$$

such that

$$x_{ij} < \overline{m-r}$$
 for all  $j$  and for  $i < r$ ,  
 $x_{rj} \ge \overline{m-r}$  for all  $j$ .

Then, these groups have solitonic behaviour.

In order to prove Theorem 4.2, we look at the highest weight system and operate the Rmatrix on the system until completion. If r < m - 1, we can reduce it to the proof given by Yamada (Yamada, 2004). For the case r = m - 1, we perform a detailed technical analysis of the evolution using similar ideas from (Yamada, 2004), where the details will be made available in the subsequent paper.

**Example 4.3.** Consider system of height r = 2, with m = n = 3. Example of solitonic behaviour

| t = 0 | • | • | • | • | $\overline{2}$ $2$ | $\overline{2}$<br>1 | •                  | •                   | $\overline{3}$<br>1 | •                   | •                | •                  | •                   | •                | •                | •                | •                   | • | • |  |
|-------|---|---|---|---|--------------------|---------------------|--------------------|---------------------|---------------------|---------------------|------------------|--------------------|---------------------|------------------|------------------|------------------|---------------------|---|---|--|
| t = 1 | • | • | • | • | •                  | •                   | $\overline{2}$ $2$ | $\overline{2}$<br>1 | •                   | $\overline{3}$<br>1 | •                | •                  | •                   | •                | •                | •                | •                   | • | • |  |
| t = 2 | • | • | • | • | •                  | •                   | •                  | •                   | $\overline{2}$ $2$  | $\overline{2}$ 1    | $\overline{3}$ 1 | •                  | •                   | •                | •                | •                | •                   |   | • |  |
| t = 3 | • | • | • | • | •                  | •                   | •                  | •                   | •                   | •                   | $\overline{2}$ 1 | $\overline{2}$ $2$ | $\overline{3}$<br>1 | •                | •                | •                | •                   | • | • |  |
| t = 4 | • | • | • | • | •                  | •                   | •                  | •                   | •                   | •                   | •                | $\overline{2}$ $2$ | •                   | $\overline{2}$ 1 | $\overline{3}$ 1 | •                | •                   | • | • |  |
| t = 5 | • | • | • | • | •                  | •                   | •                  | •                   | •                   | •                   | •                | •                  | $\overline{2}$ 1    | •                | •                | $\overline{1}$ 2 | $\overline{1}$<br>1 | • | • |  |



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Example 4.4. Example of non solitonic behaviour.

| t = 0 | • | • | $\overline{1}$<br>2 | $\overline{1}$<br>1 | •                  | •                   | •                | $\overline{1}$<br>1 | •                   | •                                   | •                                   | •                                   | •                                   | •                                   | •      | •                                   | •      | •                                   |
|-------|---|---|---------------------|---------------------|--------------------|---------------------|------------------|---------------------|---------------------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|--------|-------------------------------------|--------|-------------------------------------|
| t = 1 | • | • | •                   | •                   | $\overline{1}$ $2$ | $\overline{1}$<br>1 | •                | •                   | $\overline{1}$<br>1 | •                                   | •                                   | •                                   | •                                   | •                                   | •      | •                                   | •      | •                                   |
| t = 2 | • | • | •                   | •                   | •                  | •                   | $\overline{1}$ 2 | $\overline{1}$<br>1 | •                   | $\overline{1}$<br>1                 | •                                   | •                                   | •                                   | •                                   | •      | •                                   | •      |                                     |
| t = 3 | • | • | •                   | •                   | •                  | •                   | •                | •                   | $\overline{1}$ 2    | $\frac{\overline{3}}{\overline{1}}$ | 1<br>1                              | $\frac{\overline{2}}{\overline{1}}$ | •                                   | •                                   | •      | •                                   | •      | •                                   |
| t = 4 | • | • | •                   | •                   | •                  |                     |                  | •                   | •                   | $\frac{\overline{3}}{\overline{1}}$ | $\frac{\overline{2}}{\overline{1}}$ | $\overline{3}$<br>2                 | 1<br>1                              | $\frac{\overline{2}}{\overline{1}}$ | •      | •                                   |        | •                                   |
| t = 5 | • | • | •                   | •                   | •                  | •                   | •                | •                   | •                   | •                                   | $\frac{\overline{3}}{\overline{1}}$ | $\frac{\overline{2}}{\overline{1}}$ | •                                   | $\overline{3}$<br>2                 | 1<br>1 | $\frac{\overline{2}}{\overline{1}}$ | •      | •                                   |
| t = 6 | • | • | •                   | •                   | •                  | •                   | •                | •                   | •                   | •                                   | •                                   | $\frac{\overline{3}}{\overline{1}}$ | $\frac{\overline{2}}{\overline{1}}$ | •                                   | •      | $\overline{3}$ 2                    | 1<br>1 | $\frac{\overline{2}}{\overline{1}}$ |

## 5 Discussion and Conclusion

In this report, we described a discrete dynamical system known as the box-ball system, unveiling its rich combinatorial and crystal structure. We then generalised the super box-ball system devised by Hikami–Inoue using the KR crystals from Kwon–Okado in analogy to the generalised BBS of Yamada (Yamada, 2004). We attempted to define the structure of solitons within this system. Although we were unable to define a soliton generally within the system, we have proved some cases in which solitonic behaviour occurs in Theorem 4.2.In the future, we wish to prove the theorem for case m = r and such that there is a "split" at height k rather than height r. Further research could be done into how to define rigged configurations for  $U_q(\widehat{\mathfrak{gl}}(m|n))$  by exploiting their relationship with soliton cellular automata from (Liu and Scrimshaw, 2019). As well as comparing this system with its non-discrete counterpart, the super KdV equation.

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