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# Limit Theorems for the Curie-Weiss Model

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## Abstract

In this report we extend on some of the limit theorems from Ellis and Newman [1978]. Namely, we study the limiting distributions of the sum of spins, $S_n$, with respect to the Curie-Weiss model in the case when the inverse temperature, $\beta$, is given by $\beta = \beta_n := 1/(1 + \alpha n^{-\gamma})$. When $\gamma > \frac{1}{2}$ and for all $\alpha \in \mathbb{R}$, $S_n/n^{3/4}$ converges in distribution to a density proportional to $\exp(-\frac{x^4}{12})$. Conversely, we obtain Gaussian behavior for $S_n/n^{\frac{\gamma}{2}+\frac{1}{2}}$ when $\gamma \in [0, \frac{1}{2})$ and $\alpha > 0$ and analogous results for when $\gamma = \frac{1}{2}$. We also obtain variance results for each of these cases.



## Acknowledgements

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# Contents

# 1 Introduction

Consider a game where we have $n$ coins placed on a table where we flip these coins and count the number of heads we observe. Let a head correspond to observing the value 1 and a tail correspond to observing the value $-1$. Then in counting the number of heads we observe, we can sum the observed values for each of these coins with $-n$ corresponding to no heads (all tails) and the sum of $n$ corresponding to all heads (no tails). We note that the outcome of one coin doesn't affect the outcome of another and that each coin has the same distribution where $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$. Formally we would consider these coins to be independent and identically distributed (i.i.d.) random variables.

In this situation the number of heads we observe (corresponding to a value between $-n$ and $n$) follows what is called a binomial distribution. Figure 1 shows this distribution for $n = 40$. It seems like the distribution is

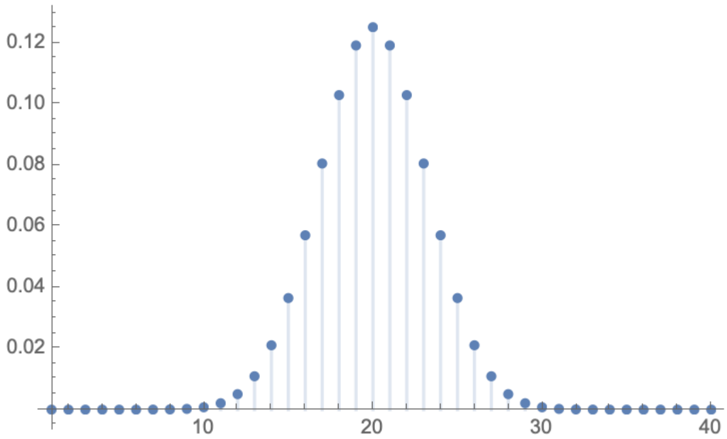


Figure 1: Binomial distribution for the number of heads observed when $n = 40$

approaching a simple curve like the Gaussian curve (commonly known as a Normal curve or the bell curve). This is true in a sense. If we correctly scale our distribution for the number of heads we do, in fact, converge to a Normal curve. This is because of the Central Limit Theorem (see appendix). Because the coins are i.i.d. random variables then the Central Limit Theorem tells us that the sum of these random variables (after appropriate scaling) converges to a Normal distribution. What happens when we modify this situation so that the coins are instead dependent on each other in a particular kind of way? The situation becomes a whole lot more complex. The Curie-Weiss model is an example of such a situation. This report studies the limiting distributions for sums of these random variables that are defined by the Curie-Weiss model.

The Curie-Weiss model is defined on a complete graph of $n$ vertices where each vertex takes a spin value $\omega_i \in \{-1, 1\}$. In physics, each of these vertices is taken to represent the magnetic dipole moment of an atom. We let any such configuration of spins, $\omega$, to be random such that

$$\mathbb{P}(\omega) = \exp\left(\frac{\beta}{2n}\sum_{i,j=1}\omega_i\omega_j\right)\Big/\sum_{\omega\in\Omega^n}\exp\left(\frac{\beta}{2n}\sum_{i,j=1}\omega_i\omega_j\right) \tag{1.1}$$

where $\Omega^n := \{-1,1\}^n$ is the set of all possible spin configurations for the $n$ vertices and $\beta$ is the inverse temperature of the system. The value of $\beta$ affects the distribution of spin configurations in a fundamental way.

We also define $S_n = \sum_{\omega_i\in\{-1,1\}}\omega_i$ which is known as the magnetisation (analogous to summing the results of the coin flips). Since each $\omega_i$ is a random variable, the magnetisation is just the sum of $n$ identically distributed (but now dependent) random variables. $S_n$ has a symmetric distribution and, in particular, $\mathbb{E}(S_n) = 0$. A crucial observation to make is that $\sum_{i,j=1}\omega_i\omega_j = S_n^2$. This means we can change equation (1.1) to

$$\mathbb{P}(\omega) = \exp\left(\frac{\beta}{2n}S_n^2\right)\Big/\sum_{\omega\in\Omega^n}\exp\left(\frac{\beta}{2n}S_n^2\right). \tag{1.2}$$

This makes analysing the magnetisation a much simpler task. The remainder of this report will be to develop limit theorems for the magnetisation.

### 1.1 Statement of Authorship

The Curie-Weiss model has been heavily studied and so not all of the ideas in this report are original. Section 2 outline well known results by Ellis and Newman [1978] and Simon and Griffiths [1973]. Section 3 is work that Tim, Eric and I did. Tim and Eric gave me a strong sense of direction for pursuing this work. They also assisted me when in solving a few problems that arose. I was responsible for writing up the detailed arguments and proofs, although they were very similar in flavour to those by Ellis [2006]. We are unaware if these results have been obtained previously.

## 2 Limit Theorems for fixed $\beta$

**Theorem 2.1** see e.g. Theorem V.9.4 [Ellis, 2006]. Fix $0 < \beta < 1$ and define $\sigma^2(\beta) = (1-\beta)^{-1}$. Then as $n \to \infty$,

$$\frac{S_n}{n^{1/2}} \to N(0, \sigma^2), \text{ in distribution.}$$

We note that this is analogous to the Central Limit Theorem for i.i.d. random variables since $\mathbb{E}[\omega_i] = \mu = 0$. We also observe that $\sigma^2(\beta)$ diverges when $\beta = 1$, thus we expect the central limit theorem type result to fail in this case. The next theorem which was discovered by Simon and Griffiths [1973] gives us different limit theorem for the case when $\beta = 1$.

**Theorem 2.2** see e.g. Theorem V.9.5 [Ellis, 2006]. Fix $\beta = 1$. Then there exists a random variable $X$ with density proportional to $\exp(-\frac{1}{12}x^4)$ such that as $n \to \infty$,

$$\frac{S_n}{n^{3/4}} \to X, \text{ in distribution.}$$

We call $\beta = 1$ the critical value for the Curie-Weiss model. In the case when $\beta > 1$ the dependencies are too strong, and the above limit theorems fail. In the next section we will set $\beta = \beta_n := 1/(1 + \alpha n^{-\gamma})$ which converges to the critical value of $\beta = 1$ as $n \to \infty$ and see how this effects the limiting distributions for the magnetisation.

## 3 Limit Theorems when $\beta = \beta_n = 1/(1 + \alpha n^{-\gamma})$

**Theorem 3.1** Let $\beta_n = \frac{1}{1+\alpha n^{-\gamma}}$ be the inverse-temperature and $S_n = \sum_{\omega_i \in \{-1,1\}} \omega_i$ be the magnetisation for the Curie-Weiss model with $n$ spins. For all $\gamma > \frac{1}{2}$ and $\alpha \in \mathbb{R}$, there is a random variable $X$ with density proportional to $\exp(-\frac{1}{12}x^4)$ such that,

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{3}{4}}} \to X, \text{ in distribution}$$

and

$$\mathrm{Var}(S_n^2) \sim n^{\frac{3}{2}} \int_{-\infty}^{\infty} x^2 \exp(-\frac{1}{12}x^4)\, dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{12}x^4)\, dx.$$

This defines what we will call the Critical Scaling Window. To prove this, we will show that,

$$\lim_{n\to\infty} \mathbb{E}\Big(\exp\Big(r\frac{S_n}{n^{\frac{3}{4}}}\Big)\Big) = \int_{-\infty}^{\infty} \exp(rx - \frac{1}{12}x^4)\, dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{12}x^4)\, dx := g(r). \tag{3.1}$$

The theorem then follows from the MGF Continuity Theorem (see appendix). We will have that $\mathbb{E}\big(\exp(r\frac{S_n}{n^{\frac{3}{4}}})\big) \to g(r)$ as $n \to \infty$. Clearly $\lim_{r\to 0} g(r) = g(0) = 1$. This then tells us that $S_n n^{-\frac{3}{4}}$ converges in distribution to a distribution defined by $P(x) = \exp(-\frac{1}{12}x^4)\Big/\int_{\infty}^{\infty} \exp(-\frac{1}{12}x^4))$. We will start by proving the following lemma.

**Lemma 3.2** Let $t_n$ be an be any sequence and $S_n$ be the magnetisation for the Curie-Weiss model. Then for any $\beta > 0$,

$$\mathbb{E}(\exp(t_n S_n)) = \frac{\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\frac{n}{\beta}t_n^2 + \frac{x t_n n}{\beta} - \frac{1}{2}\frac{n}{\beta}x^2 + n\log(\cosh(x)))\, dx}{\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\frac{n}{\beta}x^2 + n\log(\cosh(x)))\, dx}.$$

*Proof Lemma 3.2.* By definition of expected value.

$$\mathbb{E}(\exp(t_nS_n)) = \sum_{\omega\in\Omega^n} \exp(t_nS_n)\frac{\exp(\frac{\beta}{2n}S_n^2)}{\sum_{\omega\in\Omega^n}\exp(\frac{\beta}{2n}S_n^2)}$$
$$= \frac{\sum_{\omega\in\Omega^n}\exp(t_nS_n)\exp(\frac{1}{2}(\sqrt{\frac{\beta}{n}}S_n)^2)}{\sum_{\omega\in\Omega^n}\exp(\frac{1}{2}(\sqrt{\frac{\beta}{n}}S_n)^2)}.$$

We can then use the identity $\exp(\frac{1}{2}x^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(yx - \frac{1}{2}y^2)\,dy$ to replace the term $\exp(\frac{1}{2}(\sqrt{\frac{\beta}{n}}S_n)^2)$. From this we obtain

$$\mathbb{E}(\exp(t_nS_n)) = \frac{\sum_{\omega\in\Omega^n}\int_{-\infty}^{\infty}\exp((t_n+\sqrt{\frac{\beta}{n}}x)S_n - \frac{1}{2}x^2)\,dx}{\sum_{\omega\in\Omega^n}\int_{-\infty}^{\infty}\exp(\sqrt{\frac{\beta}{n}}xS_n - \frac{1}{2}x^2)\,dx}.$$

Next we use linearity of the integral and the definition of $S_n$ to obtain

$$\mathbb{E}(\exp(t_nS_n)) = \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\sum_{\omega\in\Omega^n}\exp((t_n+\sqrt{\frac{\beta}{n}}x)S_n)\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\sum_{\omega\in\Omega^n}\exp(\sqrt{\frac{\beta}{n}}xS_n)\,dx}$$
$$= \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\sum_{\omega\in\Omega^n}\prod_{i=1}^n\exp((t_n+\sqrt{\frac{\beta}{n}}x)\omega_i)\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\sum_{\omega\in\Omega^n}\prod_{i=1}^n\exp(\sqrt{\frac{\beta}{n}}x\omega_i)\,dx}$$
$$= \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\prod_{i=1}^n(\exp(t_n+\sqrt{\frac{\beta}{n}}x)+\exp(-(t_n+\sqrt{\frac{\beta}{n}}x))\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)\prod_{i=1}^n(\exp(\sqrt{\frac{\beta}{n}}x)+\exp(-\sqrt{\frac{\beta}{n}}x))\,dx}.$$

Multiplying the numerator and the denominator by $1/2^n$ then yields,

$$\mathbb{E}(\exp(t_nS_n)) = \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2 + n\log(\cosh(\sqrt{\frac{\beta}{n}}x+t_n))\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2 + n\log(\cosh(\sqrt{\frac{\beta}{n}}x))\,dx}$$
$$= \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}\frac{n}{\beta}t_n^2 + \frac{xt_nn}{\beta} - \frac{1}{2}\frac{n}{\beta}x^2 + n\log(\cosh(x)))\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}\frac{n}{\beta}x^2 + n\log(\cosh(x)))\,dx}$$

which holds for all $\beta > 0$.

□

Now we will prove Theorem 3.1 which involves many ideas used in the proof of Theorem 2.2. See e.g. Theorem V.9.5 [Ellis, 2006].

*Proof Theorem 3.1.* From Lemma 3.2, letting $\beta = \beta_n$ we get

$$\begin{aligned}\mathbb{E}(\exp(t_nS_n)) &= \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}n(1+\alpha n^{-\gamma})t_n^2 + xt_nn(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{1-\gamma}x^2 - \frac{1}{2}nx^2 + n\log(\cosh(x)))\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}\alpha n^{1-\gamma}x^2 - \frac{1}{2}nx^2 + n\log(\cosh(x)))\,dx} \\ &= \frac{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}n(1+\alpha n^{-\gamma})t_n^2 + xt_nn(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{1-\gamma}x^2 - nG(x))\,dx}{\int_{-\infty}^{\infty}\exp(-\frac{1}{2}\alpha n^{1-\gamma}x^2 - nG(x))\,dx}\end{aligned} \tag{3.2}$$

where $G(x) = \frac{1}{2}x^2 - \log(\cosh(x))$. We observe that the Taylor series of $G(x)$ around $x = 0$ is $G(x) = \frac{1}{12}x^4 + \frac{G^6(c)}{6!}x^6$ for some $c \in (0, x)$ and for all $|x| < A$, where $A$ is the radius of convergence. From this we can see that $nG(\frac{x}{n^{\frac{1}{4}}}) = \frac{1}{12}x^4 + \frac{G^6(c)n^{-\frac{1}{2}}}{6!}x^6 \to \frac{1}{12}x^4$ as $n \to \infty$. We also notice for $\gamma > \frac{1}{2}$, $-\frac{1}{2}\alpha n^{1-\gamma}(\frac{x}{n^{\frac{1}{4}}})^2 = -\frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 \to 0$ as $n \to \infty$. This motivates the change of variables $x \to \frac{x}{n^{\frac{1}{4}}}$. Applying this to the numerator and the denominator in (3.2) we get

$$\mathbb{E}(\exp(t_nS_n)) = \frac{\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}n(1+\alpha n^{-\gamma})t_n^2 + xt_nn^{\frac{3}{4}}(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG\Big(\frac{x}{n^{\frac{1}{4}}}\Big)\Big)\,dx}{\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG\Big(\frac{x}{n^{\frac{1}{4}}}\Big)\Big)\,dx}. \tag{3.3}$$

To obtain a non trivial limit for the $x$ term we then set $t_n = \frac{r}{n^{\frac{3}{4}}}$ which gives

$$\mathbb{E}\Big(\exp\Big(r\frac{S_n}{n^{\frac{3}{4}}}\Big)\Big) = \frac{\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}n^{-\frac{1}{2}}(1+\alpha n^{-\gamma})r^2 + xr(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG\Big(\frac{x}{n^{\frac{1}{4}}}\Big)\Big)\,dx}{\int_{-\infty}^{\infty}\exp\Big(-\frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG\Big(\frac{x}{n^{\frac{1}{4}}}\Big)\Big)\,dx}. \tag{3.4}$$

We notice that for the function $G(x)$, there exist positive real numbers $\epsilon$ and $A$ such that

a) $G(x) \geq \epsilon x^4$ for $|x| \leq A \Rightarrow \exp(-nG(\frac{x}{n^{\frac{1}{4}}})) \leq \exp(-\epsilon x^4)$ for $|x| \leq An^{\frac{1}{4}}$;

b) $G(x) \geq \epsilon x^2$ for $|x| > A \Rightarrow \exp(-nG(\frac{x}{n^{\frac{1}{4}}})) \leq \exp(-\epsilon x^2 n^{\frac{1}{2}})$ for $|x| > An^{\frac{1}{4}}$.

from a) we define a sequence of functions $f_n(x)$ as the integrand in the numerator of equation (3.4) and let $\mathbb{I}$ be the indicator function. Then we see that

$$\begin{aligned}f_n(x)\mathbb{I}(|x| \leq An^{\frac{1}{4}}) &:= \exp(-\frac{1}{2}n^{-\frac{1}{2}}(1+\alpha n^{-\gamma})r^2 + xr(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG(\frac{x}{n^{\frac{1}{4}}}))\mathbb{I}(|x| \leq An^{\frac{1}{4}}) \\ &\leq \exp((1+\alpha)xr - \epsilon x^4)\end{aligned} \tag{3.5}$$

which is an integrable function over the real line. From b) we get

$$\begin{aligned}f_n(x)\mathbb{I}(|x| > An^{\frac{1}{4}}) &:= \exp\Big(-\frac{1}{2}n^{-\frac{1}{2}}(1+\alpha n^{-\gamma})r^2 + xr(1+\alpha n^{-\gamma}) - \frac{1}{2}\alpha n^{\frac{1}{2}-\gamma}x^2 - nG\Big(\frac{x}{n^{\frac{1}{4}}}\Big)\Big)\mathbb{I}(|x| > An^{\frac{1}{4}}) \\ &\leq \exp((1+\alpha)xr - \epsilon x^2n^{\frac{1}{2}}).\end{aligned} \tag{3.6}$$

The expression in equation (3.6) converges to 0 as $n \to \infty$. We also see for $\gamma > \frac{1}{2}$, we get that $f_n(x) \to \exp(rx - \frac{1}{12}x^4), \forall x \in \mathbb{R}$ since $An^{\frac{1}{4}} \to \infty$. Using this alongside the results in (3.5) and (3.6) and Lebesgue's dominated convergence theorem (see appendix), then $\int_{-\infty}^{\infty} f_n(x)\,dx \to \int_{-\infty}^{\infty} \exp(rx - \frac{1}{12}x^4)\,dx$ as $n \to \infty$. Applying this argument to the denominator in equation (3.4) yields the result in equation (3.1). We also get from the MGF Continuity Theorem that for inside the scaling window

$$\mathbb{E}\Big(\Big(\frac{S_n}{n^{\frac{3}{4}}}\Big)^j\Big) \to \int_{-\infty}^{\infty} x^j \exp(-\frac{1}{12}x^4)\,dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{12}x^4)\,dx.$$

Setting $j = 2$, then by linearity we obtain

$$\mathbb{E}(S_n^2) \sim n^{\frac{3}{2}} \int_{-\infty}^{\infty} x^2 \exp(-\frac{1}{12}x^4)\,dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{12}x^4)\,dx$$

which yields asymptotics for the variance of $S_n$ since $\mathbb{E}(S_n) = 0$. This completes the proof.

□

**Theorem 3.3** Let $\beta_n = \frac{1}{1+\alpha n^{-\gamma}}$ be the inverse-temperature and $S_n = \sum_{\omega_i \in \{-1,1\}} \omega_i$ be the magnetisation for the Curie-Weiss model with $n$ spins. For $0 \leq \gamma < \frac{1}{2}$ and $\alpha \in (0, \infty)$,

$$\lim_{n\to\infty} \frac{\alpha^{\frac{1}{2}} S_n}{n^{\frac{1}{2}+\frac{\gamma}{2}}} \to N(0,1), \text{ in distribution}$$

and

$$\mathrm{Var}(S_n^2) \sim \frac{n^{1+\gamma}}{\alpha}.$$

Similarly to Theorem 3.2, we will show that

$$\lim_{n\to\infty} \mathbb{E}\Big(\exp\Big(r\frac{\alpha^{\frac{1}{2}} S_n}{n^{\frac{1}{2}+\frac{\gamma}{2}}}\Big)\Big) = \int_{-\infty}^{\infty} \exp(rx - \frac{1}{2}x^2)\,dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2)\,dx. \tag{3.7}$$

The result for theorem 3.3 will again follow from the MGF Continuity Theorem.

*Proof Theorem 3.3.* The proof of this is similar to that of Theorem 3.1. As such we will begin by looking at equation (3.3). Since we observe a Central Limit Theorem ($\lim_{n\to\infty} S_n/n^{\frac{1}{2}} \to N(0, \frac{1}{1-\beta})$) for fixed $\beta < 1$ then we expect for small $\gamma$ (slow convergence to the critical value $\beta = 1$) to again observe a Gaussian limiting distribution. For $0 \leq \gamma < \frac{1}{2}$, the term $\frac{1}{2}\alpha n^{\frac{1}{2}-\gamma} x^2$ dominates the term $nG(xn^{-\frac{1}{4}})$ in (3.3). We then make a change of variables $x \to x\alpha^{-\frac{1}{2}} n^{\frac{\gamma}{2}-\frac{1}{4}}$ to make the $x^2$ term have a non-trivial limit as $n \to \infty$. This yields from equation (3.3),

$$\mathbb{E}(\exp(t_nS_n)) = \frac{\int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}n(1+\alpha n^{-\gamma})t_n^2 + xt_n\alpha^{-\frac{1}{2}}n^{\frac{\gamma}{2}+\frac{1}{2}}(1+\alpha n^{-\gamma}) - \frac{1}{2}x^2 - nG\Big(\frac{x}{\alpha^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\gamma}{2}}}\Big)\Big)\,dx}{\int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}x^2 - nG\Big(\frac{x}{\alpha^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\gamma}{2}}}\Big)\Big)\,dx}. \quad (3.8)$$

We then pick $t_n$ so as to give a non trivial limit for the linear $x$ term in equation (3.8). We set $t_n = r\alpha^{\frac{1}{2}}n^{-\frac{\gamma}{2}-\frac{1}{2}}$ to obtain

$$\mathbb{E}\Big(\exp\Big(r\frac{\alpha^{\frac{1}{2}}S_n}{n^{\frac{1}{2}+\frac{\gamma}{2}}}\Big)\Big) = \exp(-\frac{\alpha}{2}r^2n^{-\gamma}(1+\alpha n^{-\gamma}))\frac{\int_{-\infty}^{\infty} \exp\Big(rx(1+\alpha n^{-\gamma}) - \frac{1}{2}x^2 - nG\Big(\frac{x}{\alpha^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\gamma}{2}}}\Big)\Big)\,dx}{\int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}x^2 - nG\Big(\frac{x}{\alpha^{\frac{1}{2}}n^{\frac{1}{2}-\frac{\gamma}{2}}}\Big)\Big)\,dx}. \quad (3.9)$$

We note that $nG(x\alpha^{-\frac{1}{2}}n^{\frac{\gamma}{2}-\frac{1}{2}}) \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$ when $0 \leq \gamma < \frac{1}{2}$. Clearly $\exp(rx(1+n^{-\gamma}) - \frac{1}{2}x^2 - nG(xn^{\frac{\gamma}{2}-\frac{1}{2}})) < \exp(2rx - \frac{1}{2}x^2)$ which is integrable over $\mathbb{R}$. Using Lebesgue's dominated convergence theorem, we find in taking the limit of the result in equation (3.9) yields equation (3.7). Analogously to Theorem 3.1 we get from the MGF Continuity Theorem

$$\mathbb{E}\Big(\big(\frac{\alpha^{\frac{1}{2}}S_n}{n^{\frac{1}{2}+\frac{\gamma}{2}}}\big)^j\Big) \to \int_{-\infty}^{\infty} x^j \exp(-\frac{1}{2}x^2)\,dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2)\,dx = 1$$

Setting $j = 2$, then by linearity we obtain

$$\mathbb{E}(S_n^2) \sim \frac{n^{1+\gamma}}{\alpha}$$

which yields the expression asymptotics of the variance since $\mathbb{E}(S_n) = 0$.

□

Finally we have a theorem for the case when $\gamma = \frac{1}{2}$. The case when $0 \leq \gamma < \frac{1}{2}$ and $\alpha < 0$ will be discussed in the following sections.

**Theorem 3.4** Let $\beta_n = \frac{1}{1+\alpha n^{-\gamma}}$ be the inverse-temperature and $S_n = \sum_{\omega_i \in \{-1,1\}} \omega_i$ be the magnetisation for the Curie-Weiss model with $n$ spins. For all $\gamma = \frac{1}{2}$ and $\alpha \in \mathbb{R}$, there is a random variable $X$ with density proportional to $\exp(-\frac{1}{2}\alpha x^2 - \frac{1}{12}x^4)$ such that,

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{3}{4}}} \to X, \text{ in distribution}$$

and

$$Var(S_n^2) \sim n^{\frac{3}{2}} \int_{-\infty}^{\infty} x^2 \exp(-\frac{1}{2}\alpha x^2 - \frac{1}{12}x^4)\,dx \Big/ \int_{-\infty}^{\infty} \exp(-\frac{1}{2}\alpha x^2 - \frac{1}{12}x^4)\,dx.$$

The proof of this is analogous to that of Theorem 3.1 and 3.3. One just needs to analyse equation (3.4) taking $\gamma = \frac{1}{2}$. This yields

$$\mathbb{E}\Big( \exp\big(r\frac{S_n}{n^{\frac{3}{4}}}\big)\Big) = \frac{\int_{-\infty}^{\infty} \exp(-\frac{1}{2}n^{-\frac{1}{2}}(1+\alpha n^{-\frac{1}{2}})r^2 + xr(1+\alpha n^{-\frac{1}{2}}) - \frac{1}{2}\alpha x^2 - nG(\frac{x}{n^{\frac{1}{4}}}))\,dx}{\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\alpha x^2 - nG(\frac{x}{n^{\frac{1}{4}}}))\,dx}.$$

We can then use Lebesgue's dominated convergence theorem in a similar way done in theorems 3.1 and 3.3. The MGF Continuity Theorem will then give you the result.

## 4 Further Study

The one case that we have omitted from this report is the case when $0 \leq \gamma < \frac{1}{2}$ and $\alpha < 0$, outside the critical scaling window but approaching the critical value $\beta = 1$ from above. This means we are in the low temperature region. In the Law of Large Numbers scaling for $S_n$ (taking $S_n/n$) we obtain a *mixture* of point masses in the limit as $n \to \infty$. This tells us not to expect a central limit theorem (In the case when $\beta \leq 1$, $S_n/n$ converges to a single point mass and so a central limit theorem type result may be expected). From Figure 2, we can see

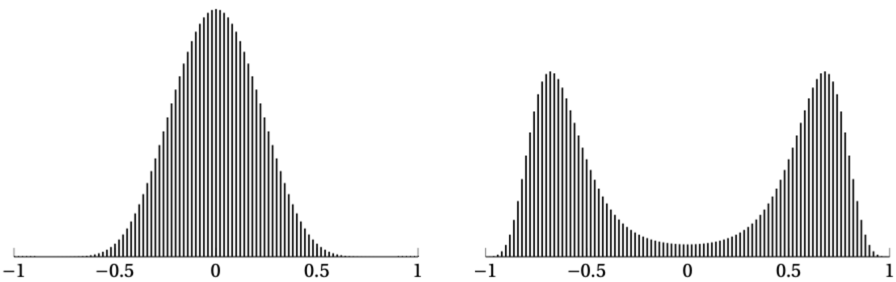


Figure 2: The distribution for $S_n/n$ when $n = 100$ for $\beta = 0.8$ and $\beta = 1.2$, respectively. Figure from Friedli and Velenik [2017]

that in the low temperature region we observe a bi-modal distribution for the magnetisation when $n$ is large. In a paper by Ellis et al. [1980], it was shown that, in the fixed $\beta > 1$ case, if you condition on the magnetisation being near one of these positive peaks, then we again obtain a central limit theorem type result. More clearly, when we condition on $S_n/n \in [m-a, m+a]$ where m is the value corresponding to the position of the peak of the distribution in Figure 2, then we find that $S_n/n^{1/2}$ converges, in distribution, to a Gaussian.

In the first attempt at solving this for $\beta = \beta_n = 1/(1 + \alpha n^{-\gamma})$, we attempted to utilise some of the techniques that we developed by Ellis et al. [1980]. What we later realised was that because we are converging to to the critical value $\beta = 1$, then the two peaks at $m$ and $-m$ were also converging to 0 (they are in fact sequences of real numbers converging to 0 so can be denoted $m_n$). This means that in a region of radius $a$ around one of these peaks, in the limit there will be a point where we are actually conditioning over 2 peaks. This ultimately leads to an incorrect result. Part of some further study which we would like to tackle is refining some of the techniques from Ellis et al. [1980] to enable us to limit our calculation to being conditioned around just a single peak. To do this we will condition on $S_n > 0$ rather than centering around $m_n$. This seems like an interesting problem to tackle.

## 5 Conclusion

In modifying the Curie-Weiss Model to accommodate a $\beta_n$ that is asymptotic to the critical value $\beta = 1$ we were able to obtain limit theorems that were in many ways analogous to those by Simon and Griffiths [1973] and Ellis and Newman [1978]. With $\beta = \beta_n = 1/(1+\alpha n^{-\gamma})$, we found that when $\gamma > \frac{1}{2}$ that we obtained the same result as for the fixed $\beta = 1$ case. This tells us that for fast convergence to the critical value, the modal is unable to tell the difference between the asymptotic $\beta = \beta_n$ and the fixed value $\beta = 1$ when considering the distribution of the magnetisation, $S_n$. With slower convergence ($0 \leq \gamma < \frac{1}{2}$), when we approach the critical value from below we find that that the behaviour of the magnetisation is much like that for the high temperature case and obtain a Gaussian limit for the magnetisation (after appropriately scaling). The low temperature scaling window is considerably more difficult but a strategy for solving this problem has been outlined.

# Appendix

## The Central Limit Theorem

See e.g. A.8.5 [Ellis, 2006].

Let $\{X_i\colon\ i = 1, 2, 3...\}$ be a sequence of i.i.d. random variables and define $S_n = \sum_{i=1}^{n} X_i$. If $\mathbb{E}(X_1) = \mu$ and $Var(X_1) = \sigma^2$ are finite and $\sigma^2 > 0$, then

$$\frac{S_n - n\mu}{n^{1/2}} \to N(0, \sigma^2), \text{ in distribution,}$$

where $N(0, \sigma^2)$ is a probability measure on $\mathbb{R}$ with density

$$(2\pi\sigma^2)^{1/2} \exp(-\frac{1}{2}x^2)$$

## Moment Generating Function (MGF) Continuity Theorem

see e.g. A.8.7 [Ellis, 2006]. Let $P_n$ be a sequence of Borel probability measures on $\mathbb{R}$, $\mathcal{M}(\mathbb{R})$. Assume that the moment generating functions $g_n(r) = \int_{-\infty}^{\infty} \exp(rx) P_n dx$ are defined for all $r$ in an interval $C$ which has nonempty interior and which contains the origin. Assume that for all $r \in C$ the limit $g(r) = \lim_{n\to\infty} g_n(r)$ exists and that $g(r) \to g(0) = 1$ as $r \in C$, $r \neq 0$, converges to 0. Then the following conclusions hold.

a) $P_n$ converges weakly to some $P \in \mathcal{M}(\mathbb{R})$ and $g(r) = \int_{-\infty}^{\infty} \exp(rx) P\, dx$

b) If in addition $0 \in \text{int}C$, then for any $j \in \{1, 2, \dots\}$, $\int_{-\infty}^{\infty} x^j P_n\, dx \to \int_{-\infty}^{\infty} x^j P\, dx$

## Lebesgue's Dominated Convergence Theorem

Suppose $f_n : \mathbb{R} \to [-\infty, \infty]$ are (Lebesgue) measurable functions such that the point wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is bounded by some integrable function $g$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, d\mu = \int_{\mathbb{R}} f\, d\mu$$

## References

Richard S Ellis. *Entropy, large deviations, and statistical mechanics*, volume 1431. Taylor & Francis, 2006.

Richard S Ellis and Charles M Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete (Journal for Probability Theory and Related Areas)*, 44(2):117–139, 1978.

Richard S Ellis, Charles M Newman, and Jay S Rosen. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete (Journal for Probability Theory and Related Areas)*, 51(2):153–169, 1980.

S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, 2017. ISBN 9781107184824. URL https://books.google.com.au/books?id=h_o6DwAAQBAJ.

Barry Simon and Robert B Griffiths. The $(\varphi^4)_2$ field theory as a classical ising model. *Communications in Mathematical Physics*, 33(2):145–164, 1973.