AMSI VACATION RESEARCH SCHOLARSHIPS 2020–21

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# Distribution of Maximal and Minimal Deviation Points in Multivariate Chebyshev Approximation Problem

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# 1 Abstract

This project takes a closer look at what connections the points of deviation between approximations and functions could arise. The points of minimal and maximal deviation in two dimensions have been plotted in python, shown to have a clear structure and be inseparable by a polynomial of the same degree as the approximation.

# 2 Introduction

Chebyshev approximation is the best polynomial approximation of a continuous function in supremum norm. This is done by constructing a polynomial of a given degree such that the deviation of maximal and minimal points is as small as possible.

In one dimension the optimal solution is characterised by the alternation condition. The alternation theorem states "Let

$$f \in C[a, b].$$

The necessary and sufficient condition for the algebraic polynomial

$$p \in P_n$$

to be a polynomial of best uniform approximation for

$$f \in P_n$$

is that $p$ realizes Chebyshev alternation for $f$ in $[a, b]$". This gives an alternating sequence of point of minimal and maximal deviation.

The alternation condition states that there must be $d + 2$ points of maximal and minimal deviation in the solution from the original function of degree $d$ being approximated.

In higher dimensions it is possible to verify optimality via a separation condition, however there is no direct analogue of the one-dimensional alternation condition.

There is no longer a uniqueness to approximations in higher dimensions. See reference 1.

# 3 Statement of Authorship

- Sona Reddy developed code in Python to generate and compare graphs, produced results and wrote this report.
- Vera Roshchina supervised this project and checked accuracy of code.
- Vinesha Peiris assisted in debugging code and supervising this project.
- Vera Roshchina and Vinesha Peiris proofread this report.

# 4 Approximation Theory

## 4.1 One Variable Approximation

In Figure 1 the function $x^9$ is approximated by the polynomial of degree 5 on the segment [0,10]. From these plots we can see the points of minimal and maximum deviation which shows that this approximation is sufficient. From the second plot it is clear that the approximation is the best approximation for this degree from the number of maximal and minimal points that can be found. Refer to Figure 1 The Vandermonde matrix $V_n$ used in

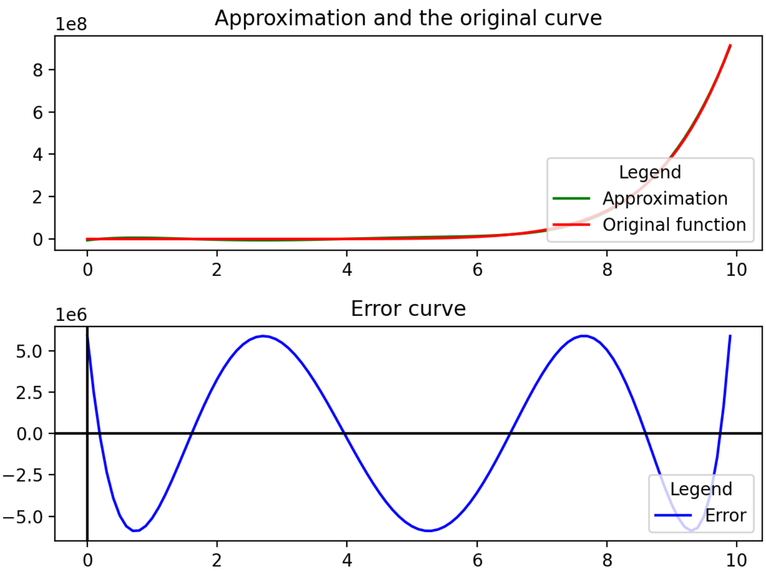


Figure 1: One variable approximation and error curves

calculations in this section is an $m \times n$ matrix with m being the number of points in the discretised domain and n being the degree plus one. Each of the m rows in this matrix is a geometric progression starting at 1 and the common ratio being $x$. A different matrix is used when dealing with two or more variables as discussed in the next section.

$$\text{minimize} \sup |f(x) - p(x)|$$

where $p$ is a polynomial of degree $d$ and $x \in C$

.

$$p(t) = a_0 + a_1 t + ... + a_d t^d = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ t \\ \vdots \\ t^d \end{bmatrix}$$

where

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix}$$

and

$$G(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^d \end{bmatrix}$$

the monomial basis.

$$\text{minimize} \sup |f(t) - aG(t)|$$

$$a \in R^{d+1}$$

$$t \in [a, b].$$

$$\text{minimize} \sup |f(x^i) - aG(x^i)|$$

$$x \in R$$

$$i \in 1, ...N.$$

$$\text{minimize}\, t$$

such that

$$|f(x^i) - aG(x^i)| \leq t.$$

$$\text{minimize}\, t$$

such that

$$|f(x^i) - aG(x^i)| \leq t.$$

$$\min \boldsymbol{c}^T \boldsymbol{x}$$

such that

$$Ax \leq B.$$

where

$$A = \begin{bmatrix} \begin{bmatrix} V_n \end{bmatrix} - 1 \\ \begin{bmatrix} -V_n \end{bmatrix} - 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} F1 \\ -F1 \end{bmatrix}$$

where

$$F1 = f(T)^T.$$

## 4.2 Two Variable Approximation

When coding the next section in python, mostly only small changes were required to account for the new $y$ variable.

For the new matrix $M_n$ that replaces the previously used Vandermonde matrix more coding was required to contain the values of monomials at all points in the specified domain. In this new matric each column contains an evaluation of the monomial basis ($G(x, y)$ as shown below for two variables which was used in this project in python code) at a point in the domain. Hence all of the columns together evaluate all the points in the domain. Numpy package as well as the cvxopt package in python were used to solve the optimization problem defined above which is used to generate plots. The package matplotlib.pyplot was used to create the plots shown below.

$$\text{minimize} \sup |f(x, y) - p(x, y)|$$

where

$$p \in P^n$$

and

$$(x, y) \in C.p(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + ... + a_{1(n-1)}xy^{n-1} + a_{0n}y^n$$

where

$$a = \begin{bmatrix} a_{00} \\ a_{10} \\ \vdots \\ a_{0n} \end{bmatrix}$$

and

$$G(x, y) = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ \vdots \\ x^n \\ \vdots \\ y^n \end{bmatrix}.$$

$$\text{minimize} \sup |f(x, y) - aG(x, y)|$$
$$a \in R^M$$
$$(x, y) \in C.$$

$$\text{minimize} \sup |f(x^i, y^i) - aG(x^i, y^i)|$$
$$a \in R^M$$
$$i \in 1, ...N.$$

$$\text{minimize}\, t$$

such that

$$|f(x^i, y^i) - aG(x^i, y^i)| \leq t.$$

$$\text{minimize}\, t$$

such that

$$|f(x^i, y^i) - aG(x^i, y^i)| \leq t.$$

$$\min \boldsymbol{c}^T \boldsymbol{x}$$

such that

$$Ax \leq B.$$

where

$$A = \begin{bmatrix} \begin{bmatrix} M_n \end{bmatrix} - 1 \\ \begin{bmatrix} -M_n \end{bmatrix} - 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} F1 \\ -F1 \end{bmatrix}$$

where

$$F1 = f(T)^T.$$

## 4.3 Python Plots Rectangular Domain

Various functions and different domains were then plotted. Each function shows the original function, the approximation of this function to a set constant degree, the deviation of the approximation from the original function and finally a two dimensional plot of the maximum and minimum points of deviation.

In Figure 2 the function $\tan(x) + \tan(y)$ is approximated by the polynomial of degree 3 on the plane [-1,1] on both $x$ and $y$ axes.

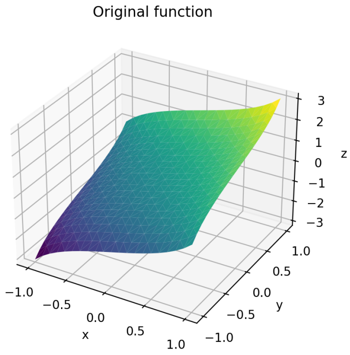


Figure 2: Plot of $\tan(x) + \tan(y)$

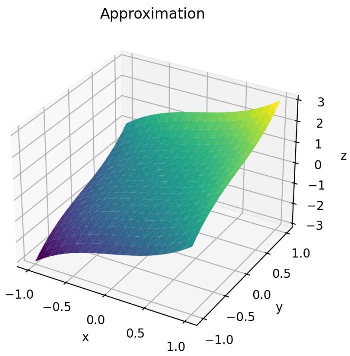


Figure 3: Approximation plot of $\tan(x) + \tan(y)$

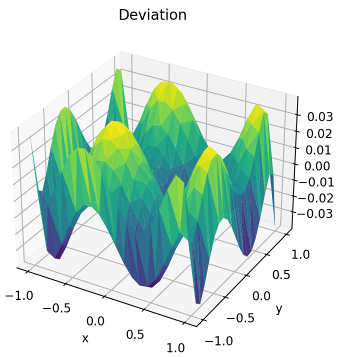


Figure 4: Deviation plot of $\tan(x) + \tan(y)$

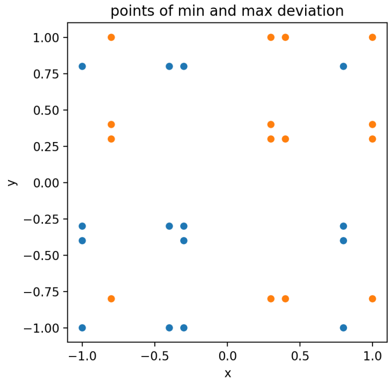


Figure 5: Points of minimum and maximum deviation of $\tan(x) + \tan(y)$

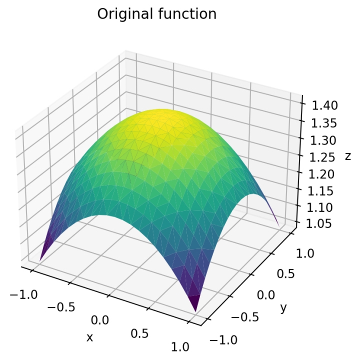


Figure 6: Plot of $(cos(x) + cos(y))^{1/2}$

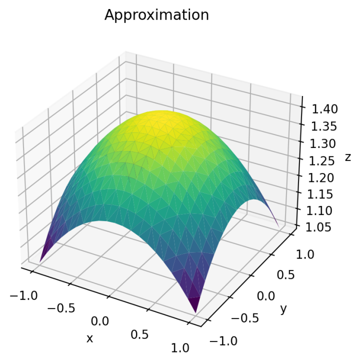


Figure 7: Approximation plot of $(cos(x) + cos(y))^{1/2}$

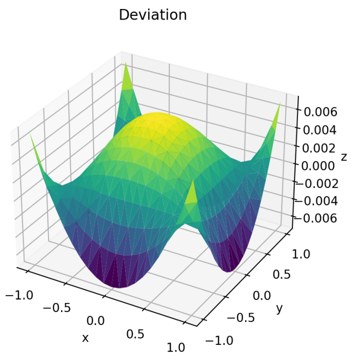


Figure 8: Deviation plot of $(cos(x) + cos(y))^{1/2}$

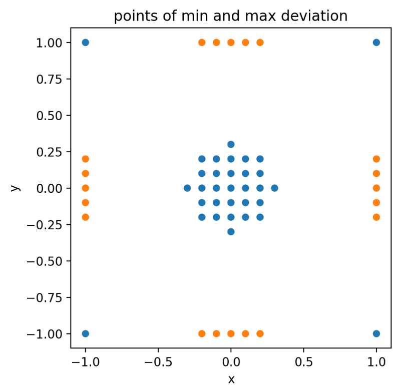


Figure 9: Points of minimum and maximum deviation of $(cos(x) + cos(y))^{1/2}$

## 4.4 Python Plots Disc Domain

To compare plots to a different domain the cosine function plot is shown below over a disc domain to show the difference in shape of the plot of points of maximum and minimum deviation.

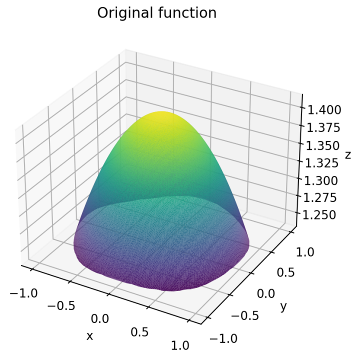


Figure 10: Plot of $(cos(x) + cos(y))^{1/2}$

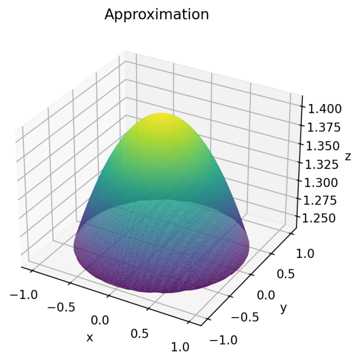


Figure 11: Approximation plot of $(cos(x) + cos(y))^{1/2}$

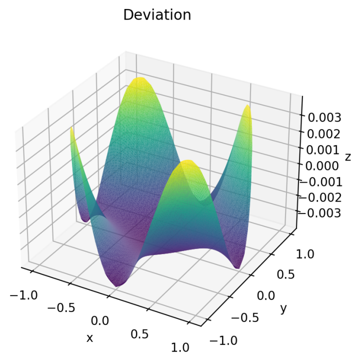


Figure 12: Deviation plot of $(cos(x) + cos(y))^{1/2}$

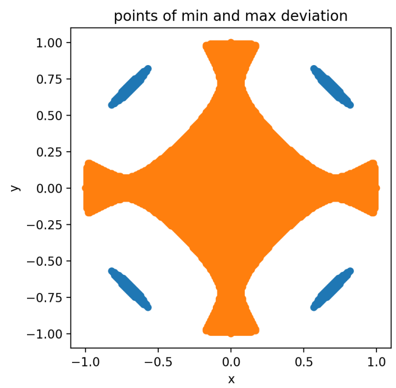


Figure 13: Points of minimum and maximum deviation of $(cos(x) + cos(y))^{1/2}$

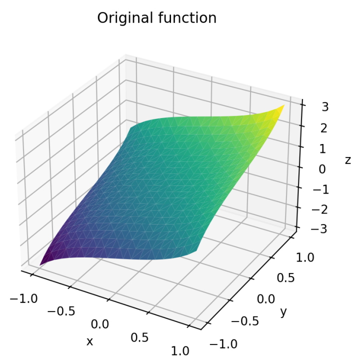


Figure 14: Plot of tan(x) + tan(y)

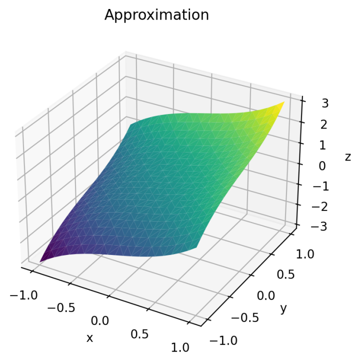


Figure 15: Approximation plot of tan(x) + tan(y)

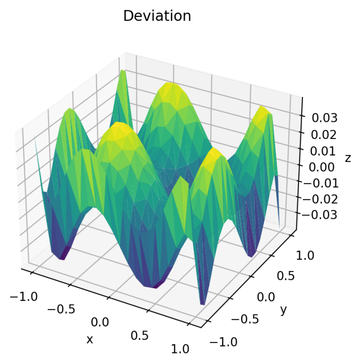


Figure 16: Deviation plot of tan(x) + tan(y)

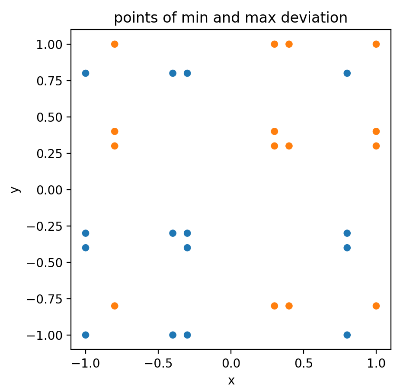


Figure 17: Points of minimum and maximum deviation of tan(x) + tan(y)

# 5 Discussion and Conclusion

This report has covered the basic approximation and theories in one dimension and ventured to higher dimensions.

There are many interesting patterns that arise when looking at plots of approximation in higher dimensions. The maximal and minimum deviation points have been graphed and compared, the difference in shapes and patterns of the points can be seen in the graphs.

The points of minimal and maximal deviation have been shown to be inseparable by a polynomial of the same degree as the approximation. It is also been shown that there is clearly structure and patterns in the distribution of points and further investigation is worth looking into to discover more patterns.

# 6 References


1. Roshchina, Sukhorukova and Ugon 2019, 'Uniqueness of solutions in multivariate Chebyshev approximation problems'

2. J. P. Thi and C. Detaille, 'On Real and Complex-Valued Bivariate Chebyshev Polynomials'