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Facts are Relative: The extended Wigner's friend experiment

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Abstract

The present report concerns the foundations of quantum mechanics. We rigorously develop and prove all the necessary mathematical tools to be able to give an axiomatic definition of quantum theory. We then develop some important concepts within quantum mechanics, including quantum entanglement, and give a timeline of developments in the field specifically related to foundational understanding. The culmination is Brukner's theorem, proved using the extended Wigner's friend thought experiment, arguing that at the quantum level, there is no such thing as observer independent facts, or facts are relative.

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1 Introduction

This AMSI VRS project aims to investigate by way of literature review some recent developments in the foundational understanding of quantum mechanics. For most of the theory's approximately 100 years, the attitude among most physicists has been one of 'shut up and calculate'. The results of such an attitude have been significant, leading to the development of devices such as lasers and transistors, which have in turn lead to wild leaps in technology and a digital revolution, with devices such as CD, DVD, smartphones, computers and laptops becoming not just possible, but widely available. But the question remains: How can we interpret the theory of quantum mechanics?

The key difference from classical mechanics, as we shall see, is due to observable quantities now being described by operators, which unlike numbers, aren't always commutative. In fact, for pairs of operators with non-zero commutator, we have an uncertainty relation, meaning both cannot be precisely known simultaneously. This non-commutativity feature gives rise to a non-classical type of probability associated with observables. States can now validly exist in linear combinations of their possible measurement outcomes, and the quantum probability associated to these different states gives the likelihood the outcome of a measurement will be that state. Quantum states evolve in time in a unitary, deterministic, and local way, just like in classical mechanics, but when a measurement is made, the transformation (or 'collapse') of the state into the observed eigenstate is non-unitary and not deterministic either. Therefore the question of how to describe the process of measurement is a new one to physics, which has not encountered such an issue in any prior theory. Superposition states and collapsing can lead to seemingly contradictory scenarios, such as Einstein's much-abhorred 'spooky action at a distance', which stems from quantum entanglement.

Some of the theory's founders, including Einstein, were in favour of making quantum mechanics as classical as possible. They were not pleased with the probability and randomness, and attempted to reconcile quantum theory in terms of classical mechanics. Others, including Bohr, saw quantum mechanics as new and more fundamental, and had no issue with the different predictions. Einstein, together with Podolsky and Rosen, devised a paradox (discussed below) to argue their case that quantum mechanics had to be incomplete.

Since then, the study of fundamental quantum mechanics has gone through phases of high interest, such as when Bell published his theorem (discussed below), and low interest. Many famous physicists have advocated for different interpretations of quantum mechanics, and the original Wigner's friend thought experiment was conceived by Wigner to espouse his own view. Recently, several papers have presented new results placing stronger conditions on the properties any interpretation of quantum mechanics is allowed to have. Brukner developed the extended Wigner's friend thought experiment and proved that a theory cannot support freedom of choice, locality, and observer-independent facts while also being in line with the predictions of quantum mechanics. The primary research goal in this project is to develop a comprehensive understanding of the mathematical, physical, and philosophical principles needed to understand the developments in this field. After giving a rigorous treatment of the mathematical principles underlying quantum mechanics, we develop the theory itself, including several worked examples. In the final section, we give a timeline of the physical and philosophical arguments that constitute the history of development in the quantum measurement problem, before finally being able to explore the extended Winger's friend thought experiment, and prove the no-go theorem of Brukner. After this, we discuss experimental verification of the no-go theorem and some possible interpretations of quantum mechanics in terms of the results.

1.1 Statement of Authorship

This report was written by Alex Paviour. All results are credited to their original authors, and where they are not, they have been derived from notes by or discussions with Adam Rennie and Alex Mundey. Original sources are cited for all ideas and concepts, but the reproduction here is the work of Alex Paviour, according to his understanding of them. Proofs given without citation are the work of Alex Paviour, sometimes prompted by discussions with Adam Rennie and Alex Mundey, although no claim is made as to whether or not such results have been previously published.

1.2 Acknowledgements

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2 Mathematical Formalism

In the first section, we will present a range of definitions, results, and notation required to rigorously build up the theory of quantum mechanics. Important points we will discuss include Hilbert space, operators, and kets and bras.

2.1 Hilbert space

This section takes its main results from the treatment of Hilbert spaces and bounded operators given in [26, Sec. 3.1, 3.2]. The first definition we need to make is that of the space in which the vectors describing quantum states will live in. It is called a Hilbert space after David Hilbert, and is a vector space with some additional properties.

Definition 2.1.1. A *Hilbert space* $\mathcal{H}$ is a complex inner product space which is complete in the norm.

One of the postulates of quantum mechanics is that each physical state of a quantum system is completely specified by its vector in its Hilbert space. The norm of some vector $\xi \in \mathcal{H}$ is given by $\|\xi\|^2 = \langle \xi, \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product. The inner product will be important to us when we come to carry out actual calculations on quantum systems. It is important to note immediately that some of the Hilbert spaces used in quantum mechanics are infinite dimensional. The requirement of completeness in the norm is important in this case, since essentially it allows the majority of the tools we are used to employing in finite dimensional spaces to carry over, provided we are careful of some caveats, such as converging infinite sums. Some of the most important examples of Hilbert spaces are given below, some of which are infinite dimensional. Note that there is no one preferred Hilbert space in the formalism of quantum mechanics, since there are multiple distinct but equivalent formulations. Each utilises a different Hilbert space.

Example 2.1.2. The vector space $\mathbb{C}^n$ equipped with the usual inner product of column vectors is a Hilbert space.

Example 2.1.3. The set

$$L^2(\mathbb{R}^3, \mathrm{d}^3x) = \left\{ f : \mathbb{R}^3 \to \mathbb{C} : \int_{\mathbb{R}^3} |f|^2(x)\, \mathrm{d}^3x < \infty \right\},$$

is a Hilbert space with the inner product given by

$$\langle f, g \rangle = \int \overline{f(x)} g(x)\, \mathrm{d}x\,.$$

Note here that the notation d^3x is a shorthand for $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and we are therefore taking a volume integral of sorts, where we integrate across the entire space. Functions which satisfy this property and are in the space L^2 are called square-integrable.

Example 2.1.4. Very similarly to Example 2.1.3, we can consider sums instead of integrals with the space

$$\ell^2(\mathbb{N}) = \left\{ (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{C},\ \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

with inner product

$$\big\langle (x_n), (y_n) \big\rangle = \sum_{n=1}^{\infty} \bar{x}_n y_n.$$

This is another Hilbert space, however that fact is unsurprising when we consider that sequences are just functions from $\mathbb{N}$ to $\mathbb{R}$, and this example is otherwise the same as Example 2.1.3.

Now that we have our vectors representing the quantum states and the inner product, we need to understand how we can extract physical information from these state vectors. This is done using operators. Another of the postulates of quantum mechanics is that each physical observable corresponds to an operator on Hilbert space. This means we need to consider the notion of bounded operators.

Definition 2.1.5. If $\mathcal{H}$ is a Hilbert space, then the set of *bounded operators* on $\mathcal{H}$ is

$$\mathcal{B}(\mathcal{H}) = \left\{ T : \mathcal{H} \to \mathcal{H} \,|\, T \text{ is linear},\ \|T\| := \sup_{\|\xi\|\neq 0} \frac{\|T\xi\|}{\|\xi\|} < \infty \right\},$$

where a linear map is defined in the usual way.

Essentially, in Definition 2.1.5, the norm of $T \in \mathcal{B}(\mathcal{H})$ is the furthest that T can stretch any vector, and we require that it can't send something infinitely far, a reasonable requirement for a definition of boundedness. Operators also have a notion of positivity.

Definition 2.1.6. For $T, S \in \mathcal{B}(\mathcal{H})$ we write $T \geq 0$ and call T a *positive operator* if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. We say $T \geq S$ if $T - S \geq 0$.

We can square root operators as long as they are positive, established in the following result from [26, Lemma 3.55].

Lemma 2.1.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be positive. Then there exists a unique $S \in \mathcal{B}(\mathcal{H})$ such that $S^2 = T$ and $S \geq 0$. We write $S = T^{1/2}$.*

We need to define a special kind of operator, called self-adjoint. The reason for this will become clear.

Definition 2.1.8. Let $T \in \mathcal{B}(\mathcal{H})$, then the *adjoint* of T is the operator $S \in \mathcal{B}(\mathcal{H})$ such that $\langle T\xi, \eta\rangle = \langle \xi, S\eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. The operator S exists and is unique, so we use the notation $S = T^*$. We call a map *self-adjoint* if $T = T^*$

We can now consider the operator norm when our Hilbert space is the complex numbers.

Example 2.1.9. If $\mathcal{H} = \mathbb{C}^n$, and T is some linear map then

$$\|T\| = \max\{\lambda_1, \dots, \lambda_n \,|\, \lambda_j \text{ are eigenvalues of } \sqrt{T^*T}\},$$

One of the postulates of quantum mechanics is that observable quantities are represented by operators on the Hilbert space. We will formalise this in Section 3.1 when we give an axiomatic definition of quantum mechanics. For now, we will simply state some examples of common operators and the observable they correspond to.

Example 2.1.10. Some examples of the most common observables and their operators include:

- Momentum given by $\frac{\hbar}{i}\frac{\partial}{\partial x}$ on $L^2(\mathbb{R})$, where $\hbar$ is a constant called the reduced Planck constant.
- Position given by X on $L^2(\mathbb{R})$ where X acts on a state vector ξ by $(X\xi)(x) = x\xi(x)$.
- Kinetic Energy given by $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ on $L^2(\mathbb{R})$. The total energy, that is kinetic plus some potential term, is called the Hamiltonian H. We can describe systems by giving their Hilbert space $\mathcal{H}$ and Hamiltonian H.

We note that actually none of the operators in Example 2.1.10 are bounded in the sense of Definition 2.1.5, however for our purposes this changes nothing. In a finite dimensional Hilbert space there are no unbounded operators, and for our main example we will be working in a 16-dimensional Hilbert space, which although large, is certainly finite.

2.2 Orthonormal bases

The following section is based on the discussion of orthonormal bases in [26, Sec. 3.1]. In order to make sense of Hilbert space, as with any vector space, we need to be able to talk about bases. Unfortunately, the infinite dimensionality of Hilbert spaces complicates things slightly. In the following section we will give a rundown of orthonormal bases of infinite dimensional spaces, leading into some notation that will prove vital to carrying out the computations relating to the extended Wigner's friend experiment.

Definition 2.2.1. Let $\mathcal{H}$ be a Hilbert space. A subset $S \subset \mathcal{H}$ is an orthonormal set if $\langle e_1, e_2\rangle = 0$ for all $e_1 \neq e_2$ where $e_1, e_2 \in S$, and $\langle e_1, e_1\rangle = 1$ for all $e_1 \in S$.

Definition 2.2.1 is identical to the definition of orthonormality in finite dimensions, however note that S can be infinite dimensional. The Gram-Schmidt algorithm [20, Ch. 6, Sec. 2] [25, Sec. 6.4] then gives us a way to take any linearly independent set $M \subset \mathcal{H}$ in our space and make an orthonormal set S such that $\text{span}(S) = \text{span}(M)$. This may seem fine, but bases are most useful when we can use them to write a vector as a sum of its components in each of the basis 'directions'. This looks like $\xi = \sum_n \langle e_n, \xi\rangle e_n$ for $\xi \in \mathcal{H}$ and $\{e_n\}, n = 1, 2, \ldots$ a (possibly infinite) orthonormal set. This corresponds, as in finite dimensions, to something like taking the 'amount' of ξ in the 'direction' of each e_n and then multiplying by that 'direction'. The sum of all these components should be the entire vector. But since the orthonormal set may be infinite, we could run into a problem if the sum isn't convergent. To show that indeed it is and that we need not be concerned about convergence, we introduce a theorem which gives an upper bound, called Bessel's inequality.

Theorem 2.2.2 (Bessel's inequality)**.** *Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}, n = 1, 2, \ldots$ a countable orthonormal set. Then for all $\xi \in \mathcal{H}$*

$$\sum_n |\langle e_n, \xi\rangle|^2 \leq \|\xi\|^2.$$

Proof. For all $n \geq 1$ we define

$$\xi_n = \xi - \sum_{j=1}^{n} \langle e_j, \xi\rangle e_j.$$

If we imagine ξ as an infinite column vector, then ξ_n is the same vector but with all the entries above n made 0. Because of this, taking the inner product of ξ_n and e_j when j is between 1 and n will yield 0. This means we have $\xi_n \perp e_j$ for $1 \leq j \leq n$, and so we can apply the Pythagorean theorem to get

$$\begin{aligned} \|\xi\|^2 &= \|\xi_n\|^2 + \left\| \sum_{j=1}^{n} \langle e_j, \xi \rangle e_j \right\|^2 \\ &= \|\xi_n\|^2 + \sum_{j=1}^{n} |\langle e_j, \xi \rangle|^2 \\ &\geq \sum_{j=1}^{n} |\langle e_j, \xi \rangle|^2, \end{aligned}$$

where the second line follows since $\|e_j\|^2 = 1$, as they are orthonormal. □

Note that the orthonormal set was specifically chosen to be countable, rather than uncountable. A countable set S is one that is no larger than the natural numbers, in the sense that there is an injection $I : S \to \mathbb{N}$. Examples of countable sets include the natural numbers, the even numbers, and the multiples of 3. Uncountable sets include the reals and the complex numbers. Countable sets are preferable, since we can apply more finite dimensional intuition to understand them. For example, the sum of components of ξ could be carried out as a straightforward sum, setting first $n = 1$ then $n = 2$ and so forth. If instead the n had to be all the real numbers for instance, it would be far less clear how best to proceed with the sum. However, with two small corollaries we can extend the result of Bessel's inequality to any set, not necessarily countable.

Corollary 2.2.3. *Given any orthonormal set $S \in \mathcal{H}$ and some vector $\xi \in \mathcal{H}$, there are at most countably many $e \in S$ such that $\langle e, \xi \rangle \neq 0$.*

Proof. For all $n \geq 1$, we define $S_n = \{e \in S : |\langle e, \xi \rangle| \geq \frac{1}{n}\}$. Then we can apply the Bessel inequality to some $\xi \in \mathcal{H}$ and all $e_j \in S_n$ to get

$$\|\xi\|^2 \geq \sum_j |\langle e_j, \xi \rangle|^2 \geq \sum_j \frac{1}{n^2}. \tag{2.1}$$

If S_n were infinite, then on the right hand side of (2.1) we would have a divergent sum, which is a contradiction since (2.1) also shows it is bounded. Therefore S_n must be finite, and since we can write $S = \cup_n S_n$, which is a countable union of finite sets, we know that S is countable. Therefore there are at most countably many non-zero inner products with ξ and elements of S. □

Corollary 2.2.4. *The condition on Theorem 2.2.2 that $\{e_n\}$ be countable is superfluous, and therefore the Bessel inequality holds for any orthonormal set.*

Proof. This follows since only countably many terms are non-zero, using the result of Corollary 2.2.3. □

Now that we know we can make sense of the notion of basis in infinite dimensional Hilbert space, we give the formal definition.

Definition 2.2.5. An *orthonormal basis* is a maximal orthonormal set.

Maximal is used in Definition 2.2.5 in the sense that any set strictly containing the basis would no longer be orthonormal. This would be because making the set too big would violate linear independence of the set. We now wish to state a theorem which lays some basic ideas about orthonormal bases of Hilbert spaces, followed by a theorem which outlines how orthonormal bases of Hilbert spaces work similarly to bases of vector spaces.

Theorem 2.2.6. *Let $\mathcal{H}$ be a Hilbert space. Then:*

- *Orthonormal bases for $\mathcal{H}$ exist.*
- *The cardinality of any two orthonormal bases of $\mathcal{H}$ is the same.*
- *Any other Hilbert space isomorphic to $\mathcal{H}$ has orthonormal bases of the same cardinality, in other words isomorphism classes of Hilbert spaces are determined by the cardinality of the orthonormal bases.*
- *Given $\psi \in \mathcal{H}$ and an orthonormal basis $\{\psi_\alpha\}$ we have*

$$\psi = \sum_\alpha \psi_\alpha \langle \psi_\alpha, \psi \rangle \quad \text{and} \quad \|\psi\|^2 = \sum_\alpha |\langle \psi_\alpha, \psi \rangle|^2.$$

The following theorem is from [26, Theorem 3.16].

Theorem 2.2.7. *If S is an orthonormal set in a Hilbert space $\mathcal{H}$ then the following are equivalent:*

1. *S is an orthonormal basis;*
2. *If $\xi \in \mathcal{H}$ and $\xi \perp S$ then $\xi = 0$;*
3. $\overline{\operatorname{span}(S)} = \mathcal{H}$*;*
4. *For all $\xi \in \mathcal{H}$, $\xi = \sum_{e \in S} \langle e, \xi \rangle e$;*
5. *If $\xi, \eta \in \mathcal{H}$ then $\langle \xi, \eta \rangle = \sum_{e \in S} \langle \xi, e \rangle \langle e, \eta \rangle$;*
6. *If $\xi \in \mathcal{H}$ then $\|\xi\|^2 = \sum_{e \in S} |\langle e, \xi \rangle|^2$ (called Parseval's theorem).*

This theorem gives us all the tools we need to be sure that our bases will do as we expect them to. Lastly for this section, we will state and prove a result regarding orthogonal bases of eigenvectors for operators. The importance of eigenvectors and eigenvalues of operators will be emphasised later. This result also shows the importance of commutativity, a property most pairs of operators do not have, causing many of the problems we will examine.

Theorem 2.2.8. *Let $T, S \in M_n(\mathbb{C})$ be self adjoint operators. Then there is an orthogonal basis of $\mathbb{C}^n$ of joint eigenvectors for T and S if and only if $TS = ST$.*

Proof. First, assume T and S commute. We will show that an orthogonal basis of $\mathbb{C}^n$ which diagonalises T also diagonalises S. Since T is self-adjoint, it is diagonalisable and so we know that an orthogonal basis of eigenvectors of T exists. This implies that we have a full set of eigenvectors for T, and so assume that $\psi \in \mathbb{C}^n$ is an eigenvector of T, with eigenvalue $t \in \mathbb{C}$, so that $T\psi = t\psi$. If $t = 0$, we have $TS\psi = ST\psi = 0$ and we are done, so assume $t \neq 0$, then we can write $\frac{1}{t}T\psi = \psi$. Now, we have

$$S\psi = S\frac{1}{t}T\psi = \frac{1}{t}ST\psi = \frac{1}{t}TS\psi,$$

and so we have $TS\psi = tS\psi$. So $S\psi$ is an eigenvector of T for the same eigenvalue, and so $S\psi$ is in the t-eigenspace of T. Thus we have $SP_t^T = P_t^T S = P_t^T S P_t^T$ for a projection P_t^T onto the t-eigenspace of T, and since S is self-adjoint, we can find a basis of $P_t^T\mathbb{C}^n$ consisting of eigenvectors of S.

Now, for the converse statement, we assume that there exists an orthonormal basis of common eigenvectors for T and S, and must prove that $TS = ST$. Let ψ be a common eigenvector, so $T\psi = t\psi$, and $S\psi = s\psi$, so that we have

$$ST\psi = St\psi = tS\psi = ts\psi.$$

Similarly, we have

$$TS\psi = Ts\psi = sT\psi = st\psi = ts\psi,$$

and so $ST\psi = TS\psi$. Since TS and ST have the same action on the orthonormal basis of eigenvectors, they have the same action on all vectors, and are therefore equal. □

2.3 The tensor product

We have so far seen how we can use Hilbert space to describe one quantum system, but now we ask what of composite systems? The answer relies on an operation called the tensor product $\otimes$, which we will now explore. We leave discussions of how these composite systems look and their internal interactions to subsequent sections. The tensor product will be widely used in our consideration of the extended Wigner's friend experiment, as we will have to consider 4 distinct quantum systems each described by $\mathbb{C}^2$, placing our computations into a 16-dimensional Hilbert space.

We will present a similar treatment of the tensor product to that given in [20, Ch. 13, Sec. 1, 2]. We want to define a product of Hilbert spaces $\otimes$ that is bilinear. That is, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $\xi, \xi' \in \mathcal{H}_1$ and $\eta, \eta' \in \mathcal{H}_2$ and $\lambda \in \mathbb{C}$ then the following three properties hold

$$\begin{aligned}(\xi + \xi') \otimes \eta &= \xi \otimes \eta + \xi' \otimes \eta \\ \xi \otimes (\eta + \eta') &= \xi \otimes \eta + \xi \otimes \eta' \\ (\lambda\xi) \otimes \eta &= \lambda(\xi \otimes \eta) = \xi \otimes \lambda\eta.\end{aligned}$$

This amounts to nothing more than linearity in its usual sense in both the variables of the product, hence the name bilinearity. This allows us to state a theorem [20, Ch. 13, Theorem 1] establishing the existence of such a map and the existence of the space of such products.

Theorem 2.3.1. *Let V, W be finite dimensional vector spaces over $\mathbb{C}$. Then there exists a finite dimensional space T over $\mathbb{C}$, and a bilinear map $V \times W \to T$ denoted by*

$$(v, w) \mapsto v \otimes w,$$

satisfying the following properties.

1. *The map $(v, w) \mapsto v \otimes w$ is bilinear.*
2. *If U is a vector space and $g : V \times W \to U$ is a bilinear map, then there exists a unique linear map $g_* : T \to U$ such that, for all pairs (v, w) with $v \in V$ and $w \in W$ we have $g(v, w) = g_*(v \otimes w)$.*
3. *If $\{v_1, \dots, v_n\}$ is a basis of V, and $\{w_1, \dots, w_n\}$ is a basis of W, then the elements $v_i \otimes w_j$ for $i, j = 1, \dots, n$ form a basis of T.*

4. *There is a unique isomorphism* $U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ *such that* $u \otimes (v \otimes w) \to (u \otimes v) \otimes w$ *for all* $u \in U$, $v \in V$ *and* $w \in W$.

Proof. The proof may be found in [20, Ch. 13 Theorem 1, 2]. □

We can make a definition from the result in Theorem 2.3.1.

Definition 2.3.2. We call the space T established in Theorem 2.3.1 the tensor product of V and W, and write $T = V \otimes W$. The element $v \otimes w$ is also called the tensor product of v and w. The inner product on the tensor product space is defined to be $\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle_{V\otimes W} = \langle v_1, v_2\rangle_V \langle w_1, w_2\rangle_W$.

Theorem 2.3.1 refers only to finite dimensional vector spaces, however the extension to infinite dimensional Hilbert space is possible, although some care must be taken with completeness. We will not be working in such spaces for the purposes of this investigation, and therefore will not concern ourselves with the details. Further details may be found in [26, Sec. 3.1.1].

From part 3 of Theorem 2.3.1 we can see that the dimension of $V \otimes W$ is simply the dimension of V multiplied by the dimension of W. We therefore have a way to 'multiply' Hilbert spaces using the tensor product, in the same sense as the direct sum of spaces is 'adding' them. It will be important that we know the tensor product of bases forms the basis of the tensor product space. Part 4 of Theorem 2.3.1 establishes that the tensor product is associative. Part 2 of Theorem 2.3.1 is the most subtle. It refers to a property of the tensor product sometimes called the universal property. Essentially, if we can draw a map from the direct product space $V \times W$ to some other space U, then there is an equivalent map from the tensor product space $V \otimes W$ to U. Therefore if we are mapping to U, we may either map directly to U, or first go via the tensor product space and then to U.

We can think of the tensor product as taking every possible combination of products of the elements of vectors, at least when we are in $\mathbb{C}^n$. An example to illustrate this will wrap up this section.

Example 2.3.3. Take $a, b, c, d \in \mathbb{C}$. We identify the tensor product of two vectors in $\mathbb{C}^2$ with a vector in $\mathbb{C}^4$ via the isomorphism

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \to \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

We can therefore make the identification $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, and may refer to this isomorphism as an equality. In our main example we will be working in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which we say is $\mathbb{C}^{16}$, although it is technically only isomorphic to $\mathbb{C}^{16}$. In the isomorphism, there is some free choice around where we put which product, but with careful specification, this is no issue. We will not be worrying so much about actually calculating tensor products, since in 16 dimensional Hilbert spaces this will get messy, and our calculations can be carried out without such large matrices. We therefore leave the worked examples of the tensor product here. An accessible introduction to tensor products with further examples can be found in [19].

2.4 Bras and kets

The next goal is to introduce some notation that we will use extensively in the analysis of the extended Wigner's friend experiment, and which is used very extensively in quantum mechanics. Bra-ket notation was introduced by Paul Dirac in 1939. In the notation, we call an object $|\xi\rangle \in \mathcal{H}$ as "ket xi" and the object $\langle\xi| \in \mathcal{H}^*$ as "bra xi".

The space $\mathcal{H}^*$ is called the dual space and will be defined shortly. Mathematically speaking, a ket is a vector, while a bra is a linear map $\langle\xi| : \mathcal{H} \to \mathbb{C}$. We will unpack what this means and justify the rules that we will then use to manipulate bras and kets.

Definition 2.4.1. If V is a vector space with a norm, then $V^* := \{\phi : V \to \mathbb{C} : \phi \text{ is linear and continuous}\}$ is called the *dual space* of V.

Lemma 2.4.2. *The space V^* is a vector space, with V^* defined as in Definition 2.4.1. The addition in V^* is given by $(\phi_1 + \phi_2)(v) = \phi_1(v) + \phi_2(v)$ for $\phi_1, \phi_2 \in V^*$ and $v \in V$, and the scalar multiplication is given by $(\lambda\phi_1)(v) = \lambda\phi_1(v)$ for $\lambda \in \mathbb{C}$. The norm is given by $\|\phi_1\| = \sup_{\|v\|=1} |\phi_1(v)|$.*

Essentially the norm of V^* is asking, if we take all elements of size 1 in V, how far can ϕ_1 possibly take them? The reason for the name dual space is about to become apparent, since it turns out that there is an isomorphism between a vector space and its dual space. Since a Hilbert space is ultimately a vector space we can also talk about the dual space of a Hilbert space. In fact, there also exists an isomorphism for the dual of Hilbert spaces, as captured in the next result, which appears in [26, Proposition 3.11].

Theorem 2.4.3 (Riesz representation theorem)**.** *If $\mathcal{H}$ is a Hilbert space, then each $\varphi \in \mathcal{H}^*$ is given by $\varphi(\xi) = \langle\eta, \xi\rangle$ for a unique $\eta \in \mathcal{H}$. Further, $\|\eta\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}^*}$.*

Proof. First, consider the case where φ is the zero map. In this case, take $\eta = 0$, and then we have $\langle\eta, \xi\rangle = \langle 0, \xi\rangle = 0$ and we have $\langle\eta, \xi\rangle = \varphi(\xi)$. Now we consider the case where φ is non-zero. Then there exists some $\tilde{\eta} \in \ker(\varphi)^\perp$. Using this, we make the definition

$$\eta = \frac{\overline{\varphi(\tilde{\eta})}}{\|\tilde{\eta}\|^2}\tilde{\eta}.$$

First note that for any $\xi \in \mathcal{H}$ we may write

$$\xi = \left(\xi - \frac{\varphi(\xi)}{\varphi(\tilde{\eta})}\tilde{\eta}\right) + \frac{\varphi(\xi)}{\varphi(\tilde{\eta})}\tilde{\eta}. \tag{2.2}$$

Note that the parenthesised term in (2.2) is in the kernel of φ, since

$$\varphi\left(\xi - \frac{\varphi(\xi)}{\varphi(\tilde{\eta})}\tilde{\eta}\right) = \varphi(\xi) - \varphi\left(\frac{\varphi(\xi)}{\varphi(\tilde{\eta})}\tilde{\eta}\right) = \varphi(\xi) - \frac{\varphi(\xi)}{\varphi(\tilde{\eta})}\varphi(\tilde{\eta}) = 0,$$

while the non-parenthesised term is in the span of $\tilde{\eta}$. We therefore check whether $\langle\eta, \xi\rangle = \varphi(\xi)$ in the cases where $\xi \in \ker(\varphi)$ and $\xi = \lambda\tilde{\eta}$ for some $\lambda \in \mathbb{C}$. Firstly, if $\xi \in \ker(\varphi)$, then $\xi \perp \tilde{\eta}$, by the very choice of $\tilde{\eta}$, and so $\langle\eta, \xi\rangle = 0$ and we are done. If instead $\xi = \lambda\tilde{\eta}$, we calculate

$$\begin{aligned}\langle\eta, \xi\rangle = \langle\eta, \lambda\tilde{\eta}\rangle = \left\langle \frac{\overline{\varphi(\tilde{\eta})}}{\|\tilde{\eta}\|^2}\tilde{\eta}, \lambda\tilde{\eta}\right\rangle = \lambda\left\langle \frac{\overline{\varphi(\tilde{\eta})}}{\|\tilde{\eta}\|^2}\tilde{\eta}, \tilde{\eta}\right\rangle = \lambda\frac{\varphi(\tilde{\eta})}{\|\tilde{\eta}\|^2}\langle\tilde{\eta}, \tilde{\eta}\rangle \\ = \lambda\frac{\varphi(\tilde{\eta})}{\|\tilde{\eta}\|^2}\|\tilde{\eta}\|^2 = \lambda\varphi(\tilde{\eta}) = \varphi(\lambda\tilde{\eta}) = \varphi(\xi).\end{aligned}$$

We have therefore shown that any arbitrary $\varphi \in \mathcal{H}^*$ can be written as $\varphi(\xi) = \langle\eta, \xi\rangle$. To show this η is unique for each φ, assume there exists $\eta_1, \eta_2 \in \mathcal{H}$ such that $\langle\eta_1, \xi\rangle = \varphi(\xi)$ and $\langle\eta_2, \xi\rangle = \varphi(\xi)$. Then we have $\langle\eta_1, \xi\rangle = \langle\eta_2, \xi\rangle$, and we can write

$$\langle\eta_1, \xi\rangle = \langle\eta_2, \xi\rangle, \quad \text{or} \quad \langle\eta_1 - \eta_2, \xi\rangle = 0,$$

and since this must hold for any choice of ξ we can choose $\xi = \eta_1 - \eta_2$, and so we have

$$\langle \eta_1 - \eta_2, \eta_1 - \eta_2 \rangle = 0, \quad \text{or} \quad \|\eta_1 - \eta_2\|^2 = 0, \quad \text{and so} \quad \|\eta_1 - \eta_2\| = 0,$$

and since the only vector whose norm is 0 is the zero vector, we know $\eta_1 - \eta_2 = 0$ and therefore $\eta_1 = \eta_2$, proving uniqueness of the η for any φ. Finally, we want to prove the claim about the equality of norms. That is, we want to show $\|\eta\| = \sup_{\|\xi\|=1} |\varphi(\xi)| = \sup_{\|\xi\|=1} |\langle \eta, \xi \rangle|$, using the first part of the lemma. The Cauchy-Schwarz inequality for inner products [see 25, Theorem 6.37] says that $|\langle \eta, \xi \rangle| \leq \|\eta\| \|\xi\|$ for all $\eta, \xi \in \mathcal{H}$, with equality if and only if η is a multiple of ξ. We therefore have

$$\sup_{\|\xi\|=1} |\langle \eta, \xi \rangle| \leq \|\eta\|, \tag{2.3}$$

since $\|\xi\| = 1$ by choice. Indeed we have equality in (2.3) since we are taking the supremum, and any other ξ giving a smaller value of the inner product must cause the right hand side of (2.3) to also become smaller. ☐

Corollary 2.4.4. *There is a conjugate linear isomorphism $\mathcal{H} \to \mathcal{H}^*$ given by $\varphi \mapsto \eta$.*

With the understanding of the relationship between the dual space and the Hilbert space itself, we introduce the "bra-ket" notation. For an element $\xi \in \mathcal{H}$, we now write $|\xi\rangle$. We can write any element of the dual space as the inner product with ξ, and so we separate the notation and write $\varphi = \langle \eta|$, assigning to φ its equivalent element in $\mathcal{H}$ and using notation reminiscent of the 'first half' of the inner product notation. Then we can write the action of the map $\langle \eta|$ on ξ as $\langle \eta| (\xi) = \langle \eta|\xi\rangle$, and we also know it makes sense to write the inner product as $\langle \eta|\ |\xi\rangle = \langle \eta|\xi\rangle$. Note that it also makes sense to transform a ket into its equivalent bra, and we will need to do this during our calculations. If we take the product of a ket and then a bra, like $|\xi\rangle\langle\eta|$, we get something called the *outer product*, which is an object living in the space of operators on $\mathcal{H}$. The true beauty of the bra-ket notation can be summed up with the following result

$$(|\xi\rangle \langle \eta|) |\alpha\rangle = |\xi\rangle \langle \eta|\alpha\rangle, \tag{2.4}$$

where $\alpha \in \mathcal{H}$. The left-hand side of (2.4) is an operator acting on an element of $\mathcal{H}$, while the right hand side is a number multiplied by an element of $\mathcal{H}$. We used the result of Lemma 2.4.3 to rewrite the action of $\langle \eta|$ on $|\alpha\rangle$ as an inner product. When using the bra-ket notation, it is easy to think that nothing of note happened in (2.4), since it looks obvious that the equality holds, however we needed to do some work to be able to even make that statement. That is the power of bras and kets.

As discussed in [29, Sec. 1.3], if we consider that a ket is a column vector, then its equivalent bra can just be the same vector written as a row, with each element turned into its complex conjugate, called the conjugate transpose. Row vectors do indeed act as linear maps on column vectors, since a row times a column using matrix multiplication is a number. Also, a column times a row with matrix multiplication will give us a matrix, in line with our expectation of the outer product.

We now make one last definition, merely notational. For the outer product, we may sometimes write $\Theta_{\xi,\eta}$ for the operation $|\xi\rangle\langle\eta|$. This is because there exists an isomorphism from the space $\mathcal{H} \otimes \mathcal{H}^*$ to the space of all traceclass operators. We move now to introduce the trace.

2.5 Trace and partial trace

The trace is an operation we have been taught to perform on matrices, however it can also be carried out on a general operator which has yet to be written down in a particular orthonormal basis. Perhaps more interestingly, the trace is independent of the choice of orthonormal bases, and obeys many of the nice properties that linear functions also obey.

Definition 2.5.1. Let $\mathcal{H}$ be a Hilbert space, and let $(\phi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $T \geq 0$. Then we define the *trace of* T to be the operation $\mathrm{Tr}^\phi : \mathcal{B}(\mathcal{H})_+ \to [0, \infty]$ given by

$$\mathrm{Tr}^\phi(T) := \sum_{j\in\mathbb{N}} \langle \phi_j, T\phi_j\rangle.$$

There are some technical points to make here. Definition 2.5.1 requires that T be a positive operator. By Lemma 2.1.7, we can write $T = (T^{1/2})^2$, an expression we will use in the next result, which shows that the trace is independent of the choice of orthonormal bases.

Theorem 2.5.2. *If $(\phi_j)_{j\in\mathbb{N}}$ and $(\psi_j)_{j\in\mathbb{N}}$ are two orthonormal bases for $\mathcal{H}$, and T is an operator, then $\mathrm{Tr}^\phi(T) = \mathrm{Tr}^\psi(T)$.*

Proof. First we realise that the adjoint of $T^{1/2}$ is $T^{1/2}$ itself. Then we calculate

$$\begin{aligned}\mathrm{Tr}^\phi(T) &= \sum_j \langle \phi_j, T\phi_j\rangle = \sum_j \langle T^{1/2}\phi_j, T^{1/2}\phi_j\rangle = \sum_j\sum_k \langle T^{1/2}\phi_j, \psi_k\rangle\langle \psi_k, T^{1/2}\phi_j\rangle = \sum_j\sum_k |\langle T^{1/2}\phi_j, \psi_k\rangle|^2 \\ &= \sum_k\sum_j |\langle T^{1/2}\phi_j, \psi_k\rangle|^2 = \sum_k\sum_j \langle T^{1/2}\psi_k, \phi_j\rangle\langle \phi_j, T^{1/2}\psi_k\rangle = \sum_k \langle T^{1/2}\psi_k, T^{1/2}\psi_k\rangle = \mathrm{Tr}^\psi(T).\end{aligned}$$

We are allowed to exchange the summation signs due to a measure theoretic result called Tonelli's Theorem [see 27, Ch. 12, Sec. 4]. □

There are many further results to do with the trace, some of which are stated in the following result.

Theorem 2.5.3. *If T, S are positive operators, then the following hold*

1. $\mathrm{Tr}(T + S) = \mathrm{Tr}(T) + \mathrm{Tr}(S)$*;*
2. $\mathrm{Tr}(\lambda T) = \lambda\,\mathrm{Tr}(T)$ *for all* $\lambda \in [0, \infty)$*;*
3. *If* $0 \leq T \leq S$ *then* $\mathrm{Tr}(T) \leq \mathrm{Tr}(S)$*;*
4. $\mathrm{Tr}(SS^*) = \mathrm{Tr}(S^*S)$*;*
5. *If U is a unitary matrix, then $\mathrm{Tr}\big(UTU^{-1}\big) = \mathrm{Tr}(T)$, in other words the trace is invariant under unitary transformations.*

We will now briefly discuss the details of the classification of those bounded operators for which the trace is finite and independent of choice of basis. In Definition 2.5.1 we said T needs to be a positive operator, and to extend the definition to non-positive T, we require that T is traceclass.

Definition 2.5.4. Let T be an operator in $\mathcal{B}(\mathcal{H})$. Then T is called *traceclass* if $|T| := \sqrt{T^*T}$ has finite trace. We write $\mathcal{L}^1(\mathcal{H})$ for the set of all trace class operators. That is, $\mathcal{L}^1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathrm{Tr}(|T|) < \infty\}$.

In fact, the subspace of trace class operators is special within the larger space of bounded operators.

Theorem 2.5.5. *If $\mathcal{H}$ is a Hilbert space, the trace class operators $\mathcal{L}^1(\mathcal{H})$ form a two-sided $*$-ideal in the bounded operators $\mathcal{B}(\mathcal{H})$. In particular, this means we have the following*

1. *$\mathcal{L}^1(\mathcal{H})$ is a vector space;*
2. *For $T \in \mathcal{L}^1(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, $TS, ST \in \mathcal{L}^1(\mathcal{H})$;*
3. *$T \in \mathcal{L}^1(\mathcal{H})$ implies that $T^* \in \mathcal{L}^1(\mathcal{H})$.*

We also wish to introduce here the partial trace, which is used in quantum mechanics to dissect from a composite system the important details about only one of the composing systems. Examples of the usage of this will be given later.

Definition 2.5.6. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Let $\xi \in \mathcal{H}_2$. Then the partial trace of T on $\mathcal{H}_1$ is an operator on $\mathcal{H}_2$ given by

$$\mathrm{Tr}_1(T)\xi = \sum_{j,k} |\psi_k\rangle \Big\langle\, |\phi_j\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\phi_j\rangle \otimes |\xi\rangle\,\big)\Big\rangle,$$

where $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_1$ and $(\psi_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_2$. Similarly, if instead $\xi \in \mathcal{H}_1$ then the partial trace of T on $\mathcal{H}_2$ is

$$\mathrm{Tr}_2(T)\xi = \sum_{j,k} |\phi_j\rangle \Big\langle\, |\phi_j\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\xi\rangle \otimes |\psi_k\rangle\,\big)\Big\rangle.$$

The name partial trace is used since we are essentially summing over a basis of one of the spaces, and are therefore tracing out one of the spaces, which is also why we are left with an operator on the other space. The last important result in this section comes as no surprise in light of what we have seen.

Theorem 2.5.7. *The partial trace is independent of the choice of orthonormal bases.*

Proof. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $(\phi_n)_{n\in\mathbb{N}}$ and $(\Phi_m)_{m\in\mathbb{N}}$ be orthonormal bases for $\mathcal{H}_1$, while $(\psi_j)_{j\in\mathbb{N}}$ and $(\Psi_k)_{k\in\mathbb{N}}$ are orthonormal bases for $\mathcal{H}_2$. The partial trace over $\mathcal{H}_1$ of some operator $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ expressed in the respective first bases is

$$\begin{aligned}
\mathrm{Tr}_1(T)\xi &= \sum_{j,k} |\psi_k\rangle \Big\langle\, |\phi_j\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\phi_j\rangle \otimes |\xi\rangle\,\big)\Big\rangle \\
&= \sum_{j,k,l} |\psi_k\rangle \Big\langle\, |\Phi_l\rangle\langle\Phi_l|\phi_j\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\phi_j\rangle \otimes |\xi\rangle\,\big)\Big\rangle \\
&= \sum_{j,k,l} |\psi_k\rangle \Big\langle\, |\Phi_l\rangle \otimes |\psi_k\rangle \,\Big|\, \langle\phi_j|\Phi_l\rangle\, T\big(\,|\phi_j\rangle \otimes |\xi\rangle\,\big)\Big\rangle \\
&= \sum_{j,k,l} |\psi_k\rangle \Big\langle\, |\Phi_l\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\phi_j\rangle\langle\phi_j|\Phi_l\rangle \otimes |\xi\rangle\,\big)\Big\rangle \\
&= \sum_{k,l} |\psi_k\rangle \Big\langle\, |\Phi_l\rangle \otimes |\psi_k\rangle \,\Big|\, T\big(\,|\Phi_l\rangle \otimes |\xi\rangle\,\big)\Big\rangle,
\end{aligned}$$

which is an equivalent expression in the second basis of $\mathcal{H}_1$. We now change basis for $\mathcal{H}_2$

$$\begin{aligned}
\mathrm{Tr}_1(T)\xi &= \sum_{k,l} |\psi_k\rangle \Big\langle |\Phi_l\rangle \otimes |\psi_k\rangle \Big| T\big(|\Phi_l\rangle \otimes |\xi\rangle\big) \Big\rangle \\
&= \sum_{k,l,m} |\Psi_m\rangle \langle \Psi_m|\psi_k\rangle \Big\langle |\Phi_l\rangle \otimes |\psi_k\rangle \Big| T\big(|\Phi_l\rangle \otimes |\xi\rangle\big) \Big\rangle \\
&= \sum_{k,l,m} |\Psi_m\rangle \Big\langle |\Phi_l\rangle \otimes |\psi_k\rangle \langle \psi_k|\Psi_m\rangle \Big| T\big(|\Phi_l\rangle \otimes |\xi\rangle\big) \Big\rangle \\
&= \sum_{k,l,m} |\Psi_m\rangle \Big\langle |\Phi_l\rangle \otimes |\Psi_m\rangle \Big| T\big(|\Phi_l\rangle \otimes |\xi\rangle\big) \Big\rangle,
\end{aligned}$$

as required. The argument is entirely symmetric for the partial trace over $\mathcal{H}_2$. □

3 Physical Theory

We now move to applying the mathematics we have developed to the understanding of quantum systems. For this, we will cover a number of definitions and examples. We begin with the core postulates of quantum theory.

3.1 Quantum states

Postulate 3.1.1. A *quantum mechanical system* is a pair $(\mathcal{H}, H)$, where $\mathcal{H}$ is a Hilbert space, and H is a self-adjoint linear operator on $\mathcal{H}$ called a *Hamiltonian*.

We saw in Example 2.1.10 that the Hamiltonian can represent the total energy, but it is important as it governs the time evolution of the system. In particular, if $|\psi\rangle_0$ is some state vector in $\mathcal{H}$ initially, then at time t the vector $|\psi\rangle_t$ is given by $|\psi\rangle_t = e^{-iHt}|\psi\rangle_0$. An alternate view point is that we may consider the Hamiltonian to be the object which evolves operators in time, rather than state vectors, in which case an operator A_0 will become $A_t = e^{-iHt}A_0e^{iHt}$ at time t.

Postulate 3.1.2. Each physical state of a quantum system $(\mathcal{H}, H)$ is specified by a norm-1 vector $|\psi\rangle \in \mathcal{H}$, and is called a *pure* state.

As mentioned in Postulate 3.1.2 above, each vector $|\psi\rangle \in \mathcal{H}$ represents a possible state of the quantum system. We always ask that $\| |\psi\rangle \| = 1$, so that our vector is *normalised*, equivalent to the total probability it is somewhere being 1, or certain. Technically, scalar multiples of a vector represent the same quantum system, and so we choose the unit vector in that direction, since it allows us to interpret quantum mechanics in a sensible probabilistic way, referred to as the Born interpretation.

Postulate 3.1.3. Each physical observable is represented by a linear, self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$.

Associated to each observable T is a measure, and the eigenvalues (or spectrum) of T are the values which may be observed when a measurement is made of T in a state $|\psi\rangle$. We can expand state vectors into the eigenbasis of the operator we are looking at, as $|\psi\rangle = \sum_{k=1}^{n} c_k |\psi_k\rangle$, and the square of the coefficient $|c_k|^2$ in front of eigenstate $|\psi_k\rangle$ is the probability the system will be measured to be in that state. The eigenvalue equation $T|\psi_k\rangle = \lambda_k |\psi_k\rangle$ then guarantees the measured value of T in this state will be the eigenvalue λ_k.

Often a fourth postulate is included, being that the states and Hamiltonian obey the Schrödinger equation. In reality, the equation is a form of eigenvalue equation for the Hamiltonian, and we are not so interested in time-evolution for the Wigner's friend experiment, and so we will leave discussion of this here. Further discussion on the Schrödinger equation may be found in any standard quantum mechanics textbook, see for example [29, Sec. 4.3], [18, Sec. 1.1], [14, Sec. 7.2].

With the basic postulates of the theory established, we now want to dig a little deeper into some of the results we can prove from here. As it turns out, the pure state vectors in $\mathcal{H}$ don't encapsulate all the possible quantum states. We call states that are not pure states *mixed* states. The latter are states which may be in any of a number of pure states, with some classical probability attached to each state it may be in. This is not a quirky quantum mechanical phenomenon, a mixed state simply represents the case where we are ignorant of the state. We cannot write a mixed state as a single vector, since we are considering the possibility of lots of different state vectors, so we must introduce something new, called a density matrix, our next definition.

Definition 3.1.4. A *density matrix* is a trace-class operator ρ of the form $\rho = T^*T$ with T any operator, such that $\mathrm{Tr}(\rho) = 1$. The most general form of a density matrix is $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ where the ψ_j are pure states and λ_j is the probability the state will be $|\psi_j\rangle$, so therefore $\sum_j \lambda_j = 1$.

This brings about an easier way to distinguish pure and mixed states; a state is pure if and only if the density matrix is of the form $\rho = |\psi\rangle\langle\psi|$. Note also that we can use the notation defined at the end of section 2.4 in this case to write $\rho = \Theta_{\psi,\psi}$. Since there exists an isomorphism from $\mathcal{H} \otimes \mathcal{H}^*$ to the space of density matrices, it is also correct to write $\rho = |\psi\rangle \otimes |\psi\rangle^*$. The most useful way to distinguish pure and mixed states is the following result.

Lemma 3.1.5. *A density matrix ρ represents a pure state if and only if it is idempotent, that is $\rho^2 = \rho$.*

Proof. If ρ is a pure state then $\rho = |\psi\rangle\langle\psi|$ for some state vector ψ. Then we have

$$\rho^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = |\psi\rangle \|\psi\|^2 \langle\psi| = |\psi\rangle\langle\psi| .$$

Now conversely assume we know $\rho^2 = \rho$. The most general form of ρ^2 is

$$\left(\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|\right)\left(\sum_k \lambda_k |\psi_k\rangle\langle\psi_k|\right) = \sum_{j,k} \lambda_j\lambda_k |\psi_j\rangle \langle\psi_j|\psi_k\rangle \langle\psi_k| = \rho^2 = \rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j| .$$

Since $\lambda_j\lambda_k = \lambda_j$ for all k, we must have every $\lambda_k = 1$ or 0, and so every corresponding $\lambda_j = 1$ or 0 since they are equal. But then we cannot have more than one term in ρ without violating the probabilistic meaning of mixed states, and so ρ must be pure. □

An important distinction as noted above is that the probabilities λ_j associated with mixed states are classical, which is different from the quantum probabilities associated with superposition states. We use mixed states to describe the case where we simply aren't sure exactly how to describe the system, and not the case where we have a system in a superposition of multiple different states at once (discussed in Section 3.3). It is like the quantum analogue of a classical ensemble of systems, as used in thermodynamics. A brief and simple explanation of mixed states can be found in the Afterword of [18], while other accounts may be found in [22, Sec. 4.4.] and [3, Sec. 6.3].

Returning now to pure states and the operators that act upon them, we wish to learn more about the values taken by the observables represented by these operators. Since we are dealing with probabilities, it makes sense to define a notion of the expectation value for an observable, this being the most likely value to measure.

Definition 3.1.6. If $\mathcal{H}$ is a Hilbert space and $|\psi\rangle \in \mathcal{H}$ is such that $\||\psi\rangle\| = 1$, then we define a function

$\mathbb{E}_{|\psi\rangle} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by

$$\mathbb{E}_{|\psi\rangle}(T) = \langle \psi, T\psi \rangle$$

The function $\mathbb{E}_{|\psi\rangle}$ gives the *expected value* of the observable T when measured in the state $|\psi\rangle$. If $|\psi\rangle$ is a mixed state of pure states $|\psi_j\rangle$, $j = 1, \ldots, N$ we have

$$\mathbb{E}_{|\psi\rangle}(T) = \sum_{j=1}^{N} \lambda_j \langle \psi_j, T\psi_j \rangle.$$

This can be demonstrated in a more concrete example.

Example 3.1.7. If $\mathcal{H} = L^2(\mathbb{R})$, A is an observable and $|\psi\rangle = \psi(x)$ is a (pure) state, then the expected value of A in the state $|\psi\rangle$ is

$$\langle \psi, A\psi \rangle = \int_{\mathbb{R}} \bar{\psi}(x) A\psi(x)\,\mathrm{d}x\,.$$

Let $A = X$, the position operator, and let the state vector $\psi(x)$ be given by

$$\psi(x) = \begin{cases} 0 & |x| \geq 22 \\ \frac{1}{\sqrt{44}} & |x| < 22 \end{cases},$$

which is normalised. Then the expected position to find a particle which is in the state ψ is

$$\langle \psi, X\psi \rangle = \int_{\mathbb{R}} \bar{\psi}(x) X\psi(x)\,\mathrm{d}x = \frac{1}{44}\int_{-22}^{22} x\,\mathrm{d}x = 0.$$

We are therefore most likely to find the particle at $x = 0$.

Next, we present a result which shows the usefulness of the density matrix and trace in light of finding the expectation value.

Theorem 3.1.8. *If ρ is a density matrix and T an observable, then* $\mathbb{E}(T) = \mathrm{Tr}(\rho T)$.

Proof. First, consider the case where we have a pure state, and so can write $\rho = |\psi\rangle\langle\psi|$. We simply carry out the computation

$$\begin{aligned}\mathrm{Tr}(\rho T) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,T) &= \sum_{j=1}^{N} \langle \psi_j|\psi\rangle \langle \psi|T\psi_j\rangle = \sum_{j=1}^{N} \langle \psi|T\psi_j\rangle \langle \psi_j|\psi\rangle \\ &= \sum_{j=1}^{N} \langle T^*\psi|\psi_j\rangle \langle \psi_j|\psi\rangle = \langle T^*\psi|\psi\rangle = \langle \psi|T\psi\rangle = \mathbb{E}(T).\end{aligned}$$

Now we consider the case where ρ is a mixed state, and so we have $\rho = \sum_{j=1}^{N} \lambda_j \langle \psi_j|\psi_j\rangle$. Once again we proceed by direct computation

$$\begin{aligned}\operatorname{Tr}(\rho T) = \operatorname{Tr}\left(\sum_{j=1}^{N} \lambda_j \, |\psi_j\rangle\langle\psi_j| \, T\right) &= \sum_{j=1}^{N} \operatorname{Tr}(\lambda_j \, |\psi_j\rangle\langle\psi_j| \, T) = \sum_{j=1}^{N} \lambda_j \operatorname{Tr}(|\psi_j\rangle\langle\psi_j| \, T) \\ &= \sum_{j=1}^{N} \lambda_j \, \langle\psi_j|T\psi_j\rangle = \mathbb{E}(T),\end{aligned}$$

where the result for the pure case proved above was used. □

3.2 Further examples

We now give some more concrete examples with their worked solution, to become more familiar with the calculation side of quantum mechanics.

Example 3.2.1. Let the operator $A \in \mathcal{B}(\mathbb{C}^3)$ be given by

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

expressed in its eigenbasis (which diagonalises it), and let A have eigenvectors ψ_1, ψ_2, ψ_3, the standard basis vectors of $\mathbb{C}^3$. Consider a state vector $\psi \in \mathbb{C}^3$, which of course is such that $\|\psi\|^2 = 1$. Then ψ can be expanded in the eigenbasis to look like

$$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3,$$

for some scalar coefficients $c_i \in \mathbb{C}$. We then have that the norm of ψ is

$$\|\psi\|^2 = |c_1|^2 + |c_2|^2 + |c_3|^2 = 1.$$

Finally, the expected value of the operator A is

$$\begin{aligned}\langle\psi, A\psi\rangle &= \langle c_1\psi_1 + c_2\psi_2 + c_3\psi_3, 3c_1\psi_1 + 2c_2\psi_2 + c_3\psi_3\rangle \\ &= 3|c_1|^2 + 2|c_2|^2 + |c_3|^2.\end{aligned}$$

Example 3.2.2. This time, we will consider a more realistic and useful example, the harmonic oscillator. We use $\mathcal{H} = L^2(\mathbb{R})$. The Hamiltonian for the harmonic oscillator is given by $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$, where we have set the constants m and ω to 1 for simplicity. The problem proceeds as we may expect, with the main aim being to first diagonalise the operator, in this case the Hamiltonian. We begin by stating the eigenvalue problem

$$\begin{aligned}H\psi &= \lambda\psi \\ -\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + x^2\psi(x) &= \lambda\psi(x).\end{aligned}$$

We will not be solving this explicitly, however the solution does exist, as the harmonic oscillator is one of the few solvable quantum systems. The solutions are given by

$$\lambda = 2n+1, \quad n = 0,1,2,\dots$$
$$\psi_n = h_n e^{-x^2/2},$$

where h_n is the n-th Hermite polynomial [more details in 30]. In particular, $e^{-x^2/2}$ is the eigenvector for eigenvalue 1. Therefore when we diagonalise H we will have $1,3,5,\dots$ along the diagonal. This is clearly an infinite dimensional matrix. In the context of the harmonic oscillator, we can make a useful definition. Let

$$a = \frac{\mathrm{d}}{\mathrm{d}x} + x, \quad \text{and} \quad a^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + x.$$

These two operators are adjoints, and are sometimes called ladder operators, raising and lowering operators or annihilation and creation operators. They satisfy the properties

$$a^\dagger a = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 - 1 = H - 1, \quad \text{and} \quad aa^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + 1 = H + 1.$$

This leads to the following result. Let ψ be an eigenvector for eigenvalue λ. Then the vector $a\psi$ satisfies

$$Ha\psi = (aa^\dagger - 1)a\psi = a(a^\dagger a - 1)\psi = a(H-2)\psi = a(\lambda - 2)\psi = (\lambda - 2)a\psi,$$

and we find that the action of the operator a has been to decrease the eigenvalue of the state. Similarly, $a^\dagger$ will increase the eigenvalue. This justifies the various names.

A further question is whether there is a lower limit to the decrease in eigenvalues. A mathematical argument is that since $\langle a^\dagger a\xi, \xi\rangle = \langle a\xi, a\xi\rangle \geq 0$, we know $a^\dagger a$ is a non-negative operator, meaning it is bounded below, and since also $a^\dagger a = H - 1$, we know that H is bounded below and therefore there is a lower limit to the energy levels. A physical argument is that if there were no lower bound the system could lose energy indefinitely, leading to infinite energies, which is nonsensical. For this to happen, we must have some ψ_0 such that $a\psi_0 = 0$. In this case, every eigenvector can be written in the form $\psi_n = (a^\dagger)^n \psi_0$. This leads directly to the same result for the eigenvalues as we had above.

As a finale for this section, we introduce a concept, some operators and notation to go along.

Example 3.2.3. Particles have an intrinsic quantum property called *spin* (a discussion of this may be found in any textbook on the subject, for instance [29, Ch. 14]). It is not strictly correct to view spin as a particle really rotating about its axis, however it is related to angular momentum. When we measure the spin in any given direction, we get one of only two outcomes, which we call up and down. We use $\mathcal{H} = \mathbb{C}^2$ to describe the spin states of a particle, and the standard orthonormal basis of $\mathbb{C}^2$ represents the cases of up or down by

$$\text{“spin up”} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \text{“spin down”} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We introduce some notation to make this easier to work with

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As with any system, there are operators, and in this case the operators for measurement of the spin along the three coordinate directions x, y and z are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

respectively. These σ operators are called Pauli matrices. Note that the Pauli matrix σ_z has eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$. Simply carrying out computations gives the following results

$$\begin{aligned} \sigma_x |\uparrow\rangle &= |\downarrow\rangle \qquad & \sigma_x |\downarrow\rangle &= |\uparrow\rangle \\ \sigma_y |\uparrow\rangle &= i\,|\downarrow\rangle \qquad & \sigma_y |\downarrow\rangle &= -i\,|\uparrow\rangle \\ \sigma_z |\uparrow\rangle &= |\uparrow\rangle \qquad & \sigma_z |\downarrow\rangle &= -\,|\downarrow\rangle\,. \end{aligned}$$

When considering two particles in a state of spin up, we would have the state vector $|\uparrow\rangle \otimes |\uparrow\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$, which we notationally simplify by writing $|\uparrow\uparrow\rangle$. Since $|\uparrow\rangle$ and $|\downarrow\rangle$ are orthonormal bases we have $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$ and $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$. Lastly, we can combine all the Pauli matrices into the Pauli vector by simply writing a triple of matrices as

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

The dot product of this vector with a generic unit vector $\hat{a} = (a_1, a_2, a_3)$ is

$$\begin{aligned} \vec{\sigma} \cdot \hat{a} &= \sigma_x a_1 + \sigma_y a_2 + \sigma_z a_3 \\ &= \begin{pmatrix} 0 & a_1 \\ a_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ia_2 \\ ia_2 & 0 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \end{aligned} \tag{3.1}$$

This dot product with a unit vector gives the operator for measuring the spin in the direction of the unit vector.

3.3 Quantum superposition

We now wish to discuss special kinds of states which are created by linear combinations of two other states. Mathematically, if we take some state, say $|\uparrow\rangle$, and another state, say $|\downarrow\rangle$, we can write a linear combination of the two as

$$a\,|\uparrow\rangle + b\,|\downarrow\rangle\,, \tag{3.2}$$

which is still in our Hilbert space. In fact, so long as we have $|a|^2 + |b|^2 = 1$, (3.2) is a perfectly valid and pure quantum state within the framework of the theory. We call such a state a *superposition*. Consider an example.

Example 3.3.1. If we take the superposed state (3.2), we can calculate the expectation value of the spin in the

z direction using results from Example 3.2.3

$$\begin{aligned}\Big\langle a\langle\uparrow| + b\langle\downarrow| \Big| \sigma_z\big(a|\uparrow\rangle + b|\downarrow\rangle\big)\Big\rangle &= \langle a\langle\uparrow| + b\langle\downarrow|\,|a|\uparrow\rangle - b|\downarrow\rangle\,\rangle \\ &= \langle a\uparrow|a\uparrow\rangle - \langle a\uparrow|b\downarrow\rangle + \langle b\downarrow|a\uparrow\rangle - \langle b\downarrow|b\downarrow\rangle \\ &= |a|^2\langle\uparrow|\uparrow\rangle - a\bar{b}\langle\uparrow|\downarrow\rangle + b\bar{a}\langle\downarrow|\uparrow\rangle - |b|^2\langle\downarrow|\downarrow\rangle \\ &= |a|^2 - |b|^2.\end{aligned}$$

If $a = b = \frac{1}{\sqrt{2}}$, meaning each state is equally likely, then the expected value is 0, halfway between 1 and -1.

Superposition is a straightforward concept from a mathematical point of view, but how do we physically interpret a particle in a state like this? The answer is that the particle is considered to be in both states until such a time as a measurement determining which state it is really in is made. In the case of state (3.2), the probability the measurement will show spin up is $|a|^2$, and the probability the measurement will show spin down is $|b|^2$ (hence why we required their sum be 1). How can we be sure the particle is in both states simultaneously?

A famous experiment, known as the double slit experiment, requires it. In the experiment, a beam of electrons is fired at a wall. The wall has two slits cut into it, and behind it there is another wall which can detect an electron hitting it. An intensity pattern may be measured from the electrons striking the wall, showing the likelihood of the electron striking different parts of the wall. If the intensity through slit 1 is I_1 and the intensity through slit 2 is I_2, then we might expect the total intensity to be $I = I_1 + I_2$, being that the electrons either travel through slit 1 or slit 2. In reality, the observed pattern includes an interference term I_i, so we have $I = I_1 + I_2 + I_i$, and so we have to conclude that the electron travelled through both slits. Only when a measurement is made at the back wall does the superposition of slit 1 and slit 2 'collapse' to a local position, and we see it land somewhere with some probability. The strangest part happens if we place a measuring device at the location of the slits. The interference pattern disappears! The electrons choose a slit before travelling through it and we see a pattern $I = I_1 + I_2$ as if they travelled through a definite slit. For a more detailed discussion of this experiment, see [16, Ch. 6].

The question then is what constitutes a measurement? Alas, the answer remains unknown, although many theories exist. The EPR paper [13] (discussed in Section 4.1 below) aimed to show that entangled superposition states can cause issues, and Schrödinger proposed his famous cat thought experiment [28] to show his displeasure with the idea of a superposition state. Wigner proposed the original Wigner's friend thought experiment (discussed in Section 4.3) as an argument for consciousness being involved in measurements. We will discuss these ideas further in later sections.

3.4 Quantum entanglement

The titular definition for this section is also one of the most important for the analysis of the extended Wigner's friend experiment we will be carrying out in the next section. We give the definition and an example before discussing it further.

Definition 3.4.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces with orthonormal bases $\{\psi_i^{(1)}\}$ and $\{\psi_j^{(2)}\}$ respectively. A state $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ given by $\psi = \sum_{i,j} c_{ij}\psi_i^{(1)} \otimes \psi_j^{(2)}$ is called *separable* if there exists vectors $(c_i^1) \in \ell^2(\mathbb{N})$ and $(c_j^2) \in \ell^2(\mathbb{N})$ such that $c_{ij} = c_i^1 c_j^2$. Otherwise ψ is called inseparable or *entangled.*

The idea in this definition is that if the vectors (c_i^1) and (c_j^2) do exist then we can split the system easily back into its two composite states within their own separate Hilbert spaces. In reality, this virtually never happens, as all systems tend to get themselves entangled with their environment (like a child attempting to cook), although the degree of entanglement can be very small. It is not necessarily easy to show that there do not exist the required vectors. Here is an easy example.

Example 3.4.2. Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, and consider spin states of a pair of electrons. Consider an equally weighted superposition of the states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. This is

$$\psi = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big). \tag{3.3}$$

This is in fact an entangled state. Observe that if it were instead separable, we could write it as

$$\begin{aligned}\psi &= \big(a\,|\uparrow\rangle + b\,|\downarrow\rangle\big) \otimes \big(c\,|\downarrow\rangle + d\,|\uparrow\rangle\big)\\ &= ac\,|\uparrow\rangle \otimes |\downarrow\rangle + ad\,|\uparrow\rangle \otimes |\uparrow\rangle + bc\,|\downarrow\rangle \otimes |\downarrow\rangle + bd\,|\downarrow\rangle \otimes |\uparrow\rangle\\ &= ac\,|\uparrow\downarrow\rangle + ad\,|\uparrow\uparrow\rangle + bc\,|\downarrow\downarrow\rangle + bd\,|\downarrow\uparrow\rangle \end{aligned} \tag{3.4}$$

for some $a, b, c, d \in \mathbb{C}$. But to equate (3.4) with (3.3), we find that we need $ad = bc = 0$. If either of a or c is 0 then the first term in (3.3) is zero and a similar issue arises for the second term if either b or d is zero. So without one of the terms being zero we cannot have a separable system, and it is therefore entangled.

The next result gives us an easier way to see whether a given state is entangled or not.

Theorem 3.4.3. *A pure composite state Ψ is entangled if and only if one of the partial traces of ρ_Ψ is a mixed state.*

Proof. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}_1$ and $(\psi_m)_{m\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}_2$. Consider a generic state $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ given by

$$|\Psi\rangle = \sum_{n,m} c_{nm}\,|\phi_n\rangle \otimes |\psi_m\rangle\,, \tag{3.5}$$

for $c_{nm} \in \mathbb{C}$. The corresponding density matrix ρ_Ψ is

$$\rho_\Psi = |\Psi\rangle\langle\Psi| = \left(\sum_{n,m} c_{nm}\,|\phi_n\rangle \otimes |\psi_m\rangle\right)\left(\sum_{p,q} c_{pq}\,|\phi_p\rangle^* \otimes |\psi_q\rangle^*\right) = \sum_{n,m,p,q} c_{nm}\bar{c}_{pq}\,|\phi_n\rangle\langle\phi_p| \otimes |\psi_m\rangle\langle\psi_q|\,.$$

We can now calculate the partial trace over $\mathcal{H}_1$ applied to some vector $\xi \in \mathcal{H}_2$

$$\begin{aligned}\mathrm{Tr}_1(\rho_\Psi)\xi &= \sum_{i,j} |\psi_j\rangle \Big\langle\, |\phi_i\rangle \otimes |\psi_j\rangle \,\Big|\, \rho_\Psi\big(|\phi_i\rangle \otimes |\xi\rangle\big)\Big\rangle\\ &= \sum_{i,j,n,m,p,q} |\psi_j\rangle \Big\langle\, |\phi_i\rangle \otimes |\psi_j\rangle \,\Big|\, c_{nm}\bar{c}_{pq}\,|\phi_n\rangle\langle\phi_p|\phi_i\rangle \otimes |\psi_m\rangle\langle\psi_q|\xi\rangle\Big\rangle\\ &= \sum_{i,j,n,m,p,q} c_{nm}\bar{c}_{pq}\,|\psi_j\rangle\langle\phi_i|\phi_n\rangle\langle\phi_p|\phi_i\rangle\langle\psi_j|\psi_m\rangle\langle\psi_q|\xi\rangle\,.\end{aligned}$$

At this point we have three inner products of orthonormal basis sets, and so the only non-zero terms will occur when $i = p = n$ and $j = m$, so removing these indices from the sum we get

$$\mathrm{Tr}_1(\rho_\Psi)\xi = \sum_{n,m,q} c_{nm}\bar{c}_{nq}\,|\psi_m\rangle\langle\psi_q|\xi\rangle = \sum_n \left(\sum_m c_{nm}\,|\psi_m\rangle\right) \otimes \left(\sum_q c_{nq}\,|\psi_q\rangle\right)^* = \sum_n \left|\tilde{\Psi}_n\right\rangle\left\langle\tilde{\Psi}_n\right|, \tag{3.6}$$

where each $\tilde{\Psi}_n$ is some vector in $\mathcal{H}_2$. Equation (3.6) represents a pure state if and only if there exists some $n_0 \in \mathbb{N}$ such that $c_{nm} = 0$ unless $n = n_0$. A single n gives us a non-zero term if and only if the original state (3.5) can be written as

$$|\Psi\rangle = \sum_{n,m} c_{nm} |\phi_n\rangle \otimes |\psi_m\rangle = \sum_m c_{n_0 m} |\phi_{n_0}\rangle \otimes |\psi_m\rangle = |\phi_{n_0}\rangle \otimes \sum_m c_{n_0 m} |\psi_m\rangle ,$$

which is a separable state. We have therefore shown that a composite state is entangled if and only if the first partial trace is mixed, and the argument for the other partial trace is analogous. □

The real puzzle of entanglement is not necessarily obvious from the mathematical definition of an entangled state. Consider the state (3.3) from Example 3.4.2, containing one spin up particle and one spin down particle. Such a system can be prepared with relative ease by simply ensuring the total angular momentum initially is 0, and since angular momentum is conserved and spin is a type of angular momentum, the final total spin must be 0, which means the particles have opposite spin. The particles may then be space-like separated as far as we like, this fact will be unchanged.

Most disturbingly, when we measure the spin of one of the particles, we immediately learn not only its spin state, but its entangled pair's spin state, since the superposition state collapses, equivalent to us being able to rule out one of the terms in (3.3) and therefore having no uncertainty about the state of the other particle. Once again, it matters not whether the particle is on the other side of the solar system or in the Andromeda galaxy, we still learn information about it much faster than it would ever be able to travel to us. Einstein famously disliked this very much, calling it 'spooky action at a distance', and it is what lead him to the work in the EPR paper that eventually lead to others analysing the Wigner's friend experiment. A much more detailed discussion of this scenario is given in Section 4.1.

We will now compute the partial trace of the spin entangled state, to see how such a computation works, and also to validate the result of Theorem 3.4.3.

Example 3.4.4. We wish to compute the partial trace of the same state (3.3) we saw in Example 3.4.2 above

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

This is a four dimensional system, but we can't expect that that will mean it is tame. We firstly compute the density matrix for this state.

$$\begin{aligned}
\Theta_{\psi,\psi} = |\psi\rangle\langle\psi| &= \frac{1}{2}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)(\langle\uparrow| \otimes \langle\downarrow| - \langle\downarrow| \otimes \langle\uparrow|) \\
&= \frac{1}{2}(|\uparrow\rangle \otimes |\downarrow\rangle)(\langle\uparrow| \otimes \langle\downarrow|) - \frac{1}{2}(|\uparrow\rangle \otimes |\downarrow\rangle)(\langle\downarrow| \otimes \langle\uparrow|) - \frac{1}{2}(|\downarrow\rangle \otimes |\uparrow\rangle)(\langle\uparrow| \otimes \langle\downarrow|) + \frac{1}{2}(|\downarrow\rangle \otimes |\uparrow\rangle)(\langle\downarrow| \otimes \langle\uparrow|) \\
&= \frac{1}{2}\Theta_{|\uparrow\rangle\otimes|\downarrow\rangle,\langle\uparrow|\otimes\langle\downarrow|} - \frac{1}{2}\Theta_{|\uparrow\rangle\otimes|\downarrow\rangle,\langle\downarrow|\otimes\langle\uparrow|} - \frac{1}{2}\Theta_{|\downarrow\rangle\otimes|\uparrow\rangle,\langle\uparrow|\otimes\langle\downarrow|} + \frac{1}{2}\Theta_{|\downarrow\rangle\otimes|\uparrow\rangle,\langle\downarrow|\otimes\langle\uparrow|}.
\end{aligned}$$

This expression is unpleasant to say the least. Unfortunately it only gets worse. We can now apply the formula for the partial trace, with ξ some state vector

$$\mathrm{Tr}_1(\Theta_{\psi,\psi})\xi = \sum_{n,j} \psi_j \langle \phi_n \otimes \psi_j | \Theta_{\psi,\psi} \phi_n \otimes \xi \rangle .$$

We saw in Theorem 2.5.7 above that the choice of orthonormal bases doesn't matter, and so we will use the most obvious one, that is for $\mathcal{H}_1 = \mathbb{C}^2$ choose the orthonormal basis $\{\psi_1, \psi_2\} = \{|\uparrow\rangle, |\downarrow\rangle\}$ and for $\mathcal{H}_2 = \mathbb{C}^2$ choose the orthonormal basis $\{\phi_1, \phi_2\} = \{|\uparrow\rangle, |\downarrow\rangle\}$. With this choice we can expand out the partial trace to get

$$\begin{aligned}\mathrm{Tr}_1(\Theta_{\psi,\psi})\xi &= \sum_{n,j} \psi_j \langle \phi_n \otimes \psi_j | \Theta_{\psi,\psi} \phi_n \otimes \xi \rangle \\ &= |\uparrow\rangle \Big\langle |\uparrow\rangle \otimes |\uparrow\rangle \Big| \Theta_{\psi,\psi} |\uparrow\rangle \otimes \xi \Big\rangle + |\uparrow\rangle \Big\langle |\downarrow\rangle \otimes |\uparrow\rangle \Big| \Theta_{\psi,\psi} |\downarrow\rangle \otimes \xi \Big\rangle \\ &\quad + |\downarrow\rangle \Big\langle |\uparrow\rangle \otimes |\downarrow\rangle \Big| \Theta_{\psi,\psi} |\uparrow\rangle \otimes \xi \Big\rangle + |\downarrow\rangle \Big\langle |\downarrow\rangle \otimes |\downarrow\rangle \Big| \Theta_{\psi,\psi} |\downarrow\rangle \otimes \xi \Big\rangle.\end{aligned}$$

It now gets very unpleasant as we expand each term separately using the density matrix computed above. This can be done more quickly by realising which terms are zero but without the benefit of experience we will brute force it. At this point, we will remove the explicit tensor products to declutter the notation, and we will also write $\langle|\uparrow\rangle|$ as simply $\langle\uparrow|$, consistent with Corollary 2.4.4. The first term becomes

$$\begin{aligned}|\uparrow\rangle \langle\uparrow\uparrow|\Theta_{\psi,\psi}|\uparrow\xi\rangle\rangle &= |\uparrow\rangle \Big\langle \uparrow\uparrow \Big| \Big(\frac{1}{2}\Theta_{|\uparrow\downarrow\rangle,\langle\uparrow\downarrow|} - \frac{1}{2}\Theta_{|\uparrow\downarrow\rangle,\langle\downarrow\uparrow|} - \frac{1}{2}\Theta_{|\downarrow\uparrow\rangle,\langle\uparrow\downarrow|} + \frac{1}{2}\Theta_{|\downarrow\uparrow\rangle,\langle\downarrow\uparrow|} \Big) |\uparrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \uparrow\uparrow \Big| \Big(\frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow| \Big) |\uparrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \uparrow\uparrow \Big| \frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow|\uparrow\xi\rangle - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow|\uparrow\xi\rangle - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow|\uparrow\xi\rangle + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|\uparrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \uparrow\uparrow \Big| \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow|\xi\rangle - \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow|\xi\rangle \Big\rangle \\ &= \frac{1}{2}|\uparrow\rangle\langle\uparrow\uparrow|\uparrow\downarrow\rangle\langle\downarrow|\xi\rangle - \frac{1}{2}|\uparrow\rangle\langle\uparrow\uparrow|\downarrow\uparrow\rangle\langle\downarrow|\xi\rangle,\end{aligned}$$

and since tensor products of orthonormal bases are orthonormal bases of the tensor product space, both terms are 0. The second term is then

$$\begin{aligned}|\uparrow\rangle \langle\downarrow\uparrow|\Theta_{\psi,\psi}|\downarrow\xi\rangle\rangle &= |\uparrow\rangle \Big\langle \downarrow\uparrow \Big| \Big(\frac{1}{2}\Theta_{|\uparrow\downarrow\rangle,\langle\uparrow\downarrow|} - \frac{1}{2}\Theta_{|\uparrow\downarrow\rangle,\langle\downarrow\uparrow|} - \frac{1}{2}\Theta_{|\downarrow\uparrow\rangle,\langle\uparrow\downarrow|} + \frac{1}{2}\Theta_{|\downarrow\uparrow\rangle,\langle\downarrow\uparrow|} \Big) |\downarrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \downarrow\uparrow \Big| \Big(\frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow| \Big) |\downarrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \downarrow\uparrow \Big| \frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow\downarrow|\downarrow\xi\rangle - \frac{1}{2}|\uparrow\downarrow\rangle\langle\downarrow\uparrow|\downarrow\xi\rangle - \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow\downarrow|\downarrow\xi\rangle + \frac{1}{2}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|\downarrow\xi\rangle \Big\rangle \\ &= |\uparrow\rangle \Big\langle \downarrow\uparrow \Big| -\frac{1}{2}|\uparrow\downarrow\rangle\langle\uparrow|\xi\rangle + \frac{1}{2}|\downarrow\uparrow\rangle\langle\uparrow|\xi\rangle \Big\rangle \\ &= -\frac{1}{2}|\uparrow\rangle\langle\downarrow\uparrow|\uparrow\downarrow\rangle\langle\uparrow|\xi\rangle + \frac{1}{2}|\uparrow\rangle\langle\downarrow\uparrow|\downarrow\uparrow\rangle\langle\uparrow|\xi\rangle \\ &= \frac{1}{2}|\uparrow\rangle\langle\uparrow|\xi\rangle = \frac{1}{2}|\uparrow\rangle\!\langle\uparrow|\,|\xi\rangle.\end{aligned}$$

This term is non-zero. The calculations for the remaining two terms are much the same, and one is 0, while the other is $\frac{1}{2}|\downarrow\rangle\!\langle\downarrow|\,|\xi\rangle$. Our partial trace is therefore

$$\mathrm{Tr}_1(\Theta_{\psi,\psi})\xi = \frac{1}{2}|\uparrow\rangle\!\langle\uparrow|\,|\xi\rangle + \frac{1}{2}|\downarrow\rangle\!\langle\downarrow|\,|\xi\rangle = \Big(\frac{1}{2}|\uparrow\rangle\!\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\!\langle\downarrow|\Big)|\xi\rangle,$$

which is an operator as we expected, and in fact if we consider that it is a density matrix then we can see it is a mixed state, which was once again expected. It becomes clearer to see if we write the states $|\uparrow\rangle$ and $|\downarrow\rangle$ in terms of a standard basis. Using the vectors $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, we can then calculate the outer products

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$
$$|\downarrow\rangle\langle\downarrow| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}.$$

Finally we can write the partial trace matrix, which is how we are used to representing operators. Operating on the vector $|\xi\rangle$ it becomes

$$\begin{aligned}\mathrm{Tr}_1(\Theta_{\psi,\psi})\xi &= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}|\xi\rangle + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}|\xi\rangle\\ &= \begin{pmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{pmatrix}|\xi\rangle.\end{aligned}$$

Note here that the partial trace is trace-preserving, that is we have $\mathrm{Tr}^{\mathcal{H}_2}(\mathrm{Tr}_1(\Theta_{\psi,\psi})) = \mathrm{Tr}^{\mathcal{H}_1\otimes\mathcal{H}_2}(\Theta_{\psi,\psi})$. This is true generally, and goes some way to explain the name partial trace.

4 Wigner's friend

In this last main section, we give a timeline of the important developments in foundational understanding of quantum mechanics, focusing on EPR and then Bell. After this we get to the main result, analysing the extended Wigner's friend scenario to give a detailed proof of Brukner's no-go theorem on observer independent facts.

4.1 The EPR experiment

The debate about the interpretation of quantum mechanics began to heat up when, in 1935, Einstein, Podolsky and Rosen (EPR) [13] published their thought experiment to argue quantum mechanics was incomplete. Here, we will present a version of the paradox constructed by David Bohm in 1951 [4]. Accessible accounts of the paradox may be found in [18, Ch. 12] and [23, Ch. 5], while a more rigorous account can be found in [3, Sec. 6.1.4]. Suppose we have two spin-entangled particles, that is their state is given by our favourite spin entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big). \tag{4.1}$$

In the thought experiment, we can suppose such a state is created by a spin-zero pion decaying into an electron and positron, each with spin magnitude one half. Now suppose we give the electron to Alice, and the positron to Bob, and we allow them to travel very far from one another, realistically 10 or so metres but ideally we would send them as far as possible. The result is independent of distance so importantly nothing would be different if they were a light year apart.

We now permit Alice and Bob to make measurements of the spin in a direction of their choice on their particle. If the particles are spin entangled, then they are spin entangled along all axes. Particularly, this means we find a spin entanglement along both the z and x directions. Mathematically we have

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_z |\downarrow\rangle_z - |\downarrow\rangle_z |\uparrow\rangle_z \Big) = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_x |\downarrow\rangle_x - |\downarrow\rangle_x |\uparrow\rangle_x \Big).$$

Say Alice makes her measurement first, chooses the z direction, and gets spin up. The state is then collapsed into

$$\left|\tilde{\Psi}\right\rangle = |\uparrow\rangle_z |\downarrow\rangle_z ,$$

and she can be certain that Bob will measure spin down along the z direction if he chooses to measure that. Experimental verification shows that this really happens, and that Bob will measure with certainty spin down. Say, however, that Alice chose the x direction, and gets spin up. The state is then collapsed to

$$\left|\tilde{\Psi}\right\rangle = |\uparrow\rangle_x |\downarrow\rangle_x ,$$

and she can say with certainty that Bob will measure down along the x direction. Bob's particle therefore 'knows' which direction Alice has measured along. If she makes a measurement along the z direction, then the outcome of a measurement of his along the z direction has only one outcome, with probability unity, while a measurement of his along the x direction has two possible outcomes, each with probability one half. If she instead chose to measure x, it will be opposite for Bob.

Despite being so far away that not even light could propagate that correlation between the particles, Bob's particle seemingly knows which direction Alice has chosen, and adjusts its quantum state accordingly. How do we know this collapse is instant and doesn't just travel at light speed or slower? If Alice makes a measurement along z and gets spin up, and then Bob makes a measurement along z before news of Alice's measurement reaches him, then there is a one half chance he will also get spin up, a violation of the conservation of angular momentum we assumed at the beginning.

Experimentation has shown without any doubt that the collapse is instantaneous, no matter the distance of separation. This is seemingly a violation of locality, allowing information to be sent instantaneously as far as we like. Thus, EPR concluded, they had reached a contradiction and quantum mechanics had to be incomplete. There had to be something else allowing this information to be transmitted.

What EPR were trying to advocate for were hidden variables. They thought there were some other unknowns pulling the strings which quantum mechanics did not describe, hence their conclusion of incompleteness. Boiled down, the EPR paradox essentially says either the universe is nonlocal (contrary to the predictions of relativity theory) or there are hidden variables and a better theory is needed to describe them.

4.2 Bell's theorem

In 1964, John Bell published a simple argument that no local, hidden-variable theory could match the predictions of quantum mechanics [2]. This is often given as Bell's theorem. Before stating and proving the theorem, we derive the Bell inequality.

Consider once more the spin entangled state (4.1), and the scenario of EPR from the preceding section. This time, the detectors at each end (electron and positron) are allowed to rotate, so that the spin measurement is made along different directions on either end. Let the unit vector $\vec{a}$ represent the direction of the detector at the electron, and let the unit vector $\vec{b}$ represent the same at the positron.

For the derivation, we make two assumptions. The first is that there are hidden variables which characterise the complete state of both particles. Call the hidden variables λ, and note we make no assumptions about the nature of λ, for it may be a scalar or vector or even a collection of functions. The second assumption is that the outcome of Alice's measurement is independent of the choice of $\vec{b}$, since $\vec{b}$ (positron end) may be chosen just before Alice measures the electron, so that there is no time for information to travel below light speed back to Alice's electron. We have therefore assumed locality holds. With these assumptions, we can say there exists functions $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ such that A determines the outcome of Alice's measurement of the electron along the axis $\vec{a}$, and B determines the outcome of Bob's measurement of the positron along the axis $\vec{b}$. Since spin is only up or down, both functions are either of ± 1. In particular, when the detectors are aligned, we have

$$A(\vec{a},\lambda) = -B(\vec{a},\lambda). \tag{4.2}$$

We wish to consider the products of the outcomes of the measurements. In particular, the average of the products is

$$\mathrm{P}(\vec{a},\vec{b}) = \int \rho(\lambda) A(\vec{a},\lambda) B(\vec{b},\lambda)\,\mathrm{d}\lambda\,,$$

where $\rho(\lambda)$ is the probability distribution for the hidden variable(s). We make no assumptions about it beyond the usual conditions for a probability distribution. Using Equation (4.2), we can write

$$\mathrm{P}(\vec{a},\vec{b}) = -\int \rho(\lambda) A(\vec{a},\lambda) A(\vec{b},\lambda)\,\mathrm{d}\lambda\,.$$

Then, if $\vec{c}$ is some other unit vector, we can write

$$\begin{aligned}
\mathrm{P}(\vec{a},\vec{b}) - \mathrm{P}(\vec{a},\vec{c}) &= -\int \rho(\lambda)\Big(A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda)\Big)\,\mathrm{d}\lambda \\
&= -\int \rho(\lambda)\Big(1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\Big)A(\vec{a},\lambda)A(\vec{b},\lambda)\,\mathrm{d}\lambda\,,
\end{aligned}$$

where the last line follows from the fact that $\Big(A(\vec{b},\lambda)\Big)^2 = 1$. We similarly have $|A(\vec{a},\lambda)A(\vec{b},\lambda)| = 1$, and $\rho(\lambda)\Big(1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\Big) \geq 0$, and so we can take an absolute value and apply Hölder's inequality to get

$$\begin{aligned}
|\,\mathrm{P}(\vec{a},\vec{b}) - \mathrm{P}(\vec{a},\vec{c})| &= \left|\int \rho(\lambda)\Big(1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\Big)A(\vec{a},\lambda)A(\vec{b},\lambda)\,\mathrm{d}\lambda\right| \\
&\leq \int \left|\rho(\lambda)\Big(1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\Big)A(\vec{a},\lambda)A(\vec{b},\lambda)\right|\mathrm{d}\lambda \\
&= \int \rho(\lambda)\Big(1 - A(\vec{b},\lambda)A(\vec{c},\lambda)\Big)\,\mathrm{d}\lambda \\
&= \int \rho(\lambda)\,\mathrm{d}\lambda - \int \rho(\lambda)A(\vec{b},\lambda)A(\vec{c},\lambda)\,\mathrm{d}\lambda \\
&= 1 + \int \rho(\lambda)A(\vec{b},\lambda)B(\vec{c},\lambda)\,\mathrm{d}\lambda \\
&= 1 + \mathrm{P}(\vec{b},\vec{c}),
\end{aligned}$$

and we have the famous Bell inequality

$$|\,\mathrm{P}(\vec{a},\vec{b}) - \mathrm{P}(\vec{a},\vec{c})| \leq 1 + \mathrm{P}(\vec{b},\vec{c}), \tag{4.3}$$

which holds for any local, hidden-variable theory. We can now prove Bell's theorem.

Theorem 4.2.1 (Bell's theorem)**.** *No local, hidden-variable theory is consistent with quantum mechanics.*

Proof. We wish to show that quantum mechanics makes predictions which violate the Bell inequality. First, we need to know the form of the probability $\mathrm{P}(\vec{a},\vec{b})$ predicted by quantum mechanics to test the inequality. Consider the spin entangled state (4.1) and a spin measurement along the axis of $\vec{a}$ for the first particle and $\vec{b}$ for the second. The respective spin operators are given by the dot product with the Pauli vector as in Equation (3.1), and so the expected value is

$$
\begin{aligned}
\mathrm{P}(\vec{a},\vec{b}) &= \left\langle \Psi \middle| \left((\sigma\cdot\vec{a}) \otimes (\sigma\cdot\vec{b}) \right) \Psi \right\rangle \\
&= \left\langle \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \Big) \middle| \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \otimes \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix} \frac{1}{\sqrt{2}} \Big(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \Big) \right\rangle \\
&= \frac{1}{2} \left\langle \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \otimes \begin{pmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{pmatrix} \left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \right) \right\rangle \\
&= \frac{1}{2} \left\langle \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} a_3 \\ a_1 + ia_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 - ib_2 \\ -b_3 \end{pmatrix} - \begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix} \otimes \begin{pmatrix} b_3 \\ b_1 + ib_2 \end{pmatrix} \right\rangle \\
&= \frac{1}{2} \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} a_3 \\ a_1 + ia_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix}0\\1\end{pmatrix} \middle| \begin{pmatrix} b_1 - ib_2 \\ -b_3 \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix}0\\1\end{pmatrix} \middle| \begin{pmatrix} a_3 \\ a_1 + ia_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} b_1 - ib_2 \\ -b_3 \end{pmatrix} \right\rangle \\
&\quad - \frac{1}{2} \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix}0\\1\end{pmatrix} \middle| \begin{pmatrix} b_3 \\ b_1 + ib_2 \end{pmatrix} \right\rangle + \frac{1}{2} \left\langle \begin{pmatrix}0\\1\end{pmatrix} \middle| \begin{pmatrix} a_1 - ia_2 \\ -a_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix}1\\0\end{pmatrix} \middle| \begin{pmatrix} b_3 \\ b_1 + ib_2 \end{pmatrix} \right\rangle \\
&= \frac{1}{2} a_3(-b_3) - \frac{1}{2}(a_1 + ia_2)(b_1 - ib_2) - \frac{1}{2}(a_1 - ia_2)(b_1 + ib_2) + \frac{1}{2}(-a_3)b_3 \\
&= -a_3 b_3 - a_2 b_2 - a_1 b_1 \\
&= -\vec{a}\cdot\vec{b}.
\end{aligned}
$$

We assumed here nothing about the orientation of the vectors $\vec{a}$ and $\vec{b}$. We can choose any vectors, so let $\vec{a}$ be the x direction, and $\vec{b}$ be the z direction, and $\vec{c}$ lie in the xz plane at an angle of $\pi/4$ to both, and of course being normalised. Then $\vec{a} = (1,0,0)^T$, $\vec{b} = (0,0,1)^T$, and $\vec{c} = (1/\sqrt{2}, 0, 1/\sqrt{2})^T$. The average value of the spins are predicted by quantum mechanics to be

$$
\begin{aligned}
\mathrm{P}(\vec{a},\vec{b}) &= -\vec{a}\cdot\vec{b} = 0 \\
\mathrm{P}(\vec{a},\vec{c}) &= -\vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}} \\
\mathrm{P}(\vec{b},\vec{c}) &= -\vec{b}\cdot\vec{b} = -\frac{1}{\sqrt{2}},
\end{aligned}
$$

and the Bell inequality (4.3) is clearly violated, since $1/\sqrt{2} \not\leq 1 - \sqrt{2}$. This concludes the proof. ☐

Bell's inequality has been experimentally verified to be violated. Aspect et al. [1] found experimental results in agreement with quantum theory and in violation of Bell's inequality by 5 standard deviations.

So what does Bell's theorem tell us? It says that no theory can have all three properties: agreement with quantum mechanics, locality, and hidden variables. Different interpretations of quantum mechanics remove different properties. A famous example of a hidden variable theory which is extremely non-local is the de Broglie-Bohm pilot wave formulation of quantum mechanics [5], which inspired Bell to see if he could eliminate the non-locality, and lead him to develop his theorem.

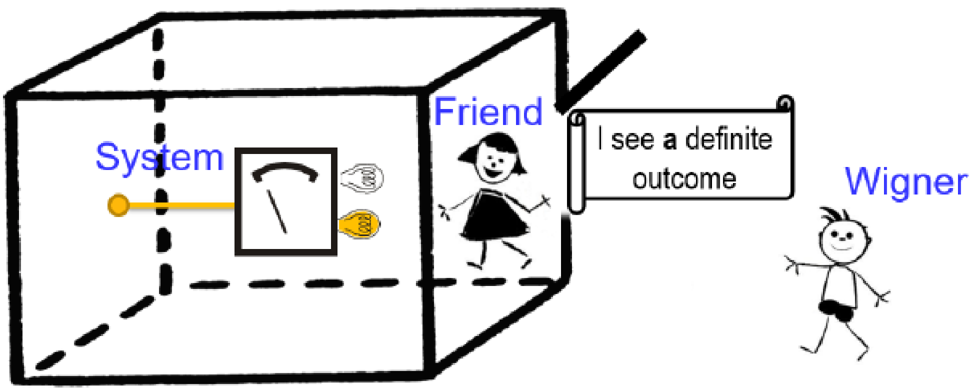

Figure 1: A diagram of the Deutsch version of the Wigner's friend thought experiment. Taken from [8, Figure 1].

In truth however, it is widely accepted that quantum mechanics has to be right, due to the overwhelming empirical evidence as such, and also that locality should hold, because of the equally successful theory of relativity. We are therefore lead to conclude from Bell's theorem that quantum mechanics cannot be a hidden variable theory, and the uncertainties and probabilities really are just that. They aren't due to our lack of information, they are truly uncertain. Indeed, this is further evidence for the argument that a superposition state is truly a superposition, unlike a mixed state which corresponds to a lack of information.

So then what of the EPR paradox? Didn't it prove that there either had to be hidden variables or non-locality? And in light of Bell's theorem doesn't this mean we're back to non-locality again? It is now widely accepted that the EPR paradox in fact is not a paradox, and doesn't violate locality at all. This is because no information is sent faster than light speed from Alice to Bob. Although there is correlation between the measurement outcomes, it isn't causal, meaning it can't carry information to affect Bob, and therefore doesn't violate the theory of relativity. This is analogous to a shadow projected onto a huge screen. A small movement from the object will cause the shadow to move very fast, and a large enough screen can cause the shadow to move faster than light, but no information can be carried on a shadow, and so there is no violation of locality. For Alice to make use of her knowledge of Bob's particle's state, she would need to send him a message, or meet him in the middle, and these obey the laws of relativity as we understand them. We have correlation, not causation, and so, after all that, there really was no paradox.

However, what we learned from all this was that wavefunction collapse really does happen instantaneously across the entire universe, and there are no hidden variables making it happen. This begs the question, when does the wavefunction actually collapse? Wigner introduced his thought experiment to try and find out.

4.3 Wigner's friend

Eugene Wigner proposed his Wigner's friend thought experiment in 1961 [31], reformulating the Schrödinger's cat thought experiment [28]. The Schrödinger's cat thought experiment was designed to show the absurdity of quantum superposition by using a macroscopic object, and Wigner's friend is also designed to show the absurdity of superposition, and also to argue Wigner's belief that consciousness was involved in the collapse of the wave function, a view shared by John von Neumann. A diagram of the Wigner's friend thought experiment is shown in Figure 1.

In the experiment, Wigner is outside a lab, inside which there is a particle and measuring device. His friend is also inside the lab, and she is isolated from the outside. The friend makes a measurement of the particle, while Wigner makes a measurement of the particle-lab system as a composite system. Before any measurements are made, the friend knows the particle to be in a superposition state $a\left|\uparrow\right\rangle_P + b\left|\downarrow\right\rangle_P$, and Wigner knows the state is $(a\left|\uparrow\right\rangle_P + b\left|\downarrow\right\rangle_P) \otimes \left|0\right\rangle_F$, where $\left|0\right\rangle_F$ is the state of the friend. Then, the friend is free to make her measurement, 'collapsing' her superposition into a definite outcome, say $\left|\uparrow\right\rangle_P$. However, with no knowledge of whether she has yet made her measurement or what her outcome was, if any, the state Wigner sees remains a superposition $a\left|\uparrow\right\rangle_P \otimes \left|\uparrow\right\rangle_F + b\left|\downarrow\right\rangle_P \otimes \left|\downarrow\right\rangle_F$.

In a further refinement of the thought experiment due to David Deutsch [12], the friend can pass a message to Wigner indicating that she has measured a definite outcome, but without telling him what it is. In this case, his superposition state still doesn't collapse, despite knowing she has seen the definite outcome. The question is then how can these two contradictory views of the same particle be rectified?

Wigner believed in the 'consciousness causes collapse' interpretation of quantum mechanics, otherwise called the von Neumann-Wigner interpretation. In his view, a measurement is simply an 'impression' on an observer's mind, and we can show then that there is no paradox. The measurement of the friend causes an impression on her consciousness, which is when her superposition collapses. However, until she tells Wigner exactly what her outcome was, his consciousness is not impressed upon and therefore his superposition is not collapsed. Wigner argued that in quantum mechanics, the role of a conscious observer is different to that of an inanimate object such as the measuring device. Wigner discusses these views in his paper 'Remarks on the Mind-Body Question' [31].

With this all in mind, we move towards the final result, built off an extended version of this Wigner's friend scenario. The result will be a no-go theorem for observer independent facts, showing that it really may be the case that facts cannot be universal in a sense, and must be relative to each observer. If this is true, then we would have a rectification of the of the paradox in the Wigner's friend thought experiment. Before moving to the extended Wigner's friend thought experiment, we need a generalisation of the Bell inequality, which will be used in the proof of the no-go theorem.

4.4 The CHSH inequality

The CHSH inequality (which stands for Clauser-Horne-Shimony-Holt, and was first derived in [11]) is a Bell-type inequality involving four measurement settings, rather than two as in the original Bell inequality. Now Alice and Bob are given a choice of 2 measurements each, and have the freedom to select which they use. We also assume as before that locality holds, so that the outcome of a measurement made by Alice can't impact Bob's measurements. The last main assumption we make is that the four observables about which we care have a joint probability distribution.

The precise nature of these observables and the physical interpretation of the existence of the joint distribution will be discussed at length later, but for now suffice it to say this roughly translates to the observables being compatible with one another. Call the observables A_1, A_2, B_1 and B_2, and we will assume that they take the values ± 1, corresponding to either a positive or negative outcome (or spin up or spin down). The same result can be derived for a continuous spectrum $[-1, 1]$, see [10]. The expectation value for the pair of observables $A_1 B_1$ is then

$$\langle A_1 B_1 \rangle = \sum_{a,b=-1,1} ab\, \mathrm{P}(A_1 = a, B_1 = b), \tag{4.4}$$

where $\mathrm{P}(A_1 = a, b_1 = b) = \sum_{c,d=-1,1} \mathrm{P}(A_1 = a, A_2 = c, B_1 = b, B_2 = d)$, and similarly for the other pairs of one A and one B. We can see that the expectation value is simply summing over all the double probabilities weighted by their value, and the double probability is simply the full joint probability distribution for all four observables summed over the two we are not considering. We had to assume here that even writing down the full joint distribution makes sense. If it does, then we can derive the following result about the four double expectation values

$$S := \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \leq 2. \tag{4.5}$$

To derive the inequality (4.5), we begin by making the substitution for the definition of the double expectations as given in equation (4.4). We use additional superscripts for clarity

$$\begin{aligned} S = &\sum_{a^1,b^1=-1,1} a^1 b^1 \,\mathrm{P}(A_1 = a^1, B_1 = b^1) + \sum_{a^1,b^2=-1,1} a^1 b^2 \,\mathrm{P}(A_1 = a^1, B_2 = b^2) \\ &+ \sum_{a^2,b^1=-1,1} a^2 b^1 \,\mathrm{P}(A_2 = a^2, B_1 = b^1) - \sum_{a^2,b^2=-1,1} a^2 b^2 \,\mathrm{P}(A_2 = a^2, B_2 = b^2). \end{aligned}$$

We can now make the substitution for the definition of the double probability distributions to get

$$\begin{aligned} S = &\sum_{a^1,b^1=-1,1} \sum_{c^2,d^2=-1,1} a^1 b^1 \,\mathrm{P}(A_1 = a^1, A_2 = c^2, B_1 = b^1, B_2 = d^2) \\ &+ \sum_{a^1,b^2=-1,1} \sum_{c^2,d^1=-1,1} a^1 b^2 \,\mathrm{P}(A_1 = a^1, A_2 = c^2, B_1 = d^1, B_2 = b^2) \\ &+ \sum_{a^2,b^1=-1,1} \sum_{c^1,d^2=-1,1} a^2 b^1 \,\mathrm{P}(A_1 = c^1, A_2 = a^2, B_1 = b^1, B_2 = d^2) \\ &- \sum_{a^2,b^2=-1,1} \sum_{c^1,d^1=-1,1} a^2 b^2 \,\mathrm{P}(A_1 = c^1, A_2 = a^2, B_1 = d^1, B_2 = b^2). \end{aligned} \tag{4.6}$$

We want to factorise (4.6) into two terms, and to do so we relabel some dummy variables. Note that the first two terms share the same variables for the A operators, but different for the B ones. We can make them the same by simply changing the label d^2 into a b^2 in the first term, and by changing d^1 into b_1 in the second term. This is allowed because d^1 and d^2 are simply dummy variables, and neither d^2 nor b^2 appear outside the probability in the first term, while neither d^1 nor b^1 appear outside the probability in the second term, and so making the change of label doesn't change the sum in any way. In a similar fashion, we make the third and fourth terms factorisable by changing the label d^2 to b^2 in the third term and d^1 to b^1 in the fourth term. We end up with

$$\begin{aligned} S = &\sum_{a^1,b^1=-1,1} \sum_{c^2,b^2=-1,1} a^1 b^1 \,\mathrm{P}(A_1 = a^1, A_2 = c^2, B_1 = b^1, B_2 = b^2) \\ &+ \sum_{a^1,b^2=-1,1} \sum_{c^2,b^1=-1,1} a^1 b^2 \,\mathrm{P}(A_1 = a^1, A_2 = c^2, B_1 = b^1, B_2 = b^2) \\ &+ \sum_{a^2,b^1=-1,1} \sum_{c^1,b^2=-1,1} a^2 b^1 \,\mathrm{P}(A_1 = c^1, A_2 = a^2, B_1 = b^1, B_2 = b^2) \\ &- \sum_{a^2,b^2=-1,1} \sum_{c^1,b^1=-1,1} a^2 b^2 \,\mathrm{P}(A_1 = c^1, A_2 = a^2, B_1 = b^1, B_2 = b^2) \\ = &\sum_{a^1,b^1,c^2,b^2=-1,1} a^1 (b^1 + b^2) \,\mathrm{P}(A_1 = a^1, A_2 = c^2, B_1 = b^1, B_2 = b^2) \\ &+ \sum_{a^2,b^1,c^1,b^2=-1,1} a^2 (b^1 - b^2) \,\mathrm{P}(A_1 = c^1, A_2 = a^2, B_1 = b^1, B_2 = b^2), \end{aligned}$$

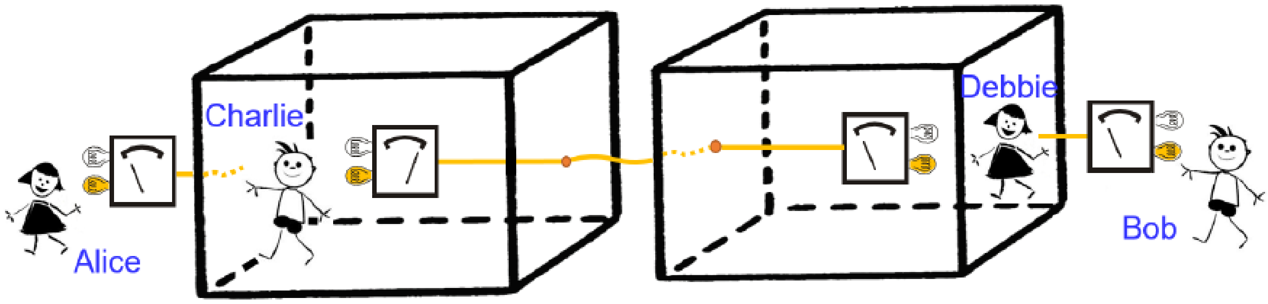

Figure 2: A diagram of the extended Wigner's friend thought experiment. Taken from [8, Figure 2]

Finally, to factorise these into a single summation, we need only to relabel c^2 as a^2 in the first term, and c^1 as a^1 in the second, which is allowed for the same reasons as above. This gives us

$$\begin{aligned} S = &\sum_{a^1,b^1,a^2,b^2=-1,1} a^1(b^1+b^2)\,\mathrm{P}(A_1=a^1,A_2=a^2,B_1=b^1,B_2=b^2) \\ &+ \sum_{a^2,b^1,a^1,b^2=-1,1} a^2(b^1-b^2)\,\mathrm{P}(A_1=a^1,A_2=a^2,B_1=b^1,B_2=b^2) \\ = &\sum_{a^1,a^2,b^1,b^2=-1,1} \left(a^1(b^1+b^2)+a^2(b^1-b^2)\right)\mathrm{P}(A_1=a^1,A_2=a^2,B_1=b^1,B_2=b^2). \end{aligned} \tag{4.7}$$

From here, we can see that this expression is smaller than or equal to 2. This is because $a^1(b^1+b^2)+a^2(b^1-b^2) \leq 2$ since if $b^1+b^2=2$ then $b^1-b^2=0$ and vice versa, and the a can only be ±1, so the largest we can manufacture is $+2$. Then the sum of all the probabilities must equal 1, and so we have derived the CHSH inequality $S \leq 2$, with the assumption that the four variables in question are independent in the sense that a joint probability distribution may be defined. Note that a scenario in which quantum mechanics violates the CHSH inequality can be cooked up almost as easily as we found one for the Bell inequality. We however, are not concerned with this generic violation, and instead look to put it to use in our no-go theorem.

4.5 Extended Wigner's friend

We now describe the extended Wigner's friend experiment, first proposed in [7] and refined to its current form in [17]. A diagram of the thought experiment is shown in Figure 2. This time, we use two laboratories, each isolated from the outside and one another, and each with an observer inside, called Charlie and Debbie respectively. A pair of entangled particles is produced, and each sent into one of the labs, where Charlie and Debbie are free to take a measurement of the spin state, which they then record on a measurement device inside their lab. Outside the labs are placed two more observers, called superobservers. We name the superobserver of Charlie's lab Alice, and the superobserver of Debbie's lab Bob (so that we have observers A and B on the outside and C and D on the inside respectively). After Charlie and Debbie have made their measurements, they can tell Alice or Bob respectively that they have measured a definite outcome, and Alice and Bob are left with a choice to proceed to make one of two measurements each. The measurements in some way correspond to either measuring the particle, or measuring the particle and the lab (including the measuring device) all together, referred to by Brukner as 'friend-type' and 'Wigner-type' [8].

We note that the original Wigner paradox is still present here. For example, Charlie can make a measurement and have a definite outcome, and can even tell Alice he has one, but that doesn't change the fact that Alice is yet to have any idea which it might be. Hence we have a sneaking suspicion that there may be contradiction of viewpoints here. In the next section we will rigorously formalise the mathematics and show that we can in fact produce a violation of the CHSH inequality, showing there cannot exist a joint probability distribution for the measurements of Alice and Bob, and we will subsequently discuss the physical implications.

4.6 Violating the CHSH inequality

This section is a detailed and rigorous proof of the scenario given in [7], described above. Initially, a pair of entangled spin-1/2 particles S_1 and S_2 are produced, such that Charlie has particle S_1 and Debbie has particle S_2. The entangled state of the particles is

$$\left|\psi^-\right\rangle_{S_1,S_2} = \frac{1}{\sqrt{2}}\big(\left|z+\right\rangle_{S_1} \otimes \left|z-\right\rangle_{S_2} - \left|z-\right\rangle_{S_1} \otimes \left|z+\right\rangle_{S_2}\big), \tag{4.8}$$

corresponding to a spin entangled state just like (3.3) in Example 3.4.2. Here, $\left|z+\right\rangle_{S_1}$ represents the state where particle S_1 has spin-up measured along the z-direction, and so on. We then act on this with a rotation $\mathbb{1} \otimes e^{-i\theta\sigma_y}$, with $\mathbb{1}$ being the identity operator. This takes the particle in the second position, which belongs to Debbie, and rotates the state by an angle θ, so Debbie now makes her spin measurement on an angle θ to Charlie. As before, σ_y is the Pauli matrix corresponding to the y direction spin operator, as introduced in Example 3.2.3. Since the exponential has a convergent Taylor series, we can write the expansion

$$\begin{aligned}
\exp(-i\theta\sigma_y) &= \sum_{n=0}^{\infty} \frac{1}{n!}(-i\theta\sigma_y)^n \\
&= \mathbb{1} - i\theta\sigma_y - \frac{1}{2!}\theta^2\mathbb{1} + \frac{1}{3!}i\theta^3\sigma_y + \frac{1}{4!}\theta^4\mathbb{1} - \dots \\
&= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots)\mathbb{1} - i(\theta - \frac{1}{3!}\theta^3 + \dots)\sigma_y \\
&= \cos(\theta)\mathbb{1} - i\sin(\theta)\sigma_y
\end{aligned} \tag{4.9}$$

since $\sigma_y^2 = \mathbb{1}$. We can now use (4.9) on the right side of the tensor products in Equation (4.8) above, to get

$$\begin{aligned}
\left|\psi\right\rangle_{S_1,S_2} &= (\mathbb{1} \otimes e^{-i\theta\sigma_y})\frac{1}{\sqrt{2}}\Big(\left|z+\right\rangle_{S_1} \otimes \left|z-\right\rangle_{S_2} - \left|z-\right\rangle_{S_1} \otimes \left|z+\right\rangle_{S_2}\Big) \\
&= \Big(\mathbb{1} \otimes \big(\cos(\theta)\mathbb{1} - i\sin(\theta)\sigma_y\big)\Big)\frac{1}{\sqrt{2}}\Big(\left|z+\right\rangle_{S_1} \otimes \left|z-\right\rangle_{S_2} - \left|z-\right\rangle_{S_1} \otimes \left|z+\right\rangle_{S_2}\Big) \\
&= \frac{1}{\sqrt{2}}\Big[\left|z+\right\rangle_{S_1} \otimes \big(\cos(\theta)\mathbb{1} - i\sin(\theta)\sigma_y\big)\left|z-\right\rangle_{S_2} - \left|z-\right\rangle_{S_1} \otimes \big(\cos(\theta)\mathbb{1} - i\sin(\theta)\sigma_y\big)\left|z+\right\rangle_{S_2}\Big].
\end{aligned} \tag{4.10}$$

We now consider how $-i\sigma_y$ acts on the spin-up and spin-down states. Using the calculations done in Example 3.2.3, we have $-i\sigma_y\left|z+\right\rangle = \left|z-\right\rangle$ and $-i\sigma_y\left|z-\right\rangle = -\left|z+\right\rangle$. Substituting these into (4.10) we get

$$\begin{aligned}
\left|\psi\right\rangle_{S_1,S_2} &= \frac{1}{\sqrt{2}}\Big[\left|z+\right\rangle_{S_1} \otimes \big(\cos(\theta)\left|z-\right\rangle_{S_2} - i\sin(\theta)\sigma_y\left|z-\right\rangle_{S_2}\big) - \left|z-\right\rangle_{S_1} \otimes \big(\cos(\theta)\left|z+\right\rangle_{S_2} - i\sin(\theta)\sigma_y\left|z+\right\rangle_{S_2}\big)\Big] \\
&= \frac{1}{\sqrt{2}}\Big[\left|z+\right\rangle_{S_1} \otimes \big(\cos(\theta)\left|z-\right\rangle_{S_2} - \sin(\theta)\left|z+\right\rangle_{S_2}\big) - \left|z-\right\rangle_{S_1} \otimes \big(\cos(\theta)\left|z+\right\rangle_{S_2} + \sin(\theta)\left|z-\right\rangle_{S_2}\big)\Big].
\end{aligned} \tag{4.11}$$

Now, employing the linearity of the tensor product, pulling out the scalars $\sin(\theta)$ and $\cos(\theta)$, and leaving the tensor product implicit, we can rewrite (4.11) in the more appealing form

$$|\psi\rangle_{S_1,S_2} = -\frac{1}{\sqrt{2}}\sin(\theta)\Big(\,|z+\rangle_{S_1}|z+\rangle_{S_2} + |z-\rangle_{S_1}|z-\rangle_{S_2}\,\Big) + \frac{1}{\sqrt{2}}\cos(\theta)\Big(\,|z+\rangle_{S_1}|z-\rangle_{S_2} - |z-\rangle_{S_1}|z+\rangle_{S_2}\,\Big). \tag{4.12}$$

It was a lot of work just to rotate the angle at which Debbie measures her spin compared to Charlie, but we are now left with an unspecified parameter θ, which we can choose at a later point to ensure the maximum violation of the CHSH inequality.

To quickly recap, we now know the state of the two entangled particles shared by Charlie and Debbie, and the measurement taken by Debbie is an angle θ from the y-axis of Charlie, simply for our own convenience. Now, we let Charlie and Debbie make their measurements of the spin of their respective particle inside their lab. We want to know the overall state for Alice and Bob outside the lab. The total state will take the form

$$|\Psi_0\rangle = |\psi\rangle_{S_1,S_2}|0\rangle_C|0\rangle_D\,,$$

where $|0\rangle_C$ and $|0\rangle_D$ are the state of Charlie and Debbie's labs. The total state is now a state vector living in a 16-dimensional Hilbert space. Alice now sees one of two possible states of the particle S_1, either $|z+\rangle_{S_1}$ or $|z-\rangle_{S_1}$, and correspondingly one of two states of Charlie's measuring device, either $|C_{z+}\rangle_C$ or $|C_{z-}\rangle_C$ depending on what Charlie measured. Bob makes an analogous measurement on S_2 and Debbie's device. The state as seen by Alice and Bob is then

$$\begin{aligned}\left|\tilde{\Psi}\right\rangle = &-\frac{1}{\sqrt{2}}\sin(\theta)\Big(\,|z+\rangle_{S_1}|C_{z+}\rangle_C|z+\rangle_{S_2}|D_{z+}\rangle_D + |z-\rangle_{S_1}|C_{z-}\rangle_C|z-\rangle_{S_2}|D_{z-}\rangle_D\Big)\\ &+\frac{1}{\sqrt{2}}\cos(\theta)\Big(\,|z+\rangle_{S_1}|C_{z+}\rangle_C|z-\rangle_{S_2}|D_{z-}\rangle_D - |z-\rangle_{S_1}|C_{z-}\rangle_C|z+\rangle_{S_2}|D_{z+}\rangle_D\,\Big).\end{aligned}$$

This superposed state then contains all the possible combinations that can be seen by Alice and Bob after they've conducted their measurements. Note that the order of the tensor products was altered, although this is unproblematic since changing the order of the tensor product doesn't change the physical state, so long as we know the order we are working with. In this case it comes with the benefit of allowing greater clarity, since we can introduce the following notation

$$\begin{aligned}|A_{up}\rangle &= |z+\rangle_{S_1}|C_{z+}\rangle_C\,, & |B_{up}\rangle &= |z+\rangle_{S_2}|D_{z+}\rangle_D\,,\\ |A_{down}\rangle &= |z-\rangle_{S_1}|C_{z-}\rangle_C\,, & |B_{down}\rangle &= |z-\rangle_{S_2}|D_{z-}\rangle_D\,,\end{aligned} \tag{4.13}$$

so we have

$$\left|\tilde{\Psi}\right\rangle = -\frac{1}{\sqrt{2}}\sin(\theta)\Big(\,|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle\,\Big) + \frac{1}{\sqrt{2}}\cos(\theta)\Big(\,|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle\,\Big).$$

The benefit of introducing (4.13) is not immediately clear, but we will introduce the operators representing the measurements made by Alice and Bob in terms of these states, and our calculations will be much simplified, with the number of dimensions effectively halving.

The operators we introduce are the following for Alice

$$A_x = \Big(|A_{up}\rangle \langle A_{down}| + |A_{down}\rangle \langle A_{up}| \Big) \otimes \mathbb{1},$$
$$A_z = \Big(|A_{up}\rangle \langle A_{up}| - |A_{down}\rangle \langle A_{down}| \Big) \otimes \mathbb{1},$$

and for Bob we have

$$B_x = \mathbb{1} \otimes \Big(|B_{up}\rangle \langle B_{down}| + |B_{down}\rangle \langle B_{up}| \Big),$$
$$B_z = \mathbb{1} \otimes \Big(|B_{up}\rangle \langle B_{up}| - |B_{down}\rangle \langle B_{down}| \Big),$$

since the measurement of Alice is on Charlie's outcome, while the measurement of Bob is on Debbie's outcome. Here $\mathbb{1}$ represents the identity operator, and since we are talking about operators rather than states it is beneficial to use the tensor notation again, to understand how the operator acts on the states.

The justification for the selection of these operators is not trivial. Firstly, they are indeed self-adjoint, and therefore meet the definition of observables. Secondly, in the basis $|A_{up}\rangle, |A_{down}\rangle$ we have $A_x = \sigma_x$ and $A_z = \sigma_z$, the Pauli matrices, and therefore these operators are analogous to performing a measurement of the spin along either the x or z directions, hence the labels affixed to them. We are effectively allowing Alice and Bob to make a choice to measure one of two different things about the system $\left|\tilde{\Psi}\right\rangle$, in a way that is analogous to measuring σ_x and σ_z.

Let A_1 and B_1 be the friend-type measurement for Alice and Bob respectively, and let A_2 and B_2 be the Wigner-type measurement for each respectively. Assuming that Alice and Bob have free choice as to which they measure, we let $A_1 = A_x$ and $A_2 = A_z$, while we let $B_1 = B_z$ and $B_2 = B_x$. We now consider how these operators act on the state $\tilde{\Psi}$.

Firstly, note that A_x and A_z will only affect the kets $|A_{up}\rangle$ and $|A_{down}\rangle$, while B_x and B_z will only affect $|B_{up}\rangle$ and $|B_{down}\rangle$. Since $|A_{up}\rangle$ and $|A_{down}\rangle$ form an orthonormal basis, the inner product of two of the same is 1 and of two different is 0, and similarly with $|B_{up}\rangle$ and $|B_{down}\rangle$. Therefore we can see that the operator A_x will simply take the vector $\left|\tilde{\Psi}\right\rangle$ and change $|A_{up}\rangle$ into $|A_{down}\rangle$ and vice versa. The operator B_x will do the same to $|B_{up}\rangle$ and $|B_{down}\rangle$. The operator A_z will then change no vectors, but change the sign in front of any term with a $|A_{down}\rangle$, and similarly for B_z.

With this knowledge of how the operators will affect our particular state, we can carry out the calculation of the expected value of the various combinations required for the CHSH inequality. Firstly we use both A_1 and B_1 to get (noting that A_1 is self-adjoint and we can therefore move it to the other side of the inner product)

$$\begin{aligned}
\mathbb{E}_{|\tilde{\Psi}\rangle}(A_1 B_1) &= \mathbb{E}_{|\tilde{\Psi}\rangle}(A_x B_z) = \left\langle \tilde{\Psi} \middle| A_x B_z \tilde{\Psi} \right\rangle = \left\langle A_x \tilde{\Psi} \middle| B_z \tilde{\Psi} \right\rangle \\
&= \Big\langle A_x [-\frac{1}{\sqrt{2}} \sin(\theta)(|A_{up}\rangle |B_{up}\rangle + |A_{down}\rangle |B_{down}\rangle) + \frac{1}{\sqrt{2}} \cos(\theta)(|A_{up}\rangle |B_{down}\rangle - |A_{down}\rangle |B_{up}\rangle)] \Big| \\
&\qquad \Big| B_z [-\frac{1}{\sqrt{2}} \sin(\theta)(|A_{up}\rangle |B_{up}\rangle + |A_{down}\rangle |B_{down}\rangle) + \frac{1}{\sqrt{2}} \cos(\theta)(|A_{up}\rangle |B_{down}\rangle - |A_{down}\rangle |B_{up}\rangle)] \Big\rangle \\
&= \Big\langle -\frac{1}{\sqrt{2}} \sin(\theta)(|A_{down}\rangle |B_{up}\rangle + |A_{up}\rangle |B_{down}\rangle) + \frac{1}{\sqrt{2}} \cos(\theta)(|A_{down}\rangle |B_{down}\rangle - |A_{up}\rangle |B_{up}\rangle) \Big| \\
&\qquad \Big| -\frac{1}{\sqrt{2}} \sin(\theta)(|A_{up}\rangle |B_{up}\rangle - |A_{down}\rangle |B_{down}\rangle) + \frac{1}{\sqrt{2}} \cos(\theta)(- |A_{up}\rangle |B_{down}\rangle - |A_{down}\rangle |B_{up}\rangle) \Big\rangle .
\end{aligned}$$

From here, we can see that taking the inner product will kill all terms which don't match exactly in their combination of *up* and *down*. We are therefore left with the following 4 products

$$\begin{aligned}\mathbb{E}_{|\tilde{\Psi}\rangle}(A_1B_1) &= +\frac{1}{2}\sin(\theta)\cos(\theta)\langle A_{up}|A_{up}\rangle\langle B_{up}|B_{up}\rangle + \frac{1}{2}\sin(\theta)\cos(\theta)\langle A_{down}|A_{down}\rangle\langle B_{down}|B_{down}\rangle \\ &\quad + \frac{1}{2}\sin(\theta)\cos(\theta)\langle A_{up}|A_{up}\rangle\langle B_{down}|B_{down}\rangle + \frac{1}{2}\sin(\theta)\cos(\theta)\langle A_{down}|A_{down}\rangle\langle B_{up}|B_{up}\rangle \\ &= +\frac{1}{2}\sin(\theta)\cos(\theta) + \frac{1}{2}\sin(\theta)\cos(\theta) + \frac{1}{2}\sin(\theta)\cos(\theta) + \frac{1}{2}\sin(\theta)\cos(\theta) \\ &= 2\sin(\theta)\cos(\theta) \\ &= \sin(2\theta)\end{aligned}$$

Next we consider the pair A_1 and B_2 to get

$$\begin{aligned}\mathbb{E}_{|\tilde{\Psi}\rangle}(A_1B_2) &= \mathbb{E}_{|\tilde{\Psi}\rangle}(A_xB_x) = \left\langle\tilde{\Psi}\middle|A_xB_x\tilde{\Psi}\right\rangle = \left\langle A_x\tilde{\Psi}\middle|B_x\tilde{\Psi}\right\rangle \\ &= \Bigg\langle A_x[-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)]\Bigg| \\ &\qquad \Bigg|B_x[-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)]\Bigg\rangle \\ &= \Bigg\langle -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{down}\rangle|B_{up}\rangle + |A_{up}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{down}\rangle|B_{down}\rangle - |A_{up}\rangle|B_{up}\rangle)\Bigg| \\ &\qquad \Bigg| -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{down}\rangle + |A_{down}\rangle|B_{up}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{up}\rangle - |A_{down}\rangle|B_{down}\rangle)\Bigg\rangle \\ &= \frac{1}{2}\sin^2(\theta) + \frac{1}{2}\sin^2(\theta) - \frac{1}{2}\cos^2(\theta) - \frac{1}{2}\cos^2(\theta) \\ &= \sin^2(\theta) - \cos^2(\theta) \\ &= -\cos(2\theta)\end{aligned}$$

Next is A_2 and B_1

$$\begin{aligned}\mathbb{E}_{|\tilde{\Psi}\rangle}(A_2B_1) &= \mathbb{E}_{|\tilde{\Psi}\rangle}(A_zB_z) = \left\langle\tilde{\Psi}\middle|A_zB_z\tilde{\Psi}\right\rangle = \left\langle A_z\tilde{\Psi}\middle|B_z\tilde{\Psi}\right\rangle \\ &= \Bigg\langle A_z[-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)]\Bigg| \\ &\qquad \Bigg|B_z[-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)]\Bigg\rangle \\ &= \Bigg\langle -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle - |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle + |A_{down}\rangle|B_{up}\rangle)\Bigg| \\ &\qquad \Bigg| -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle - |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(-|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)\Bigg\rangle \\ &= \frac{1}{2}\sin^2(\theta) + \frac{1}{2}\sin^2(\theta) - \frac{1}{2}\cos^2(\theta) - \frac{1}{2}\cos^2(\theta) \\ &= \sin^2(\theta) - \cos^2(\theta) \\ &= -\cos(2\theta)\end{aligned}$$

Finally, we consider A_2 and B_2

$$\begin{aligned}
\mathbb{E}_{|\tilde{\Psi}\rangle}(A_2B_2) &= \mathbb{E}_{|\tilde{\Psi}\rangle}(A_zB_x) = \left\langle \tilde{\Psi} \middle| A_zB_x\tilde{\Psi} \right\rangle = \left\langle A_z\tilde{\Psi} \middle| B_x\tilde{\Psi} \right\rangle \\
&= \Big\langle A_z [-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)] \Big| \\
&\qquad \Big| B_x [-\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle + |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle - |A_{down}\rangle|B_{up}\rangle)] \Big\rangle \\
&= \Big\langle -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{up}\rangle - |A_{down}\rangle|B_{down}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{down}\rangle + |A_{down}\rangle|B_{up}\rangle) \Big| \\
&\qquad \Big| -\frac{1}{\sqrt{2}}\sin(\theta)(|A_{up}\rangle|B_{down}\rangle + |A_{down}\rangle|B_{up}\rangle) + \frac{1}{\sqrt{2}}\cos(\theta)(|A_{up}\rangle|B_{up}\rangle - |A_{down}\rangle|B_{down}\rangle) \Big\rangle \\
&= -\frac{1}{2}\sin(\theta)\cos(\theta) - \frac{1}{2}\sin(\theta)\cos(\theta) - \frac{1}{2}\sin(\theta)\cos(\theta) - \frac{1}{2}\sin(\theta)\cos(\theta) \\
&= -2\sin(\theta)\cos(\theta) \\
&= -\sin(2\theta)
\end{aligned}$$

Finally we can put these into the CHSH inequality (4.5) to get

$$S = \sin(2\theta) - \cos(2\theta) - \cos(2\theta) - (-\sin(2\theta)) = 2\sin(2\theta) - 2\cos(2\theta).$$

We are now free to select the measurement angle θ. Although a seemingly peculiar selection, the choice $\theta = -5\pi/8$ gives the following result

$$\begin{aligned}
S &= 2\sin(-5\pi/4) - 2\cos(-5\pi/4) = -2\sin(5\pi/4) - 2\cos(5\pi/4) = 2\sin(\pi/4) + 2\cos(\pi/4) \\
&= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2,
\end{aligned}$$

in violation of the CHSH inequality. In fact, this is the maximum possible violation, called the Tsirelson bound [10]. The implications of this violation are numerous, and deserve another section to be discussed, however a single sentence summary is that the violation of the CHSH inequality implies there cannot exist a joint probability distribution for all four of our observables, hence implying they are somehow unable to consistently coexist for all observers, or in other words saying there is no absolute viewpoint for events.

4.7 Implications and a theorem

We will now examine a little more closely the assumptions we made to arrive at the contradiction above. This will allow us to state our no-go theorem for observer-independent facts [8].

Firstly, we assumed that quantum mechanics can be applied at all scales and describes everything from atoms to submarines to giraffes to galaxies. This means we can assume that the measurement performed by Alice (Bob) on the system containing both particle S_1 (S_2) and Charlie (Debbie) is a valid quantum mechanical measurement, particularly since it includes both a measurement of the quantum system of the particle and a measurement of the macroscopic system of the measuring device. Importantly, we have to be able to assume here that quantum mechanics can be sensibly applied to an entity which is itself applying quantum mechanics. We call this assumption 'universality of quantum theory'.

Next we had to assume that the observation made by one observer has no impact on the observation made by another observer who is space-like separated from them. This is called 'locality'. It is a rather common-sense assumption, and the theory of relativity demands it, but in a quantum world in which entangled particles can seemingly defy this locality, we must be specific. We used this assumption when we made the operators A_1, A_2, B_1 and B_2 commute.

The third assumption is 'freedom of choice'. This is the assumption that allowed Alice to choose between making the measurement A_1 or A_2, and Bob to choose between B_1 and B_2. Since they were free to choose which they wanted, we were able to assign the operators to whichever label was convenient to derive the violation of the CHSH inequality. We also assumed we were free to select the angle θ, and therefore that Debbie was free to choose a different angle to Charlie along which to measure the spin.

The final assumption, and essentially the most important for what we are trying to show, is the assumption of 'observer-independent facts'. This is the one that says that there is a joint probability distribution for all four measurements, meaning everybody's measurements can be simultaneously predicted. If we accept the other three as assumptions that must hold, then this is the faltering assumption, and the notion of observer-independent facts can't be valid, implying that *facts are relative.* To formalise, we present the following theorem due to Brukner [8, Theorem 1].

Theorem 4.7.1 (No-Go Theorem for Observer-Independent Facts)**.** *The following statements are incompatible*

1. *Universal validity of quantum theory.*
2. *Locality.*
3. *Freedom of choice.*
4. *Observer-independent facts.*

Proof. The proof is exactly the scenario derived in section 4.6 above, which uses each of assumptions 1-3 and derives a contradiction to assumption 4. □

So having conclusively shown that the four assumptions in Theorem 4.7.1 are mutually incompatible, what can we conclude about the reality of each of the four assumptions?

At this point we reach the edge of knowledge. The correct way to interpret the Theorem 4.7.1 is not known, and in truth all it can do is put constraints on properties that any particular interpretation of quantum mechanics may have, rather than tell us definitively which is the correct interpretation. The vein is somewhat similar to that of Bell, but with stronger conditions. Despite the possibility that any of the four assumptions made may be violated, the researchers who developed this work have used it to argue against observer independent facts, and indeed within the Copenhagen interpretation, this is the assumption which has to be let go.

4.8 Experimental verification and further work

The somewhat startling conclusion arrived at with Theorem 4.7.1 is not limited to merely theoretical calculations and thought experiments. Precisely this set up has been experimentally verified to violate the CHSH inequality by Proietti et al. [24] and Bong et al. [6], using polarised photons.

Proietti et al. [24] used 4 photons as the observers, and another 2 photons to behave as the measuring device, and measure the expectation values, getting $S = 2.416 \pm 0.075$, a violation of the CHSH inequality by more than 5 standard deviations. There are possible loopholes in this experiment, such as the possibility of undetected communication from photons within the set up. The absence of such communication was tested, and it is therefore reasonable to assume it negligible. Another question is whether it is sensible to use photons as observers, and if it is a valid assumption. According to quantum theory, there is no reason it shouldn't be, but if indeed it were the case that photons can't observe, this may point to the need for a newer theory which can account for this. Assuming these potential loopholes are non-problematic, this experiment provides evidence that the assumptions of Theorem 4.7.1 are incompatible, and the authors argue that a compelling accommodation of the result is Brukner's resolution, that there are no 'facts of the world' per se.

Bong et al. [6] place even stronger conditions on the no-go theorem, replacing the assumption 'observer independent facts' with 'absoluteness of observed events', which doesn't assign an outcome to a measurement which has yet to be made, and therefore only considers that observed events are absolute once a measurement outcome has been reached. The assumption of 'freedom of choice' is also made more rigorous by defining 'no-superdeterminism' to replace it. The authors show that no physical theory can satisfy 'locality', 'no-superdeterminism', 'absoluteness of observed events', and also match the predictions of quantum theory. They go on to experimentally verify their stronger no-go theorem, using once again photons as observers and measurements.

In a summary article [9], Brukner notes that the work of Bong et al. puts the strongest constraints yet on the possibility of observer-independent facts. He writes that "the result can be interpreted to imply that in quantum physics, observers are indeed entitled to their own facts", but notes also that there are interpretations of quantum mechanics which allow different assumptions to be violated.

A similar but distinct no-go theorem due to Frauchiger and Renner was also developed from the extended Wigner's friend scenario [17]. Their assumptions differ slightly from Brukner's. They find that 'compliance with quantum theory', 'single world', and 'self-consistency' are incompatible. 'Single world' refers to the premise that every measurement has a definite outcome, which for example the many-worlds interpretation [15] doesn't accept. The 'self-consistency' assumption refers to the ability of quantum theory to be logical in the outcomes of its predictions, even when multiple observers are considered. The authors consider that Bohmian mechanics [4] satisfies both 'single world' and 'compliance with quantum theory', and therefore it cannot be self-consistent, and must let go of the ability to describe the views of multiple observers consistently. Brukner argues in [8] that Frauchiger and Renner's self-consistency assumption is equivalent to his own observer independent facts assumption, and therefore Bohmian mechanics is an example of an interpretation of quantum mechanics in which there are no observer-independent facts.

5 Discussion and Conclusion

We have just derived a no-go theorem for observer independent facts. We built up, rigorously, a theorem which states that no physical theory can be compatible with 'locality', 'freedom of choice', 'observer independent facts', and the predictions of quantum theory. What Brukner has shown is that there are certain limitations on the possible interpretations of quantum mechanics. We discuss here some possible interpretations of quantum mechanics which do not assume observer-independent facts. Note that the Copenhagen interpretation also removes observer-independent facts, but due to the fact that most of this report has concerned this interpretation it will not be further elaborated on.

Possibly the most popularly known interpretation of quantum mechanics is the many-worlds, or relative state, interpretation [15]. Introduced by Hugh Everett III in his 1957 doctoral thesis, it posits that there is one large wavefunction for the entire universe, and every quantum event merely splits the wavefunction of the universe into multiple branches. There is therefore no ambiguity about the problem of why a particular outcome occurs, as all outcomes occur somewhere in a branch of the wavefunction. The separate branches are entirely disconnected, and since communication between branches would seemingly be impossible, testing this interpretation experimentally is widely considered impossible. According to Bong and co-authors, this interpretation violates their assumption of ‘absoluteness of observed events’.

Another interpretation recently gaining more notice is that of QBism, pioneered by Fuchs and Schack (see [21]), which is shorthand for Quantum Bayesianism. This interpretation willingly lets go of the notion of observer-independent facts, instead encouraging that the probabilities be considered along with the individual making the gamble on them (i.e. the measurement). The odds are then updated and everything continues. By reconsidering that the scientist doing the measurement is part of the science once again, QBism claims to rectify the measurement problem of quantum mechanics. It certainly plays very nicely with the EPR scenario (Section 4.1), since Alice having knowledge of Bob’s particle’s state is now unproblematic, it simply reflects the world-view of Alice. This interpretation is comparatively new, but has gained some well-known advocates.

The future direction of work in this field is exciting, as this renewed interest in the foundational problems of quantum mechanics increases. We are certain to see more no-go theorems in the near future, either strengthening assumptions or showing another new set of incompatible ones. The primary goal is to one day solve the quantum measurement problem, and make true sense of the process of a measurement being made, and the collapse of superposition states. Perhaps we can hope that this work will one day lead to finding the ‘correct’ interpretation of quantum mechanics, if such an interpretation exists. Perhaps there isn’t one, and we just have to come to accept that quantum mechanics simply is, and stop trying to give it a meaning that we can comprehend. With theorists such as Renner and Brukner continuing to produce work in the field, this author looks forward to seeing the next steps.

In this report, we have provided a detailed and rigorous mathematical formulation of quantum mechanics as a mathematical and physical theory. We developed all the necessary mathematical tools, drawing from linear algebra and spectral theory, to give an axiomatic definition of quantum theory. Following this, we developed some ideas in quantum mechanics using the mathematical formalism, and introduced entanglement. In the final section, we gave a thorough timeline of some important work on the quantum measurement problem, and eventually introduced and proved the no-go theorem of Brukner (Theorem 4.7.1) aimed to show that the notion of observer-independent facts is incompatible with the quantum world. We did this by careful and rigorous consideration of the extended Wigner’s friend thought experiment, which can be shown to lead to a violation of the Bell-type CHSH inequality.

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