Glasner Property for (Semi-)Group Actions

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Vacation Research Scholarships are funded jointly by the Department of Education, Skills and Employment and the Australian Mathematical Sciences Institute.

Abstract

We study the Glasner property for both the semi-group $M_n(\mathbb{Z})$ and the group $SL_n(\mathbb{Z})$ acting on the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. In the case of the integers acting on $\mathbb{R}/\mathbb{Z}$ by multiplication, a quantitative variant of the Glasner property has been previously established, giving guarantees for all large enough finite subsets being dilated by some integer factor to become ϵ-dense for small ϵ. This report seeks to achieve the same goal in the cases of higher dimensional variants of the problem. While this report establishes the quantitative Glasner property for these groups, we have yet to show if the results provided are optimal.

Introduction/Background

First, we define the Glasner property.

Definition (Glanser Property)**.** *Let G be a (semi-)group with an action on some metric space X. Then $G \curvearrowright X$ has the Glasner Property if for any infinite subset $S \subseteq X$ and any $\epsilon > 0$, there exists some $g \in G$ such that gS is an ϵ-dense subset of X.*

One example of a group and metric space that has the Glasner property is $\mathbb{Z}$ acting on $\mathbb{R}/\mathbb{Z}$ by multiplication [3]. This can be thought of as taking points on a circle and multiplying each point's angle by some integer factor to get a set with points in every ϵ neighbourhood on the circle.

While this is an interesting statement, one question that arises naturally is regarding the need for infinite sets, while the ϵ-density condition doesn't explicitly require this. If we fix $\epsilon > 0$, can we find an ϵ-dense dilation of subsets that only contain a finite number of points? If so, what further conditions do we require on this subset?

These questions have been answered thoroughly in the case of $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$ by Alon and Peres [1]. They produce what is called the *quantitative Glasner property.*

Theorem (Quantitative Glasner Property for $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$ [1])**.** *For any choice of $\alpha > 0$ fixed, there exists some $\epsilon_\alpha > 0$ such that for any $0 < \epsilon < \epsilon_\alpha$ and any $S \subseteq \mathbb{R}/\mathbb{Z}$ with cardinality at least $1/\epsilon^{2+\alpha}$, there exists a $n \in \mathbb{Z}$ such that nS is ϵ-dense.*

This statement can be proven through two different methods; harmonic analysis and the second moment method. We will look at the second moment method only.

Of interest in the above theorem is that the only condition required is that the subset of $\mathbb{R}/\mathbb{Z}$ is sufficiently large. Let $k(\epsilon)$ be the minimal cardinality such that the quantitative Glasner property is always satisfied. This theorem establishes that in the case of $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$, $k(\epsilon) \leq 1/\epsilon^{2+\alpha}$. It can be demonstrated that this is sharp up to the α term. That is,

$$k(\epsilon) \geq \Omega\left(\frac{1}{\epsilon^2}\right) \qquad ([1])$$

This is shown by considering the set of reduced fractions with denominators smaller than $1/\epsilon$. It's known that the size of this set is bounded from below by some factor of $1/\epsilon^2$. Furthermore, any integer dilation will never have points intersecting the interval $(0, \epsilon)$ by construction, so there cannot be an ϵ-dense dilation.

While this collection of statements resolves many questions regarding $\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$, it says little about other spaces where the Glasner property holds. For example, what is $k(\epsilon)$ for the case of $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$, as this is another example of where the original Glasner property holds.

In this report, we will establish the quantitative Glasner property on two higher dimensional variants; $M_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ and $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$. In these cases, we find an upper bound on $k(\epsilon)$ using a probabilistic approach similar to that of Alon and Peres original one dimensional proof. We will also discuss the lower bound on $k(\epsilon)$ as well.

Statement of Authorship

The mathematical details of this work and writing of this report have been performed by Rajchert, with guidance and supervision by Fish and Badziahin.

Upper Bound on $k(\epsilon)$

Rather than dealing with $M_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ initially, we instead find a quantitative Glasner property for $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$, and use the same bound for general matrices as a direct corollary. Although the proof for $M_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ is much simpler, it achieves the same bound as that of $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ and hence is not worth examining in addition to the $SL_n(\mathbb{Z})$ case.

It is already known that for $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$, we have $k(\epsilon) \leq 1/\epsilon^{2(a+1)n}$ where a is some constant dependent on ϵ, larger than 1 [2]. We will improve on this significantly.

Theorem (Quantitative Glasner Property for $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$)**.** *For any choice of $\alpha > 0$ fixed, there exists some $\epsilon_\alpha > 0$ such that for any $0 < \epsilon < \epsilon_\alpha$ and any $S \subseteq \mathbb{R}^n/\mathbb{Z}^n$ with cardinality at least $1/\epsilon^{2n+\alpha}$, there exists a matrix $A \in SL_n(\mathbb{Z})$ such that AS is ϵ-dense in $\mathbb{R}^n/\mathbb{Z}^n$.*

Proof. We use the second moment method to prove this with a methodology very similar to that of Alon and Peres (1992). Furthermore, for the sake of brevity, we will only prove the case for $n = 2$ here. While the statement is true for $n > 2$, the proof is currently far too cumbersome for the purposes of this report.

First assume that the given set S is finite, as otherwise the statement is obvious from the fact that $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$ has the original Glasner property. Suppose that $|S| = k$, and label the points inside it by $x_i \in S, 1 \leq i \leq k$.

Now, fix M to be some large integer, and let a_1, a_2, t be three independent and identically distributed random variables such that for any integer m between 1 and M, $P(a_1 = m) = 1/M$, and 0 otherwise. Also let b be a random vector in $\mathbb{R}^2$ such that the elements of b are independent and identically distributed continuous uniform random variables from 0 to 1. Now let $Z_i = Ax_i + b \in \mathbb{R}^2/\mathbb{Z}^2$ where

$$A = \begin{bmatrix} a_1t + 1 & t \\ a_1a_2t + a_1 + a_2 & a_2t + 1 \end{bmatrix}.$$

Clearly $\det(A) = 1$, so $A \in SL_2(\mathbb{Z})$ for any choice of a_1, a_2, t. Let I be some square in $\mathbb{R}^2/\mathbb{Z}^2$ with length ϵ edges, and let $Y_i^I = 1$ if $Z_i \in I$, otherwise $Y_i^I = 0$. Finally, let $Y^I = \sum_{i=1}^k Y_i^I$.

If there exists a matrix A of the above form such that the set of $Ax_i, 1 \leq i \leq k$ is ϵ-dense, then the probability of the Z_i being ϵ-dense must be positive given density is translation invariant. Framing this in terms of Y^I, let F be a collection of $\lceil 1/\epsilon^2 \rceil$ cubes covering $\mathbb{R}^2/\mathbb{Z}^2$. Then, we would like to show

$$P(\forall I \in F, Y^I \neq 0) > 0$$
$$\iff P(\exists I \in F, Y^I = 0) < 1$$

Let $I^* \in F$ be the square such that for any interval $I \in F$, $P(Y^{I^*} = 0) \geq P(Y^I = 0)$ (i.e. the interval that maximises the probability of $Y^I = 0$.) Then by Chebyshev's inequality we note,

$$P(\exists I \in F, Y^I = 0) \leq \frac{1}{\epsilon^2} P(Y^{I^*} = 0) \leq \frac{V(Y^{I^*})}{\epsilon^2 E(Y^{I^*})^2}$$

It follows that for ϵ density [1], it is sufficient to show

$$\frac{V(Y^{I^*})}{\epsilon^2 E(Y^{I^*})^2} < 1$$

We now work to bound the expectation and variance of Y^I, independently of the interval.

Since $Z_i = Ax_i + b$ is a uniformly distributed random variable for any fixed A, it must be uniformly distributed across $\mathbb{R}^2/\mathbb{Z}^2$ over random A. Since I has an ϵ^2 area, it follows that $E(Y^I) = kE(Y_i^I) = k\epsilon^2$.

[1] This will actually show that we have a $\sqrt{5}\epsilon$-dense set, but the constant doesn't matter as it can be incorporated into the α term later.

For the variance,

$$\begin{aligned} V(Y^I) &= \sum_{i=1}^{k} V(Y_i^I) + 2\sum_{1\le i<j\le k} COV(Y_i^I, Y_j^I) \\ &< k\epsilon^2 + 2\sum_{1\le i<j\le k} COV(Y_i^I, Y_j^I) \end{aligned}$$

Noting that $COV(Y_i^I, Y_j^I) = E(Y_i^I Y_j^I) - E(Y_i^I)E(Y_j^I) = P(Y_i^I = 1, Y_j^I = 1) - \epsilon^4$, it follows we need to bound this probability term. Making use of the total law of probability,

$$P(Z_i \in I, Z_j \in I) = \sum_{1\le a_1,a_2,t\le M} P(Ax_i + b \in I, Ax_j + b \in I | \text{The elements of A are fixed}) \frac{1}{M^3}$$

Consider each coordinate of $Ax_i + b$ and $Ax_j + b$ independently. We want the probability of some uniformly distributed random variable being in some ϵ width interval, in addition to it remaining in the ϵ width interval after a given translation by $(Ax_i)_l - (Ax_j)_l$. It follows that we simply require the uniformly distributed random variable to fall into a smaller interval, based on the size of this translation. More specifically, the probability for the l-th coordinate is given by $\psi_\epsilon((Ax_i)_l - (Ax_j)_l)$ where $\psi_\epsilon(x) = \max\{0, \epsilon - |x|\}$, repeating periodically with period 1. Hence,

$$P(Z_i \in I, Z_j \in I) = \frac{1}{M^3} \sum_{1\le a_1,a_2,t\le M} \psi_\epsilon((AD)_1)\psi_\epsilon((AD)_2)$$

We now deal with several cases.

Case 1: D_1 is irrational. Looking at the first coordinate,

$$\begin{aligned} (AD)_1 &= (a_1 t + 1)D_1 + tD_2 \\ &= a_1(tD_1) + D_1 + tD_2 \end{aligned}$$

Since $tD_1 \pmod 1$ is irrational for all t, it follows that as we vary a_1, we find $a_1(tD_1)$ is equidistributed on $\mathbb{R}/\mathbb{Z}$, and hence the first coordinate must be equidistributed for every fixed t. For the second coordinate,

$$\begin{aligned} (AD)_2 &= (a_1a_2t + a_1 + a_2)D_1 + (a_2t + 1)D_2 \\ &= a_2((a_1t + 1)D_1 + tD_2) + a_1D_1 + D_2 \end{aligned}$$

Note that for any fixed value of t, there is at most 1 value of a_1 such that $(a_1t + 1)D_1 + tD_2$ is rational. Since we iterating t and a_1 over a very large range, this has no overall effect when we average over these values, so we only need to deal with the case that it is irrational. Clearly we must then have $a_2((a_1t + 1)D_1 + tD_2)$ being equidistributed on $\mathbb{R}/\mathbb{Z}$ as we vary a_2 and keep a_1, t fixed, and it follows that the second coordinate is equidistributed.

We now have both coordinates being equidistributed. Furthermore, we have the first coordinate being equidistributed for any fixed value of t as we vary a_1, and the second coordinate being equidistributed for almost every set of fixed t and a_1, as we vary a_2. It follows by the independence of the indexing that we have equidistribution on $\mathbb{R}^2/\mathbb{Z}^2$.

Applying the Riemann integral criterion for equidistribution,

$$\begin{aligned}\lim_{M\to\infty}\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\psi_\epsilon((AD)_1)\psi_\epsilon((AD)_2) &= \int_0^1\int_0^1 \psi_\epsilon(x)\psi_\epsilon(y)dxdy \\ &= \epsilon^4\end{aligned}$$

It follows that in this case, the covariance approaches 0.

Case 2: D_2 is irrational. We can assume that D_1 is rational, as otherwise we can resort to case 1. With these assumptions, we again investigate each coordinate. The first coordinate is

$$\begin{aligned}(AD)_1 &= (a_1t+1)D_1 + tD_2 \\ &= t(a_1D_1 + D_2) + D_1\end{aligned}$$

Since $a_1D_1 + D_2$ is irrational, this is clearly equidistributed on $\mathbb{R}/\mathbb{Z}$ for all fixed a_1 as we vary t. For the second coordinate,

$$\begin{aligned}(AD)_2 &= (a_1a_2t + a_1 + a_2)D_1 + (a_2t+1)D_2 \\ &= a_2((a_1t+1)D_1 + tD_2) + a_1D_1 + D_2\end{aligned}$$

Again, we have $(a_1t+1)D_1 + tD_2$ being irrational for all a_1, t and hence we have equidistribution of the second coordinate for all fixed a_1, t as we vary a_2. Since we have each coordinate being equidistruted according to a different index, we must have equidistribution on $\mathbb{R}^2/\mathbb{Z}^2$. Applying the Riemann integral criterion gives the covariance approaching 0 as $M \to \infty$, as in case 1.

Case 3: Both D_1 and D_2 are rational, and D_1 is non-zero. We will write $D_1 = \frac{c}{d}$ where c and d are coprime positive integers. Now note that via a Fourier series,

$$\psi_\epsilon(x) = \epsilon^2 + \sum_{p=1}^{\infty} F_p\cos(2\pi px),$$
$$F_p = \frac{2\sin^2(\pi p\epsilon)}{\pi^2p^2}$$

Since this series converges uniformly, we can manipulate it in a variety of ways. Implementing this in our sum,

$$
\begin{aligned}
&\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\psi_\epsilon((AD)_1)\psi_\epsilon((AD)_2)\\
&=\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\Big(\epsilon^2+\sum_{p=1}^{\infty}F_p\cos(2\pi p(AD)_1)\Big)\Big(\epsilon^2+\sum_{q=1}^{\infty}F_q\cos(2\pi q(AD)_2)\Big)\\
&=\epsilon^4+\epsilon^2\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\Big(\sum_{p=1}^{\infty}F_p\cos(2\pi p(AD)_1)+\sum_{q=1}^{\infty}F_q\cos(2\pi q(AD)_2)\Big)\\
&+\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\Big(\sum_{p=1}^{\infty}F_p\cos(2\pi p(AD)_1)\Big)\Big(\sum_{q=1}^{\infty}F_q\cos(2\pi q(AD)_2)\Big)\\
&=\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)+\cos(2\pi p(AD)_2)\Big)\\
&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)\cos(2\pi q(AD)_2)\Big)
\end{aligned}
$$

When we take $M\to\infty$, due to absolute convergence we can take the limit inside the sums for each term and evaluate them individually. For the first sum dependent on M,

$$
\begin{aligned}
&\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)+\cos(2\pi p(AD)_2)\\
&\le\frac{1}{M^3}\Big|\sum_{1\le a_1,a_2,t\le M}\exp(2\pi ip((a_1t+1)\frac{c}{d}+tD_2))+\exp(2\pi ip((a_1a_2t+a_1+a_2)\frac{c}{d}+(a_2t+1)D_2))\Big|\\
&\le\frac{1}{M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi ip((a_1t+1)\frac{c}{d}+tD_2))\Big|+\Big|\sum_{a_1=1}^{M}\exp(2\pi ip((a_1a_2t+a_1+a_2)\frac{c}{d}+(a_2t+1)D_2))\Big|\Big)\\
&=\frac{1}{M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi ipa_1t\frac{c}{d})\Big|+\Big|\sum_{a_1=1}^{M}\exp(2\pi ipa_1(a_2t+1)\frac{c}{d})\Big|\Big)
\end{aligned}
$$

Observe that the terms of the inner sum are cyclic over a_1, a_2 and t with period d. It follows that as $M\to\infty$, the limit will approach the sum over just a single cycle. Now taking $M\to\infty$,

$$
\begin{aligned}
&\to\frac{1}{d^3}\sum_{1\le a_2,t\le d}\Big(\Big|\sum_{a_1=1}^{d}\exp(2\pi ipa_1t\frac{c}{d})\Big|+\Big|\sum_{a_1=1}^{d}\exp(2\pi ipa_1(a_2t+1)\frac{c}{d})\Big|\Big)\\
&=\frac{1}{d^2}\sum_{1\le a_2,t\le d}1_{\{pt\text{ is a multiple of }d\}}+1_{\{p(a_2t+1)\text{ is a multiple of }d\}}
\end{aligned}
$$

For pt to be a multiple of d, we simply require t to be a multiple of $d/\gcd(p,d)$ which will happen exactly $\gcd(p,d)$ times as t varies from 1 to d. For $p(a_2t+1)$ to be a multiple of d,

$$
\begin{aligned}
&p(a_2t+1)\equiv 0\pmod d\\
\iff&\frac{p}{\gcd(p,d)}(a_2t+1)\equiv 0\pmod{d/\gcd(p,d)}\\
\iff&a_2t\equiv -1\pmod{d/\gcd(p,d)}
\end{aligned}
$$

This will have a solution if and only if a_2 has a multiplicative inverse, which is equivalent to saying a_2 is coprime to $d/\gcd(p,d)$. Even so, this will lead to a unique solution for t between 1 and $d/\gcd(p,d)$, hence there is at most $\gcd(p,d)$ solutions for t. Using this bound,

$$\begin{aligned}&\frac{1}{d^2}\sum_{1\le a_2,t\le d} 1_{\{pt \text{ is a multiple of } d\}} + 1_{\{p(a_2t+1) \text{ is a multiple of } d\}}\\
&\le \frac{1}{d^2}\sum_{a_2=1}^{d} 2\gcd(p,d)\\
&= \frac{2\gcd(p,d)}{d}.\end{aligned}$$

Now for the second series, we approach this in the same manner.

$$\begin{aligned}&\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)\cos(2\pi q(AD)_2)\\
&=\frac{1}{2M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi(p(AD)_1+q(AD)_2))+\cos(2\pi(p(AD)_1-q(AD)_2))\\
&\le\frac{1}{2M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi i(p((a_1t+1)\frac{c}{d}+tD_2)+q((a_1a_2t+a_2+a_2)\frac{c}{d}+(a_2t+1)D_2))\Big|\\
&+\Big|\sum_{a_1=1}^{M}\exp(2\pi i(p((a_1t+1)\frac{c}{d}+tD_2)-q((a_1a_2t+a_1+a_2)\frac{c}{d}+(a_2t+1)D_2))\Big|\Big)\\
&=\frac{1}{2M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi i(pa_1t\frac{c}{d}+qa_1(a_2t+1)\frac{c}{d})\Big|\\
&+\Big|\sum_{a_1=1}^{M}\exp(2\pi i(pa_1t\frac{c}{d}-qa_1(a_2t+1)\frac{c}{d})\Big|\Big)\\
&=\frac{1}{2M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi ia_1(pt+q(a_2t+1))\frac{c}{d})\Big|+\Big|\sum_{a_1=1}^{M}\exp(2\pi ia_1(pt-q(a_2t+1))\frac{c}{d})\Big|\Big)\end{aligned}$$

Again, observe this is periodic over each index with period d, hence as $M\to\infty$, so as we take $M\to\infty$, we only need to consider a single period as we average.

$$\begin{aligned}&\frac{1}{2M^3}\sum_{1\le a_2,t\le M}\Big(\Big|\sum_{a_1=1}^{M}\exp(2\pi ia_1(pt+q(a_2t+1))\frac{c}{d})\Big|+\Big|\sum_{a_1=1}^{M}\exp(2\pi ia_1(pt-q(a_2t+1))\frac{c}{d})\Big|\Big)\\
&\to\frac{1}{2d^3}\sum_{1\le a_2,t\le d}\Big(\Big|\sum_{a_1=1}^{d}\exp(2\pi ia_1(pt+q(a_2t+1))\frac{c}{d})\Big|+\Big|\sum_{a_1=1}^{d}\exp(2\pi ia_1(pt-q(a_2t+1))\frac{c}{d})\Big|\Big)\\
&=\frac{1}{2d^2}\sum_{1\le a_2,t\le d}1_{\{pt+q(a_2t+1)\text{ is a multiple of } d\}}+1_{\{pt-q(a_2t+1)\text{ is a multiple of } d\}}\end{aligned}$$

We wish to see how many times $pt+q(a_2t+1)\equiv 0 \pmod d$. Fixing a_2 to be constant, we solve this for t and count the number of solutions. For this, we use a lemma:

Lemma. *Consider the equation* $pt+q(at+1)\equiv 0 \pmod d$ *for some integers* a,p,q,d*. The number of solutions for* $t\in\mathbb{Z}_d$ *is at most* $\gcd(p,q,d)$

This is not obvious, and the proof of this is provided in the appendix. Furthermore, the lemma also applies to $pt-q(a_2t+1)\equiv 0 \pmod d$ by a change of variables. Applying this to our sum,

$$\begin{aligned}
&\frac{1}{2d^2}\sum_{1\le a_2,t\le d} 1_{\{pt+q(a_2t+1)\text{ is a multiple of }d\}}+1_{\{pt-q(a_2t+1)\text{ is a multiple of }d\}}\\
&\le \frac{1}{2d^2}\sum_{a_2=1}^{d} 2\gcd(p,q,d)\\
&=\frac{\gcd(p,q,d)}{d}\le\frac{\gcd(p,d)}{d}
\end{aligned}$$

Now returning to the full probability expression, we see

$$\begin{aligned}
&\lim_{M\to\infty}\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)+\cos(2\pi p(AD)_2)\Big)\\
&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)\cos(2\pi q(AD)_2)\Big)\\
&\le\epsilon^4+2\epsilon^2\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\frac{\gcd(p,d)}{d}\\
&=\epsilon^4+2\epsilon^2\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}+\Big(\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}\Big)\Big(\sum_{q=1}^{\infty}F_q\Big)
\end{aligned}$$

Note that by simply evaluating the Fourier series at $x=0$, we find

$$\sum_{q=1}^{\infty}F_q=\psi_\epsilon(0)-\epsilon^2<\epsilon$$

For the other series, we make use of another lemma:

Lemma. *For any* $\gamma>0$*, there exists a constant* C_γ *such that for any* $d\ge 1$*, we have*

$$\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}\le C_\gamma\frac{\epsilon}{d^{1-\gamma}}$$

The proof of this is included in the appendix. Implementing this in our sum, we find that for any $\gamma>0$, there exists a C_γ such that

$$\begin{aligned}
&\epsilon^4+2\epsilon^2\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}+\Big(\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}\Big)\Big(\sum_{q=1}^{\infty}F_q\Big)\\
&\le\epsilon^4+C_\gamma\epsilon^3d^{\gamma-1}+C_\gamma\epsilon^2d^{\gamma-1}\\
&\le\epsilon^4+C'_\gamma\epsilon^2d^{\gamma-1}
\end{aligned}$$

It follows that as $M\to\infty$, the covariance is bounded by $C'_\gamma\epsilon^2d^{\gamma-1}$.

Case 4: Both D_1 and D_2 are rational, with D_2 being non-zero. We write $D_2 = c/d$ for some coprime integers c, d. We can assume that $D_1 = 0$, as otherwise we can resort to case 3. Using the same Fourier series approach,

$$\begin{aligned}
&\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\psi_\epsilon((AD)_1)\psi_\epsilon((AD)_2)\\
&=\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)+\cos(2\pi p(AD)_2)\Big)\\
&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{M^3}\sum_{1\le a_1,a_2,t\le M}\cos(2\pi p(AD)_1)\cos(2\pi q(AD)_2)\Big)\\
&=\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{M^2}\sum_{1\le a_2,t\le M}\cos(2\pi pt\frac{c}{d})+\cos(2\pi p(a_2t+1)\frac{c}{d})\Big)\\
&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{M^2}\sum_{1\le a_2,t\le M}\cos(2\pi pt\frac{c}{d})\cos(2\pi q(a_2t+1)\frac{c}{d})\Big)\\
&\to\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})+\cos(2\pi p(a_2t+1)\frac{c}{d})\Big)\\
&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})\cos(2\pi q(a_2t+1)\frac{c}{d})\Big)
\end{aligned}$$

We bound the inner sums again. Looking at the first one,

$$\begin{aligned}
&\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})+\cos(2\pi p(a_2t+1)\frac{c}{d})\\
&\le\frac{1}{d^2}\sum_{a_2=1}^{d}\Big(\Big|\sum_{t=1}^{d}\exp(2\pi ipt\frac{c}{d})\Big|+\Big|\sum_{t=1}^{d}\exp(2\pi ipa_2t\frac{c}{d})\Big|\Big)\\
&=\frac{1}{d}\sum_{a_2=1}^{d}1_{\{p\text{ is a multiple of }d\}}+1_{\{pa_2\text{ is a multiple of }d\}}\\
&=1_{\{p\text{ is a multiple of }d\}}+\frac{\gcd(p,d)}{d}\le\frac{2\gcd(p,d)}{d}.
\end{aligned}$$

For the second sum,

$$\begin{aligned}
&\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})\cos(2\pi q(a_2t+1)\frac{c}{d})\\
&=\frac{1}{2d^2}\sum_{1\le a_2,t\le d}\cos(2\pi(pt+q(a_2t+1))\frac{c}{d})+\cos(2\pi(pt-q(a_2t+1))\frac{c}{d})\\
&\le\frac{1}{d^2}\sum_{t=1}^{d}\Big|\sum_{a_2=1}^{d}\exp(2\pi iqa_2t\frac{c}{d})\Big|\\
&\le\frac{1}{d}\sum_{t=1}^{d}1_{\{qt\text{ is a multiple of }d\}}=\frac{\gcd(q,d)}{d}
\end{aligned}$$

Inserting this in the sum,

$$\begin{aligned}&\epsilon^4+\epsilon^2\sum_{p=1}^{\infty}F_p\Big(\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})+\cos(2\pi p(a_2t+1)\frac{c}{d})\Big)\\&+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\Big(\frac{1}{d^2}\sum_{1\le a_2,t\le d}\cos(2\pi pt\frac{c}{d})\cos(2\pi q(a_2t+1)\frac{c}{d})\Big)\\&\le\epsilon^4+2\epsilon^2\sum_{p=1}^{\infty}F_p\frac{\gcd(p,d)}{d}+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}F_pF_q\frac{\gcd(q,d)}{d}\end{aligned}$$

Note that this is the same sum as we observed in case 3, and we have the covariance being bounded by $C_\gamma\epsilon^2d^{\gamma-1}$ as $M\to\infty$.

Returning to the variance calculation, we are now faced with estimating the number of times we end up in cases 3 and 4. Let $\overline{h_d}$ be the number of pairs (i,j) such that $1\le i<j\le k$ and either $(x_i-x_j)_1=\frac{c}{d}$ reduced and non-zero, or $(x_i-x_j)_1=0$ and $(x_i-x_j)_2=\frac{c}{d}$ reduced and non-zero. This covers all times we end in case 3 and 4 above with denominator d.

We will also make use of proposition 1.3 from Alon and Peres, which we will use as a lemma.

Lemma. *Suppose we have a collection of k points $\{x_i|1\le i\le k\}\subset[0,1)$. Let h_d be the number of pairs (i,j) such that $1\le i<j\le k$ and $d(x_i-x_j)$ is an integer. Then for any $\alpha>0$, there exists a k sufficiently large such that for any $m\ge1$,*

$$H_m:=\sum_{d=1}^{m}h_m\le(km)^{1+\alpha}$$

Observe that by our definition, $\overline{h_d}\le 2h_d$. Also note the trivial bound $H_m<k^2$. Now using this in our variance calculation,

$$\begin{aligned}V(Y)&<k\epsilon^2+2\sum_{d=2}^{\infty}\overline{h_d}C_\gamma\epsilon^2d^{\gamma-1}\\&<k\epsilon^2+C'_\gamma\epsilon^2\sum_{d=2}^{\infty}H_d\Big(\frac{1}{d^{1-\gamma}}-\frac{1}{(d+1)^{1-\gamma}}\Big)\\&\le k\epsilon^2+C''_\gamma\epsilon^2\sum_{d=2}^{k-1}k^{1+\alpha}d^{\alpha+\gamma-1}+C'_\gamma\epsilon^2\sum_{d=k}^{\infty}k^2\Big(\frac{1}{d^{1-\gamma}}-\frac{1}{(d+1)^{1-\gamma}}\Big)\\&=k\epsilon^2+C''_\gamma k^{1+\alpha}\epsilon^2\sum_{d=2}^{k-1}d^{\alpha+\gamma-1}+C'_\gamma\epsilon^2k^{1+\gamma}\end{aligned}$$

Using an integral upper bound on the sum and redefining constants, we observe that for any $\alpha>0$, there exists a C_α such that $V(Y)\le C_\alpha k^{1+\alpha}\epsilon^2$ for sufficiently large k.

Now returning to Chebyshev's inequality, we find that to have positive probability of the set of Ax_i being ϵ-dense, it's sufficient if we have

$$\frac{C_\gamma k^{1+\alpha}\epsilon^2}{\epsilon^2(k\epsilon^2)^2} < 1$$
$$\iff k^{1-\alpha} > \frac{1}{C_\alpha \epsilon^4}$$
$$\impliedby k > \frac{1}{C'_\alpha \epsilon^{4+\alpha}}$$

Now taking ϵ to be sufficiently small, we can remove the C_α constant. Furthermore, this also guarantees the condition on k being sufficiently large to apply the lemma from Alon and Peres. This completes the proof for the case of $n = 2$ $\square$

Optimality and Further Work

We have shown that $k(\epsilon) \leq 1/\epsilon^{2n+\alpha}$ when $n = 2$ for the case of $SL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ and by a direct corollary, $M_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$. Furthermore, this is known to hold for $n > 2$. Now we must address the lower bound on $k(\epsilon)$, which is done through the use of examples that cannot be made ϵ-dense.

In the one dimensional case, the example of fractions with denominators less than $\lfloor 1/\epsilon \rfloor$ was used to give a lower bound of $\Omega(1/\epsilon^2)$, however this does not work well in higher dimensions. This is because the coordinates of points can be added together, leading to fractions that may have much larger denominators.

It is possible to show using a similar example that in the n-dimensional cases,

$$k(\epsilon) \geq \Omega\Big(\frac{1}{\epsilon^{n+1}}\Big)$$

Consider the points of the form $(\frac{a_1}{b}, \frac{a_2}{b}, ..., \frac{a_n}{b})$ where $1 \leq b \leq \lfloor 1/\epsilon \rfloor$. By construction, any linear combination of the coordinates must be of the form a/b which cannot be in the $(0, \epsilon)$ ball by choice of b. It is possible to show that the number of such points is bounded from below by some factor of $1/\epsilon^{n+1}$.

Putting all this together, we find

$$\Omega\Big(\frac{1}{\epsilon^{n+1}}\Big) \leq k(\epsilon) \leq \frac{1}{\epsilon^{2n+\alpha}}.$$

This remains a wide interval, and may possibly be improved from both sides. In regards to the lower bound, we have yet to find an example that provides ϵ-density with a large number of points, and this requires further investigation to see if such an example may possibly exist.

In regards to the upper bound, it may be possible to reduce this, however the methodology is not clear. It is certain that the second moment method will not provide a stronger upper bound as it is impossible to bound the variance by some asymptotic factor smaller than $k\epsilon^n$. Furthermore, investigation into adapting Alon and Peres' harmonic analysis proof for higher dimensions suggests that this method will also achieve the same bound as the second moment method. If this is to be improved, another method of proof is required.

Furthermore, work must also be done in streamlining the proof for higher dimensions, as even in the case of $n = 2$, this is significantly longer and more in depth than the one dimensional method.

References

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Appendix

Here we prove the lemmas used in the main theorem.

Lemma. *Consider the equation* $pt + q(at+1) \equiv 0 \pmod{d}$ *for some positive integers* a, p, q, d*. The number of solutions for* $t \in \mathbb{Z}_d$ *is at most* $\gcd(p, q, d)$*.*

Proof. First assume that $\gcd(p, q, d) \neq 1$. Then, letting $p' = p/\gcd(p, q, d)$ and similarly for q' and d', we can simplify the equation.

$$p't + q'(at+1) \equiv 0 \pmod{d'}$$

By construction, this new equation is similar to the original one, but we now know $\gcd(p', q', d') = 1$. Suppose this has S solutions for t between 1 and d'. It follows that for t between 1 and d, the number of solutions must be

$$S\frac{d}{d'} = S\gcd(p, q, d)$$

It remains to find what S is, and so we aim to solve the original equation under the assumption that $\gcd(p, q, d) = 1$.

Rearranging the equation, we must solve $t(p+aq) \equiv -q \pmod{d}$. First, write $p+aq \equiv cx$ where $c = \gcd(p+aq, d)$ and $x = (p+aq)/\gcd(p+aq, d)$. Similarly, we write $-q = dy$ where $d = \gcd(q, d)$ and $y = -q/\gcd(p, d)$. Note that since x is coprime to d,

$$\begin{aligned} tcx &\equiv dy \pmod{d} \\ \iff tc &\equiv dyx^{-1} \pmod{d}. \end{aligned}$$

Since c divides d we must have $dyx^{-1} \equiv 0 \pmod{c}$ if a solution exists. Note that $\gcd(c, d) = \gcd(q, p+aq, d) = \gcd(p, q, d) = 1$, so it follows $yx^{-1} \equiv 0 \pmod{c}$, or in other words, yx^{-1} is a multiple of c.

In the case that $c = 1$, this statement is obvious. We also see from our previous working that $t \equiv dyx^{-1} \pmod{d}$, and this solution is unique by the uniqueness of the inverse, so $S = 1$.

In the case that $c \neq 1$, we have a contradiction, since c divides d but yx^{-1} must be coprime to d by definition. This means our assumption that a solution existed was incorrect, and no solution exists, so $S = 0$.

Putting the cases together implies $S \leq 1$, and hence the overall number of solutions is bounded by $\gcd(p, q, d)$. $\square$

Lemma. *For any $\gamma > 0$, there exists a constant C_γ such that for any $d \geq 1$, we have*

$$\sum_{p=1}^{\infty} F_p \frac{\gcd(p,d)}{d} \leq C_\gamma \frac{\epsilon}{d^{1-\gamma}}$$

where $F_p = \frac{2\sin^2(\pi p \epsilon)}{\pi^2 p^2}$.

Proof. We start by noting that the series contains positive terms only, hence manipulation of the summation order is allowed. Furthermore, we can overestimate the sum by considering more terms.

First note that for any divisor of d, r, having $\gcd(p,d) = r$ implies the possible solutions for p are $p = r, 2r, 3r, \ldots$. For this reason, we can group the terms of the sum by this value.

$$\begin{aligned}
\sum_{p=1}^{\infty} \frac{2\sin^2(\pi p \epsilon)}{\pi^2 p^2} \frac{\gcd(p,d)}{d} &\leq \frac{1}{d} \sum_{r|d} r \sum_{n=1}^{\infty} \frac{\sin^2(\pi (nr)\epsilon)}{\pi^2 (nr)^2} \\
&= \frac{1}{d} \sum_{r|d} \frac{1}{r} \sum_{n=1}^{\infty} \frac{\sin^2(\pi n r \epsilon)}{\pi^2 n^2} \\
&= \frac{1}{d} \sum_{r|d} \frac{1}{r} \sum_{n=1}^{\infty} \frac{\sin^2(\pi n (r\epsilon - \lfloor r\epsilon \rfloor))}{\pi^2 n^2}
\end{aligned}$$

Now note the inner sum is the original Fourier Series for $\psi_{r\epsilon - \lfloor r\epsilon \rfloor}$ evaluated at 0 (without the leading constant term). Using this,

$$\begin{aligned}
&\frac{1}{d} \sum_{r|d} \frac{1}{r} \sum_{n=1}^{\infty} \frac{\sin^2(\pi n (r\epsilon - \lfloor r\epsilon \rfloor))}{\pi^2 n^2} \\
&< \frac{1}{d} \sum_{r|d} \frac{1}{r} \psi_{r\epsilon - \lfloor r\epsilon \rfloor}(0) \\
&= \frac{1}{d} \sum_{r|d} \frac{r\epsilon - \lfloor r\epsilon \rfloor}{r} \\
&< \frac{1}{d} \sum_{r|d} \epsilon
\end{aligned}$$

For any $\gamma > 0$, we can bound the number of divisors of d by $C_\gamma d^\gamma$. Using this, we immediately get the required bound. □