



Simplified Emissions Equivalence of Greenhouse Gases and CO₂

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Vacation Research Scholarships are funded jointly by the Department of Education,
Skills and Employment and the Australian Mathematical Sciences Institute.

Abstract

Global warming is one of the biggest threats that humans face in the 21st century, and it is imperative that action is taken to reduce emissions of greenhouse gases such as Methane and Carbon Dioxide. In order to efficiently reduce the impact that humans make on the environment, proper modelling and calculations of the environment must be undertaken, and the effects of greenhouse gases must be quantified. These calculations include the ability to quantify different gases in terms of another. This report explores the mathematics which are used to calculate the effects of greenhouse gases on the warming of the earth, and explores how accurately these effects may be approximated for simplified usage in different models and frameworks such as The Kyoto Protocol.

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1 Introduction

In order to halt the rising temperature of the earth, The Kyoto Protocol establishes six greenhouse gases (GHGs) which must be controlled, including Carbon Dioxide (CO_2), Methane (CH_4), and Nitrous Oxide (N_2O). The influence of GHGs' warming of the earth is represented by radiative forcing, which is the ability for GHGs to retain solar irradiances. The approximate atmospheric content for a perturbation of a gas at time t , $M(t)$, as defined in Wigley (1998), may be linearised for small perturbations (see Appendix A), and is calculated by the time-integrated convolution of $R(t)$ and $P(t)$, with each function representing the amount of a gas left in the atmosphere after time t (its so-called response function), and the perturbation amount respectively:

$$M_x(t) = \int_0^t R_x(t - \nu)P_x(\nu)d\nu. \quad (1)$$

Radiative forcing, $F(t)$, may then be calculated by multiplying $M(t)$ by a constant representing the specific gas' radiative efficiency, α :

$$F_x(t) = \alpha_x M_x(t). \quad (2)$$

Radiative forcing cannot be directly used to solve for equivalent emissions, and different metrics such as the Global Warming Potential (GWP) are used. GWPs offer a constant value which is defined over a time horizon that scale other GHGs into a common form: generally into CO_2 -equivalent (CO_2e). A GWP-100 value of 50 for an arbitrary gas, x , can be understood to mean that gas x contributes 50-times the amount of radiative forcing than CO_2 over 100-years. The Kyoto Protocol utilises GWP values with a time horizon of 100-years to aggregate different gases into CO_2e . These CO_2e values then go on to inform policy in congruence with the transition away from fossil fuels with the least 'cost' (both economically and environmentally) in mind.

As we will show, with the use of Laplace transformations it is possible to define emission equivalences between two gases with either a sum of exponentials in the time domain, or a high-degree rational functional in the complex frequency domain. These equivalences are mathematically complex however, and thus a simplified approximation of equivalence is needed.

In this report, we explore how the use of Laplace transformations can simplify the comparison of convolution integrals pertaining to radiative forcing, and survey different functional

forms of approximations which can be used to represent equivalent emissions of greenhouse gases, in order to improve upon the current use of GWPs. A discussion will consider the advantages and disadvantages that different approximations and their functional forms offer to determine which method is most appropriate to be used for determining approximations of emissions equivalence. Finally, we will compare our approximations by determining how well they are able to reproduce radiative equivalence over different timescales.

2 Mathematics of Greenhouse Gas Emissions

2.1 Equations of Radiative Forcing

2.1.1 Response Functions

For most LLGHGs, the amount of gas that remains in the atmosphere after a perturbation follows exponential decay, approximated using a response function:

$$R_x(t) = e^{-\frac{t}{\tau_x}}, \quad (3)$$

with t representing the time in years after the emission, and τ_x representing the lifetime of gas x . Assuming a single pulse emission, the radiative forcing can be calculated using equation (2). However, for CO_2 , the response function is more complicated, and its lifetime cannot be represented by simple exponential decay. Instead, it is approximated by a sum of exponentials:

$$R_{CO_2}(t) = \alpha_0 + \sum_{i=1}^n \alpha_i e^{-\frac{t}{\tau_i}}. \quad (4)$$

where α_0 is ‘the fraction of emissions that remain permanently in the atmosphere’ and the α_i represent ‘a fraction that is associated with a certain nominal timescale τ_i ’ (Joos et al. 2013). The exact amount of increments of sums may vary depending on both literature and purpose (albeit they must satisfy $\sum_i^n \alpha_i = 1$), however for the purpose of this report, we allow $n = 3$. The choice of constants and respective τ values are calculated in different models which determine the atmospheric CO_2 after pulse emissions of CO_2 , based on current and projected redistribution and sequestration of Carbon by the ocean, ‘land biosphere, and other principal carbon reservoirs’ (Joos et al. 2013). For the purpose of this report, we use the BERN-SAR model (see Appendix B) and its associated τ and α values.

2.1.2 Radiative Forcing Formulas

As shown in (2), the amount of radiative forcing a pulse emission of a gas contributes after t -years is calculated by multiplying its response function by a constant. This may be extended beyond a pulse function, to an arbitrary function representing a perturbation, using a convolution integral incorporating an arbitrary perturbation function, $P(t)$:

$$F_x(t) = \alpha_x M_x(t) = \alpha_x \int_0^t R_x(t - \nu) P_x(\nu) d\nu. \quad (5)$$

It is important to note that a standard rule of convolution integrals is that for a convolution of a function f and kernel δ , its convolution is simply the original function f :

$$\int_0^t f(t - \nu) \delta(\nu) d\nu = f \otimes \delta = f(t).$$

2.2 Radiative Forcing Equivalences

2.2.1 Global Warming Potentials

Global Warming Potentials (GWPs) are derived from the time-integrated radiative forcing of a pulse emission of gas x relative to that of CO_2 , over a time horizon H :

$$GWP_H = \frac{\int_0^H F_x(t) dt}{\int_0^H F_{CO_2}(t) dt} = \frac{\alpha_x}{\alpha_{CO_2}} \cdot \frac{\int_0^H R_x(t) dt}{\int_0^H R_{CO_2} dt}. \quad (6)$$

While GWP-100 is the standard definition of emissions equivalence by The Kyoto Protocol, other commonly utilised GWPs are GWP-20 and GWP-500.

2.2.2 Forcing Equivalent Index

As proposed in Wigley (1998), the Forcing Equivalent Index specifies that two forcings are equivalent if their scaled concentrations are equal at all times:

$$\alpha_x M_x(t) = \alpha_x \int_0^t R_x(t - \nu) P_x(\nu) d\nu \stackrel{FEI}{=} \alpha_y \int_0^t R_y(t - \nu) P_y(\nu) d\nu = \alpha_y M_y(t). \quad (7)$$

This equivalence offers a starting point from which we can derive emissions equivalence.

3 Emissions Equivalence

GWPs' linear regime does not capture enough information to accurately equate emissions. This is because CO_2 is at equilibrium with the climate and never truly disappears; emissions accumulate indefinitely. CH_4 , however, is imbalanced with the atmosphere, and largely disappears after a timescale of decades. It is possible to visualise this difference by comparing a constant rate of emission of CO_2 with a constant rate of emission of CH_4 , and contrasting these results with the CO_2e values calculated using CH_4 's GWP-100 value.

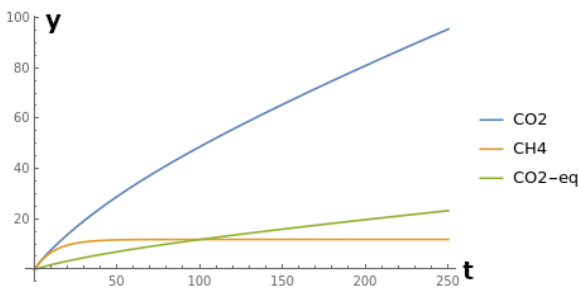


Figure 1: Gas remaining after t -years from constant CO_2 and CH_4 emissions and CO_2e emissions

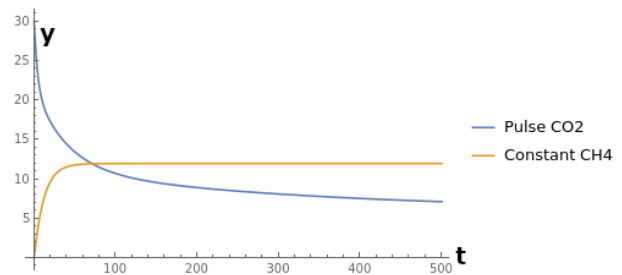


Figure 2: Gas remaining after t -years from constant CH_4 and pulse CO_2 emissions

In Figure 1, for constant emissions of CO_2 , the amount of gas in the atmosphere increases indefinitely. For constant emissions of CH_4 , the amount of gas in the atmosphere levels out after 50-years, with no increase or decrease afterwards. When applying CH_4 's GWP-100 value to constant emissions of CO_2 , the CO_2e mass of CH_4 is initially underestimated. At 100-years, as expected, the gases are equivalent. However, this graph, and thus GWP, erroneously implies that constant emissions of CH_4 would lead to a perpetual rise in the amount of CH_4 in the atmosphere, and thus radiative forcing. This phenomenon can also be seen by contrasting constant emissions of CH_4 with a one-off emission of CO_2 , such as in Figure 2. For constant emissions of an amount of CH_4 , for example 1-Tonne per year, the amount of gas in the atmosphere remains constant as the excess CH_4 is quickly cycled through the environment. Contrasting this with a single 30-Tonne release of CO_2 , it can be seen that for large values of t , the amount of gas left in the atmosphere from a single release of CO_2 behaves similarly to constant emissions of CH_4 . Given that it is clear that a single constant, such as a GWP, cannot accurately determine equivalent emissions between short-lived gases and CO_2 , this report surveys different forms of approximations which can, with low levels of error, offer emissions equivalence.

3.1 The Laplace Transformation

A function, $F(t)$, which is defined for $Re(t) \geq 0$, has a Laplace transformation of:

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} F(t) \cdot e^{-st} dt. \quad (8)$$

The Laplace transformation offers a convenient method to convert a function of a real variable into a function of a complex variable, or, from the time domain into the frequency domain respectively. The benefits of moving into the frequency domain are that convolutions are treated as multiplication, and exponential functions are treated as rational functions: Given this, we may neatly represent $M_x(t)$ as a simple multiplication:

$$\mathcal{L}\{M_x(t)\} = \tau_x(s) \cdot \rho_x(s). \quad (9)$$

As such, using Laplace transformations offer a simple framework to consider emissions equivalence using simple multiplication and division. Indeed, while it is advantageous to consider emissions in the frequency domain due to their algebraic properties, it is also favourable due to the representation of exponential decay in the time domain being converted to rational functions in the frequency domain:

$$\mathcal{L}\left\{e^{-\frac{t}{\tau_x}}\right\} = \frac{\tau_x}{\tau_x \cdot s + 1} = \frac{1}{s + \frac{1}{\tau_x}}.$$

This is particularly beneficial when it comes to approximating the ratio of two response functions as is demanded by emissions equivalences. While a sum of exponentials (or a ratio of said sum) can be approximated using, for example a Taylor's series, it is more natural to approximate the ratio of two polynomials by a related, albeit lower degree, rational function.

Another property of the Laplace Transformation is that 'information' stored in the higher domain of the t -domain are represented by inversely proportional values of s ; $1 \leq t \leq \infty$ is equivalent to $0 \leq s \leq 1$ and thus small deviations in the s -domain are represented by large deviations in the t -domain.

3.2 Equivalence Manipulation

By using the equivalence (7), we define equivalence in the s -domain using the Laplace transformation:

$$\mathcal{L}\{\alpha_x M_x\} \equiv \mathcal{L}\{\alpha_y M_y\} = \alpha_x \tau_x(s) \rho_x(s) \equiv \alpha_y \tau_y(s) \rho_y(s),$$

and allow α_x and α_y to be 1 for further analysis. Given this equivalence in the s -domain, it is simple to solve for equivalent emissions of gas x given gas y , and vice-versa, and we consider the equivalence of CO_2 and an arbitrary GHG, x , with a single lifetime:

$$p_{co_2}(s) = \frac{\tau_x(s)}{\tau_{co_2}(s)} \cdot p_x(s). \quad (10)$$

$p_{co_2}(s)$ is the equivalent perturbation of CO_2 given a perturbation of gas x , in the s -domain.

3.2.1 Response Functions in the Frequency Domain

By using standard formulae for the Laplace transformation, $\tau_{co_2}(s)$ and $\tau_x(s)$ are found.

$$\mathcal{L}\{R_x(t)\} = \tau_x(s) = \frac{\tau_x}{\tau_x \cdot s + 1}$$

$$\mathcal{L}\{R_{co_2}(t)\} = \tau_{co_2}(s) = \frac{\alpha_0}{s} + \frac{\alpha_1 \cdot \tau_1}{\tau_1 \cdot s + 1} + \frac{\alpha_2 \cdot \tau_2}{\tau_2 \cdot s + 1} + \frac{\alpha_3 \cdot \tau_3}{\tau_3 \cdot s + 1}$$

For the equivalence formula given in (10), the ratio of the two response functions being examined is algebraically equal to a rational function of degrees four. For the purpose of analysis, we introduce a new function, $\phi_x(s)$ which represents the fraction of the Laplace-transformed response functions for CO_2 referenced to a LLGHG x . In this case, $\phi(s)^{-1}$ may also be understood to be same description however of gas x -equivalent referenced to CO_2 .

$$\phi_x(s) = \frac{\tau_x(s)}{\tau_{co_2}(s)} = \frac{s(a+s)(b+s)(c+s)}{(d+s)(e+s)(f+s)(g+s)} \quad (11)$$

3.3 Methane

Methane's lifetime in the atmosphere is approximately 12-years (Wahlen 1993). Using the BERN-SAR model for lifetimes of CO_2 , we obtain the following from (11):

$$\phi_{ch_4}(s) = \frac{s \cdot (0.0000184036 + 0.00693631 \cdot s + 0.271506 \cdot s^2 + s^3)}{3.05806 \cdot 10^{-7} + 0.000231712 \cdot s + 0.0189322 \cdot s^2 + 0.277681 \cdot s^3 + s^4}.$$

The decision to restrict the domain of the function between 0 and 0.2 ($5 \leq t \leq \infty$) is due to the lack of difference in the amount of gas in the atmosphere after a perturbation of CH_4 and CO_2 . Only after 5-years does a significant difference between the amount of gas in the atmosphere for a common amount of perturbation begin to show, and it can be said that:

$$M_{ch_4}(t) \approx M_{co_2}(t) \Big|_{0 \leq t \leq 5}.$$

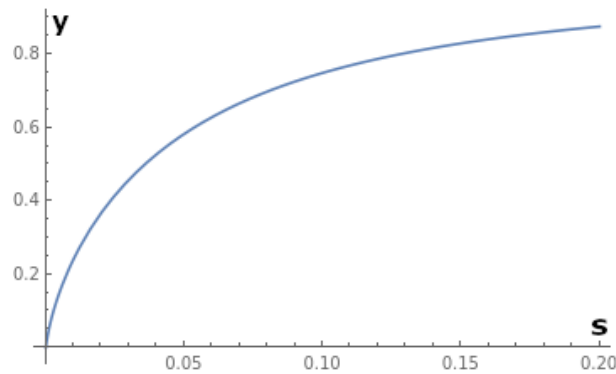


Figure 3: Exact $y = \phi_{ch4}(s)$ (11)

Following the restriction of the domain, Figure 3 clearly shows that the function is monotonically increasing with no local extrema. Consequently, it is possible to approximate this function using a lower order rational function to mimic the behaviour of the curve closely.

3.4 Forms of Approximations

Three forms (Type-A, B, and C) of approximations are surveyed: $y = a$, $y = \frac{a+b \cdot s}{c+s}$, and $y = \frac{a+b \cdot s}{1+c \cdot s+d \cdot s^2}$.

Type-A approximations are motivated by their similarities with GWPs: a constant which linearly relates different emissions. Both Form-B and Form-C were chosen by visual inspection as low order rational functions which resemble the function to be approximated.

3.5 Fitting Methodologies

With the support of Mathematica, three methods of fitting approximations are used.

3.5.1 Pade Approximants

Pade Approximants can be understood to be an approximation of a function by a rational function of a specific order, whose power series agrees with the function it is approximating. The variables for a Pade Approximant are the function to be approximated, the degrees of the approximation, and the point about which the approximation occurs.

3.5.2 The Minimax Algorithm

A Minimax algorithm approximates a function while minimising the maximum absolute error of its approximation. However, Mathematica attempts to minimise the maximum *relative* error of the approximation. The variables for Minimax minimisation are the function to be approximated, the degrees of the approximation, and the domain which the function should be approximated over. The Minimax algorithm is unable to handle functions that are undefined at any point within the fitting parameters, such as a pole. For $\phi_{ch_4}(s)$ in particular, a pole exists at $s = 0$. To overcome this, a factor of s is taken out of the function to be approximated, and the degree of the numerator of the approximation is lowered by 1. The approximation is made, and s once again multiplies the approximation.

3.5.3 The MSLE

The third method of fitting is the minimising of the mean-squared-logarithmic-error (MSLE) of the approximation. Similar to the relative error, the MSLE is mostly concerned about perceptual differences between functions and their approximations. An important quality of the minimisation of the MSLE is that it is possible to find an approximation whose reciprocal inverse agrees with the inverse of the approximated function:

$$MSLE(s) = \int (\ln \phi_{\approx ch_4}(s) - \ln \phi_{ch_4}(s))^2 ds = \int (\ln \phi_{\approx ch_4}(s)^{-1} - \ln \phi_{ch_4}(s)^{-1})^2 ds.$$

By minimising the MSLE, two approximations are found: one which approximates $\phi_{ch_4}(s)$, and one which, when its simple reciprocal inverse is taken, equally approximates $\phi_{co_2}(s)^{-1}\phi_{ch_4}(s)^{-1}$. This symmetric behaviour allows us to simply move back and forth between CO_2 and CH_4 .

3.6 Fitting in the s-domain

3.6.1 Fitting Using Pade Approximants

The Pade Approximant method is utilised around the point $s = 0.05$. The reason to approximate about this point is due to the method's behaviour in approximating around a specific point, and thus while it will approximate exactly at $s = 0.05$ ($t = 20$), it will approximate with similar accuracy at two points $s = 0.05 \pm a$ where a is some small arbitrary amount such as 0.02; $s = 0.03$ and $s = 0.07$ ($t = 33$ and $t = 14$ respectively).

3.6.2 Fitting Using Minimax

Using the Minimax method, the function is approximated over $0 \leq s \leq 1$. While it may seem counter-intuitive to fit the function over an interval that is not of particular interest ($0.2 \leq s \leq 1$), the decision to ‘extend’ the fit to $s = 1$ is to apply a bias for higher values of s , and thus lower values of t . The reasoning for applying this bias is that the Minimax method treats any interval over which it is minimising as equivalent; for example, $0 \leq s \leq 0.1$ and $0.1 \leq s \leq 0.2$ are treated equally. However, values of s are not equal in the t -domain and are not directly proportional. This means an inherent bias corresponding to longer times is approximated: while $0.1 \leq s \leq 0.2$ contains just 5-years of information, $0 \leq s \leq 0.1$ contains infinite information. By fitting to $s = 1$, the algorithm which minimises the function applies more weight to higher values of s , and thus lower values of t , balancing out the fit.

3.6.3 Minimising the MSLE

To minimise the MSLE, manual fitting is undertaken using the Minimax result’s function as a starting point.

4 Discussion of Fits in the s -domain

4.0.1 Type-A Approximation

The Type-A approximations (See [Appendix C](#); Figures 6 and 11) offer limited support in approximating ϕ_{ch_4} as constants are unable to produce a curve. This means that while they offer an approximation at one exact point, they are unable to offer meaningful emissions equivalence, and correspond more closely to the GWP.

4.0.2 Type-B Approximation

Type-B approximations (see [Appendix C](#); Figures 7, 9 and 12) do exceptionally well in approximating ϕ_{ch_4} , however different methods of fits offer relative merits. While the Pade Approximant offers a nearly perfect approximation for most of the graph, it fails to approximate as well for $s \rightarrow 0$, corresponding to longer time-frames. The Minimax and the minimisation of the MSLE methods yield approximations which only slightly deviate from the original function. However,

the latter has the advantage that it requires only a single variable, 0.0316056 which is akin to the Laplace transformation of a response function for a gas with a single lifetime.

4.0.3 Type-C Approximation

By inspection, all Type-C (see [Appendix C](#); Figures 8, 10, and 13) approximations act similarly to their respective Type-B approximations. This demonstrates that raising the order of approximation is cumbersome, and does not greatly improve on the simpler Type-B approximation.

5 Fits in the t -domain

Given our approximations of emissions equivalence are in the s -domain, it is necessary to convert them back to the t -domain using the Inverse Laplace Transformation. The standard Laplace transformation of a convolution can be once again be utilised to move back to the t -domain and define a function for emissions equivalence:

$$P_{co_2}(t) = \int_0^t \Phi_{ch_4}(t - \nu) P_{ch_4}(\nu) d\nu \Big| \Phi_{ch_4}(t) = \mathcal{L}^{-1}\{\phi_{ch_4}(s)\}. \quad (12)$$

$P_{ch_4}(t)$ can be treated as an arbitrary function corresponding to a perturbation of CH_4 . Examples include $\delta(t)$, a , and $a \cdot t$, pertaining to a pulse emission, constant emissions at a per year, and changing emissions at a rate of a per year respectively.

5.1 Type-B Forms in the t -domain

Each type of approximation when converted back into the time domain yields different forms. Due to the Type-B approximations visually yielding desirable results in the s -domain, we calculate and compare this form in the t -domain using (12) and allowing $\phi_{ch_4}(s) = \frac{a+b \cdot s}{c+s}$:

$$P_{co_2}(t) = a \int_0^t e^{-c(t-\nu)} P_{ch_4}(\nu) d\nu - b \cdot c \int_0^t e^{-c(t-\nu)} P_{ch_4}(\nu) d\nu + b \int_0^t \delta(t - \nu) P_{ch_4}(\nu) d\nu. \quad (13)$$

5.2 Goodness of Fit

To determine how well the approximations are fitted, we calculate the radiative forcing from a standard perturbation, and compare it to the equivalent radiative forcing using the approximation. As this would normally be exactly equivalent, we determine the goodness of fit of our

approximation. We compare the radiative forcing for constant emission of CH_4 , and the radiative forcing from CO_2e emissions derived using the approximated $P_{co_2}(t)$. We then compare these to the GWP-defined equivalence.

5.3 Discussion of Radiative Equivalence

Comparing the approximations' abilities to reproduce equivalent radiative forcing over a 100-year (see [Appendix D](#); Figures 14, 15, and 16) time scale offers insights into the affordances that each approximation holds. Visually, the Pade Approximant is able to reproduce equivalent radiative forcing nearly perfectly. The approximations derived from the Minimax and minimising of the MSLE yield similar results in that they are able to accurately reproduce equivalent radiative forcing with a smaller margin of error than the GWP.

It is noteworthy that although each of the Type-B approximations originate from the common form $\frac{a+b.s}{c+s}$, each of the approximations have a different amount of variables. For the Pade Approximant, all three a , b , and c variables are populated with different values, whereas for the Minimax approximation, only b and c are required. For the minimisation of the MSLE, only variable c is required. Indeed, it is clear that both the Minimax approximation and the MSLE approximation are visually quite similar, despite the former requiring 2, not 1 variable, thus indicating that to approximate emissions equivalence, it is not necessary to use a 2-variable function, but instead a 1-variable function will suffice. These three variables also correspond to constant values for (13), and as each variable is removed, the approximation becomes simpler: if $a = 0$, the first integral disappears, leaving a more simple equivalence of just one integral.

By defining $b = 1$, as in the MSLE approximation in Figure 12 (see [Appendix C](#)), there is little change in the approximation, which implies emissions equivalence can be accurately approximated as:

$$P_{co_2}(t) = P_{ch_4}(t) - c \int_0^t e^{-c(t-\nu)} P_{ch_4}(\nu) d\nu.$$

5.4 Goodness of Fit Calculations

In order to quantify the goodness of each approximation over a specific time-frame, we calculate the root mean squared relative error (RMSRE) as defined in [Despotovic et al. \(2016\)](#) (see [Appendix E](#)).

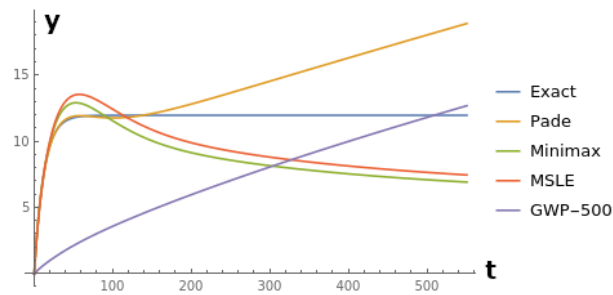


Figure 4: Approximations after 500-years.

The Pade Approximant function does much better than the other two approximations for 100-years (see [Appendix F](#)). Interestingly, the Minimax approximation does better than the other approximations for 20-years, which can likely be explained by the implicit bias applied when finding the approximation. Both the Minimax and the minimised MSLE functions have similar RMSRE values, re-affirming that these two approximations are toughly equal.

6 Discussion of Fits

6.1 Pade Approximant

While the Pade Approximant function does indeed yield a better fit over most of the function for the first 100-years, there are trade-offs that must be considered. First, a consideration for whether such an exact equivalence is necessary, and whether a complicated function is applicable for different purposes: to use this simplified form of emissions equivalence with an overall radiative forcing target in mind, three variables must be solved. Likewise as seen in Figure 4, while the function seemingly does well over $0 \leq t \leq 100$, it succumbs to the same issue that the use of the GWP metric has in that after around 150-years, forcings begin to diverge to infinity, in conflict with CH_4 's imbalance with the atmosphere. As such, the use of Pade Approximants are unlikely to be the most helpful form of approximation available for emissions equivalence on timescales of centuries.

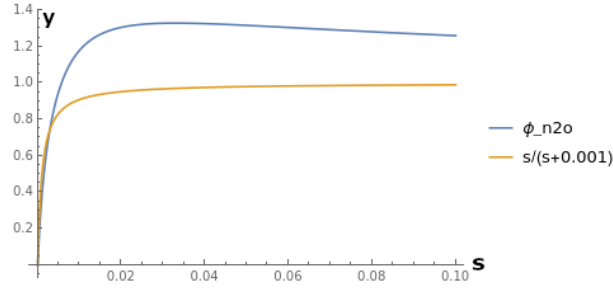


Figure 5: $y = \phi_{n_2o}$, a non-monotonic function.

6.2 Minimax and MSLE

While both the Minimax and minimisation of the MSLE handle the function similarly, the MSLE approximation has the benefit that it requires only a single variable to approximate emissions equivalence, and thus is reasonable to conclude that an approximation of the form $\phi_{ch_4} = \frac{s}{\beta+s}$ is good enough to be used for emissions equivalence between CH_4 and CO_2 . Given this, we can define emissions equivalence between CO_2 and CH_4 as the following (finally adding the constants for each gas back into the equation):

$$P_{co_2}(t) = \frac{\alpha_{ch_4}}{\alpha_{co_2}} P_{ch_4}(t) - \frac{\alpha_{ch_4}}{\alpha_{co_2}} \beta \int_0^t e^{-a(t-\nu)} P_{ch_4}(\nu) d\nu \quad (14)$$

where β is some constant which matches the behaviour of ϕ_{ch_4} depending on model for radiative forcing. This formula also has the benefit of relating pulse emissions of CH_4 to emissions of CO_2 : data which would be lost otherwise in the comparison between the two emissions.

6.2.1 Limitations

While this functional form and subsequent definition of emissions equivalence may be used for CH_4 , it is not possible to extend this for N_2O . This is because N_2O 's lifetime is approximately 110-years (Prather et al. 2015), which means the behaviour of ϕ_{n_2o} differs qualitatively from ϕ_{ch_4} (Figure 5). Notably, while ϕ_{ch_4} is strictly increasing for the domain that we are inspecting, ϕ_{n_2o} has a turning point which $\frac{s}{a+s}$ is unable to reproduce. Furthermore, the form $\frac{s}{a+s}$ has an upper limit of 1 when $a = 0$, and is unable to reproduce the behaviour of ϕ_{n_2o} within the domain we consider: it rises over 1 at approximately $s = 0.005$ ($t = 200$) and does not lower to 1 until around $s = 5$.

7 Conclusion

In this report, we have explored the possibility of improving upon the current use of GWPs by offering an alternative approximation of emissions equivalence between Carbon Dioxide and Methane. By using Laplace transformations to simplify the equations of radiative forcing, we have considered three types of functional forms to approximate the ratio of the response functions of Methane over Carbon Dioxide, and used three different methods to obtain approximations. By minimising the mean-square-error of the logarithm of the ratio of the response functions in the frequency domain, we have obtained an approximation in the form $\frac{s}{a+s}$, and subsequently identified that higher orders of approximations are superfluous. Finally, by utilising our approximation and using the inverse Laplace transformation, we have identified that with low margins of error, it is possible to approximate emissions equivalence of Carbon Dioxide and Methane using the equation (14). However, the model suggested in this report is only as accurate as the assumptions made in the calculations which were used to derive the approximation. If large deviations occur which are not projected, this model may fail, and new calculations must be made.

These approximations and their functional forms have multiple implications. Within the framework of the Kyoto Protocol, such approximations can be used to build a simplified form of emissions trading (or permit) system, allowing companies to effectively trade the rights to different gases, while accurately equating one gas to another. Such a framework requires an accurate regime which correctly values Methane in comparison to Carbon Dioxide, that a linear model such as the GWP is unfit for.

In conclusion, more research is needed to determine the best approach to an emissions trading system using the approximations derived in this report, and how well these approximations are able to fare in a framework such as that organised by the Kyoto Protocol.

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Appendix A Appendices

A.1 Appendix A

Derivation of Linear Relation of GHG

The Radiative Forcing contributed to by Carbon Dioxide is given by $\alpha \log M$ where α is some constant and M is the total mass of CO_2 in the atmosphere. M_0 is the current mass of CO_2 in the atmosphere. For small contributions (ΔM), we linearise around the M_0

$$\begin{aligned}\Delta F_{CO_2} &= F_{CO_2}(M_0 + \Delta M) - F(M_0) = \alpha \log(M_0 + \Delta M) - \alpha \log M_0 \\ &= \alpha \log\left(\frac{M_0 + \Delta M}{M_0}\right) = \alpha \log\left(1 + \frac{\Delta M}{M_0}\right)\end{aligned}$$

The Taylor's Series of $\alpha \log\left(1 + \frac{\Delta M}{M_0}\right)$ is then trivially found:

$$\Delta F_{CO_2} = \alpha \left[\frac{\Delta M}{M_0} - \frac{\Delta M^2}{2M_0} + \dots \right] \approx \alpha \frac{\Delta M}{M_0} \approx \alpha M$$

Similarly for Methane,

$$\Delta F_{CH_4} = \alpha \sqrt{M}$$

or rather

$$\Delta F_{CH_4} = \alpha \sqrt{M_0 + \Delta M} - \alpha \sqrt{M_0} = \alpha \sqrt{M_0} \left[\sqrt{1 + \frac{\Delta M}{M_0}} - 1 \right]$$

The inner square-root is then expanded yielding:

$$\Delta F_{CH_4} = \alpha \sqrt{M_0} \left[1 + \frac{\Delta M}{2M_0} + \dots - 1 \right] \approx \frac{\alpha}{2\sqrt{M_0}} \Delta M \approx \alpha M$$

A.2 Appendix B

BERN-SAR Model Coefficients from [Joos et al. \(2013\)](#)

α_0	α_1	α_2	α_3	τ_1	τ_2	τ_3
0.1994	0.1762	0.3452	0.2792	333.1	39.969	4.110

A.3 Appendix C

$\phi_{ch_4}(s)$ approximations

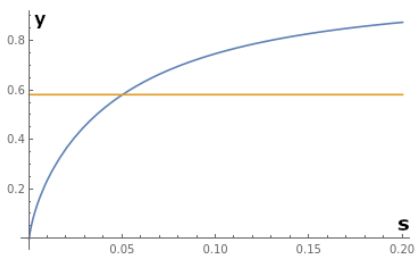


Figure 6: Type-A Pade, $y = 0.58345$

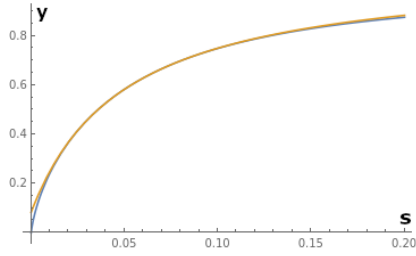


Figure 7: Type-B Pade, $y = \frac{0.0041409+1.08454s}{0.0500429+s}$

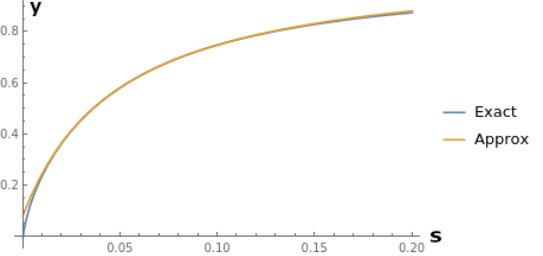


Figure 8: Type-C Pade, $y = \frac{0.200987+51.6825s}{2.39619+47.4999s+s^2}$

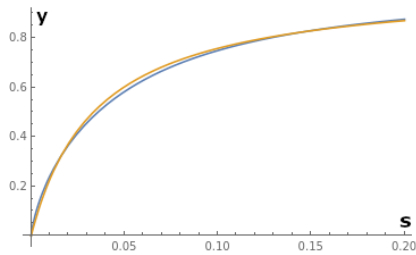


Figure 9: Type-B Minimax $y = \frac{1.02222s}{0.0347664+s}$

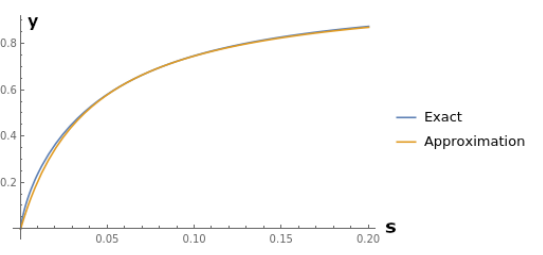


Figure 10: Type-C Minimax $y = \frac{35.8867s}{1.37857+34.0469s+s^2}$

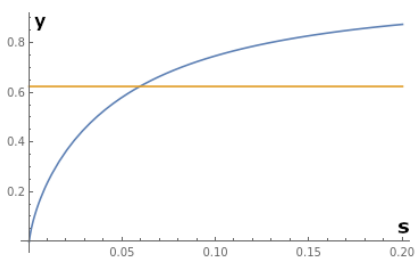


Figure 11: Type-A MSLE, $y = 0.636$

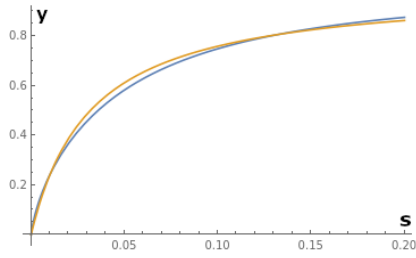


Figure 12: Type-B MSLE, $y = \frac{s}{0.0316056+s}$

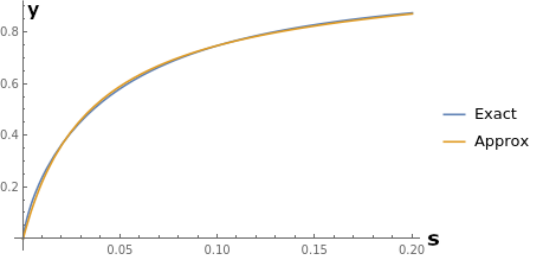


Figure 13: Type-C MSLE, $y = \frac{28.8s}{1+28.8s-4s^2}$

A.4 Appendix D

Fits against Radiative Profile

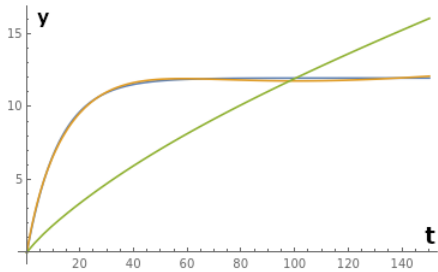


Figure 14: Type-B Pade Approximant

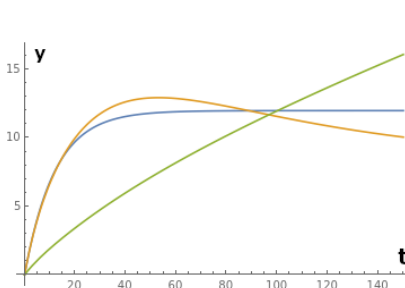


Figure 15: Type-B Minimax

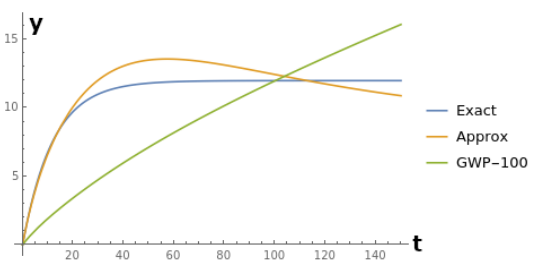


Figure 16: Type-B $Min\{MSLE\}$

A.5 Appendix E

RMSRE Formula

$$\sigma_n = \sqrt{\frac{1}{n} \int_0^n \left(\frac{F_{co_2e}(t) - F_{co_2}(t)}{F_{co_2e}(t)} \right)^2 dt.}$$

A.6 Appendix F

Goodness of Fit using RMSRE

n -years	Pade Approximant	Minimax	$Min\{MSLE\}$
20	0.02458	0.01446	0.03144
50	0.01673	0.05698	0.07761
100	0.01385	0.0545	0.08786