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Cellular Automata Processes on Trees

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Supervised by A/Prof Andrea Collevecchio and A/Prof Kais Hamza

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March 1, 2021

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Abstract

We aim to study a class of self-interacting random walks (see Sections 1.1-1.3) defined on infinite trees, i.e. graphs without cycles. The local behaviour of this class of processes is determined by a cellular automata rule (see Section 1.4) applied to Bernoulli random variables. The rules are then applied iteratively for as many time steps as desired ([Wei02]). Our aim is to study the limiting behaviour of this class of processes, and in particular under what regime they visit the root only finitely often.

1 Introduction

1.1 A brief history of random walks

The concept "random walk" was firstly introduced in a letter that Karl Pearson wrote to *Nature* in 1905. In that letter, he described a mosquito infestation in a forest using a basic probabilistic model. In this model, each mosquito moves a fixed length a, at each time step and at a randomly chosen angle, independently of the other mosquito. Pearson was interested to obtain the asymptotic distribution of the mosquitoes at large times.

Lord Rayleigh replied to the letter, after he solved a more general form of this problem in the context of sound waves in heterogeneous materials. Modeling a sound wave moving through the material is similar to adding up a sequence of random wave vectors: the wavelength of sound waves in the material is approximately constant, but the directions of sound waves are changed at scattering sites within the material.

Almost at the same time, Louis Bachelier introduced the idea of random walks in his doctoral thesis, published in 1900. In his work, these processes were connected to financial time series. This was a milestone in modern theoretical finance. Bachelier was recognised as the first to realise the connection between discrete random walks and the continuous diffusion equation.

Albert Einstein also worked in this field, in the same period. He wrote a groundbreaking paper on Brownian motion to model complicated path of a big dust particle in air driven by collisions with gas molecules. Brownian motion is the continuous counterpart of random walks.

At a glance, random walks can be applied broadly, especially in science and engineering. (see [Baz06])

1.2 Why are we interested in random walks?

We are interested in random walks for two reasons: Theoretically, they are one of the most fundamental types of stochastic processes, and a large variety of phenomena can be modeled by them: statistical physics, computer science, operations research (particularly queue theory, see e.g. [Lem76]), biology [CPB08], game theory. These processes too have also been applied to model animal locomotion and foraging, neuronal firing dynamics, decision-making in the brain, population genetics polymer chains ([Gia07])in the field of biology, stock prices in finance, ranking systems, dimension reduction and feature extraction from high-dimensional data in the field of data science. We can also predict arrival times of diseases spreading on networks by using RW. (see [MPL17])

1.3 Random walk on graph

Let $\mathcal{G} = (V, E)$ be a locally-finite graph, with a designated vertex, denoted by ρ. A random walk on $\mathcal{G}$, with starting point ρ is defined by the following dynamics. It is discrete time, it takes value on the vertices V of $\mathcal{G}$ and jumps to nearest neighbours. At each stage a neighbour of the site that is currently occupied is chosen uniformly at random, and move to this vertex.

This kind of process, by construction, satisfies the Markov property and time-reversibility (see Definition7 below). In fact, every reversible Markov chain can be viewed as random walk on $\mathcal{G}$, if we allow weighted edges. Similarly, time-reversible Markov chains can be viewed as random walks on undirected graphs, and symmetric Markov chains, as random walks on regular symmetric graphs.

The classical random walks theory deals with random walks on infinite graphs (e.g. grids, trees) and study their asymptotic behaviour: does the random walk return to its starting point with probability 1? Does it return infinitely often? One famous example is Pólya (see [Pol21])'s Theorem, which establishes recurrence/transience of simple symmetric random walk on $\mathbb{Z}^d$. If $d \leq 2$, then with probability 1, the walk returns to its starting position (recurrent). If $d \geq 3$, then with positive probability, the walk never returns to its starting position, i.e. is transient. (see [Lov93])

1.4 Introduction to cellular automata

Cellular Automata (CA) are discrete computational systems that have many useful representations in different scientific fields.

CA are composed by a finite or countable set of homogeneous and simple units called the atoms or cells. Each of these cells will be in one of a finite number of states at each step. Their status will evolve in parallel at discrete time steps, following specific state transition rules. The update of a cell state is affected by the status of neighbour cells. Furthermore, CA are abstract because we can express them in mathematical terms and implement them in physical structures. Lastly, CA are computational systems because they can solve algorithmic problems. To be more precise, they can emulate a universal Turing machine (see [Whi12]).

The simplest way to explain an automaton is like this: Imagine an automaton as a one-dimensional grid of cells, each cell can only take one of two states - either on or off. The system then follows a transition rule to evolve. At each time step, each cell updates its status in response to the states of its neighbouring cells.

Stanislaw Ulam and John von Neumann, who worked at Los Alamos National Laboratory at the time, first suggested the idea in the 1940s. Although some scholars looked into the topic in the 1950s and 1960s, it was not until the 1970s Conway introduced the Game of Life - a two-dimensional cellular automaton - that people outside academia started to recognise its importance. In the 1980s, Stephen Wolfram studied elementary cellular automata, which is one-dimensional cellular automata; his research assistant Matthew Cook showed that one of these rules is Turing-complete. In brief, a Turing Complete system means a system in which a program can be written that will find an answer (although with no guarantees regarding runtime or memory).

John von Neumann focused on self-replication specifically, seeking to explain biological development from the perspective of a reductionist theory by considering a process that creates identical copies of itself. Biology at first sight seemed to be involved with fluidity and continuous dynamics. However, Stanislaw Ulam, a colleague of von Neumann, focused on the study of a discrete, two-dimensional system. von Neumann improved his automaton with 29 states from two-state black-or-white cells with even more complicated dynamics and the ability to replicate itself. Von Neumann's CA was also the first discrete parallel computational model in history formally shown to be a universal computer, that is, capable of emulating a universal Turing machine and computing all recursive functions.

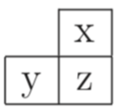
In the early 1960s, E.F. Moore and Myhill specified conditions for the existence of the Garden-of-Eden theorems, that is patterns can only appear on the lattice of a CA as initial conditions. In 1969 Gustav Hedlund investigated CA within the framework of symbolic dynamics. In 1970 John Conway introduced his infamous work Game of Life, which is probably the most well known automaton in the world, and it is one of the simplest computational models ever proved to be Turing complete, which enables it to simulate a universal constructor or any other Turing machine. In 1977 Tommaso Toffoli used CA to directly model physical laws, laying the foundations for the study of reversible CA.(see [BT17])

The Game of Life (or simply as Life) is created by the British mathematician John Horton Conway in 1970. It is a zero-player game, which ensures that its evolution is only determined by its initial state, with no extra input needed. Game of Life can be operated by creating an initial states and watching it evolve.

The domain of the Game of Life is an infinite, 2D grid of square cells, each cell is in one of two possible states - live or dead (or populated and unpopulated, respectively), and interacts with its eight neighbours, which are the cells that are horizontally, vertically, or diagonally adjacent. At each step in time, the following transitions occur:

'Any live cell with fewer than two live neighbours dies, as if by underpopulation. Any live cell with two or three live neighbours lives on to the next generation. Any live cell with more than three live neighbours dies, as if by overpopulation. Any dead cell with exactly three live neighbours becomes a live cell, as if by reproduction.' (see [BCCG82])

In the recent work of Collevecchio, Hamza, and Shi (See [CHS15]), they studied a cellular automata rule with random boundary condition. In particular they analyzed the dependence structure of the columns in the 2-dimensional grids, using a rule that any two cells in a triangular array



of the type uniquely determine the third, and the fact that the first column is identical to its top cell. They concluded that there exited a unique set of solutions for all the cells in the system.

In this project we are going to consider cellular automata rules that preserve the i.i.d structure of bernoullis, and 'recycle' random variables. There are many alternative way to construct such a rule. They are counted and studied in [CHSW20].

1.5 Statement of Authorship

Kais and Andrea formulated the project idea and model outline. Jiayu wrote the proof of the main theorem under their guidance and supervision. Project funding was provided by AMSI and the Australian Department of Education.

2 Rooted trees

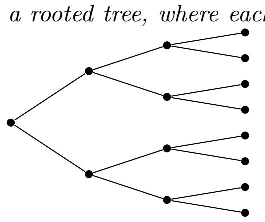
For any integer n let $[n] = \{1, 2, \dots n\}$. A graph is composed a pair (V, E) where V is the set of vertices and E is the set of edges. A tree $\mathcal{T} = (V, E)$ is a graph without cycles (see [LP17]). We are going to focus on infinite trees, because we are interested in the asymptotic behaviour of stochastic processes on this graph. For simplicity we are going to designate a special vertex ρ, and call it the root of the tree, as the starting point of the processes. We introduce a distance on this graph as follows. The distance between two vertices is given by the number of vertices in the unique self-avoiding path connecting them. We introduce the norm $\|v\|$, for $v \in V$ as the distance of v from the root ρ. Moreover, define the level sets L_n as the collection of vertices with norm n, i.e.

$$L_n := \{v \in V : \|v\| = n\}.$$

Two vertices are neighbours if they are connected by an edge. For any vertex v denote by v^{-1} its parent, i.e. the unique vertex connected by an edge to v that has norm $\|v\| - 1$, i.e. is closer to ρ. In general, denote by v^{-k} the k-th ancestor of v, i.e. the vertex at distance k from v which lies on the shortest path to the root. Moreover, define $o(v)$ the number of offspring of v, i.e. the neighbours of v different from v^{-1}. If $o(v) > 0$, let $(v_i)_{[o(v)]}$ the offspring of v.

A graph $\mathcal{T}$ is a tree if and only if between every pair of distinct vertices of $\mathcal{T}$ there is a unique path. If the tree has v vertices and e edges, then $e = v - 1$. As soon as one vertex of a tree is designated as the root, then every other vertex on the tree can be characterised by its position relative to the root. This works because there is a unique path between any two vertices in a tree. So from any vertex, we can travel back to the root in exactly one way. (see [Lev16])

Example 1. *A binary tree. It is a rooted tree, where each vertex has exactly three neighbours, with the exception of*



the root, which has two offspring.

2.1 Galton–Watson Process

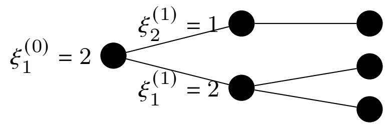
We are going to focus on a general class of random trees, called Galton-Watson (GW) trees. GW are the graphical representation of the class of stochastic processes $(Z_n)_n$ which can be defined as follows. We set $Z_0 = 1$ and define recursively

$$Z_{n+1} := \sum_{j=1}^{Z_n} \xi_j^{(n)}, \tag{2.1}$$

where $\{\xi_j^{(n)} : n, j \in \mathbb{N}\}$ is a collection of independent and identically-distributed random variables which take values on the set $\mathbb{N}$ of non-negative integers.

It can also represented by a tree (GW tree) as the example below, where Z_n can be thought of as the number of descendants in the nth generation, and $\xi_j^{(n)}$ can be thought of as the number of children of the jth of these descendants.

Example 2 (A Galton–Watson Tree)**.** *In this example consider a realisation of the random variables* $\xi_j^{(n)}$ *as follows and give a representation. Let* $\xi_1^{(0)} = 2$*,* $\xi_1^{(1)} = 2$*,* $\xi_2^{(1)} = 1$*.*



The main result regarding Galton-Watson process is a simple phase transition which depends on the offspring mean. For simplicity we exclude the case when each vertex has a deterministic number of offspring equal to one.

Theorem 3. *If* $\mathbb{E}[\xi_j] \le 1$*, the probability of ultimate extinction is 1, i.e.*

$$\mathbb{P}\left(\exists j \in \mathbb{N} \colon Z_j = 0\right) = 1.$$

Else if $\mathbb{E}[\xi_j] > 1$ *, there exists a positive probability that the population* Z_n *will never be extinct, i.e.*

$$\mathbb{P}\left(\forall j \in \mathbb{N} \colon Z_j > 0\right) > 0.$$

2.2 Markov Chains

A discrete-time Markov chain is a sequence of random variables $(X_i)_{i\in\mathbb{N}}$ that satisfies the Markov property, namely that at each time step, the chain "moves" from state i to state j with probability p_{ij}, independently of its past:

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0) = \mathbb{P}(X_{n+1} = j | X_n = i) \tag{2.2}$$

In other words, given the present X_n, the future X_{n+1} and the past $\{X_0, ..., X_{n-1}\}$ are independent.

Moreover, we deal with time-homogeneous Markov chains, which satisfy

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i),$$

for each $n \in \mathbb{N}$.

When we consider a stochastic process on an infinite state space, it is natural to study its asymptotic behaviour. In particular, the very first question is to study the probability that the process returns to its starting point. We focus on processes $\mathbf{X} = (X_n)_n$ which are defined on rooted trees, and jump on nearest neighbours. We assume that these processes start from the root, denoted by ρ. In other words, $\mathbb{P}(X_0 = \rho) = 1$.

Definition 4 (Recurrence and Transience)**.** *A stochastic process* $\mathbf{X}$ *can be either transient or recurrent. It is transient if the probability of not returning to* ρ *after its first step is a positive. It is recurrent if it is not transient.*

Remark 5. *Notice that for time-homogeneous irreducible Markov chains, recurrence (resp. transience) is equivalent to have the process visit each vertex infinitely (resp. finitely) often, with probability one.*

Example 6. *Consider the tree, denoted by* $\mathbb{Z}^+$*, consisting of the non-negative integers, where each pair of consecutive integer is connected by an edge. Identify the root* ρ *with* 0 *and let* $\mathbf{X}$ *be the simple symmetric random walk. This process is recurrent, and actually visits any vertex infinitely often. Next consider a biased random walk instead, with* *transition probabilities $P_{j,j+1} > 1/2$, where $P_{j,j+1}$ is the probability to jump from j to $j+1$ and is the same for all $j > 0$. This process is transient, and the probability to never return to the origin after the first step is equal to*

$$1 - \frac{1 - P_{j,j+1}}{P_{j,j+1}} > 0.$$

2.3 Reversible Markov Chain

Many aperiodic and irreducible Markov chains which are in steady state, show the property that the sequence of states looked backwards in time, i.e., $\ldots, \mathbf{X}_{n+1}, \mathbf{X}_n, \mathbf{X}_{n-1}, \ldots$ has the same probability structure as the sequence of states running forward in time. (see [AF02]) That is, if $\mathbf{X}_t$ is a stationary chain, then

$$(\ldots, \mathbf{X}_{n-1}, \mathbf{X}_n, \mathbf{X}_{n+1}, \ldots) \stackrel{d}{=} (\ldots, \mathbf{X}_{n+1}, \mathbf{X}_n, \mathbf{X}_{n-1}, \ldots) \tag{2.3}$$

We introduce the following general (formal) definition to illustrate time-reversibility of Markov processes. Denote by $\mathcal{G}$ a locally finite graph, and let V be its vertex set. Let $(p_{i,j})_{i,j \in V}$ be the transition kernel of the (homogeneous) aperiodic and irreducible markov chain $\mathbf{X}$.

Definition 7 (Reversible Markov Chain)**.** *We say that* $\mathbf{X}$ *is reversible if there exists a positive function* $\mathbf{g} : \mathcal{G} \mapsto (0 : \infty)$ *which satisfies*

$$g(i) p_{ij} = g(j) p_{ji}, \qquad \forall i, j \in V \tag{2.4}$$

Notice that g is unique up to a multiplicative (positive) constant.

An alternative definition of reversible markov chain is the following. To each edge e of a graph assign a positive weight a_e, and define a nearest neighbour process $\mathbf{X}$ as follows. If at time t, with $t \in \mathbb{N}$, we have that, independently of the past, the probability that the process jumps at $y \sim x$ at time $t+1$ equals

$$\frac{a_{[x,y]}}{\sum_{z:\, z \sim x} a_{[x,z]}},$$

where $[x, y]$ denotes the undirected edge connecting x to y. Notice that we can pick

$$a_{[x,y]} = g(x) P_{x,y},$$

where g is as in Definition 7.

Example 8. *Consider simple symmetric random walk on a locally finite graph $\mathcal{G}$. In this context, the process is reversible with the choice $g(x) = \text{degree(x)}$, where the degree of a vertex is the number of edges incident to it. This function is unique up to a positive multiplicative constant. Consider a random walk on $\mathbb{Z}$. In that case,*

$$\sum_x g(x) = \infty.$$

In fact, this simple random walk does not have a stationary distribution.

Thus the chain is reversible if the backward running sequence of states is statistically indistinguishable from the forward running sequence. If $\sum_i g(i) = 1$ and p_{ij} is a set of transition probabilities for an aperiodic and irreducible Markov chain, then $g(i)$ is the steady state probability.

2.4 Construction of Continuous Time Markov chain (CTMC)

Movement from state to state is determined by "competing" clocks with timers that go off at random, exponentially-distributed, times. For each state x, there is a clock associated with each state that's neighbour to x. Each time the CTMC moves to state x, clocks with timers are reset to go off at random times. Moreover, we assume that these newly set times are mutually independent and independent of the history of the CTMC prior to that transition time. By the lack-of-memory property of the exponential distribution, resetting running timers is equivalent (leaves the probability law of the stochastic process unchanged) to not resetting the times, and letting the timers continue to run. (see [Whi12])

More formally, consider a locally finite tree $\mathcal{T} = (V, E)$ and let $\mathbf{X}$ be a reversible markov chain. Denote by $(a_e)_{e\in E}$ the set of positive weights assigned to each edge. We also use the notation $a_{x,y}$ where x, y is a pair of neighbour vertices, which identifies a unique edge. For each order pair of neighbours $x, y \in V$ assign an independent Poisson process $\mathbf{W}(x, y) = (W_i(x, y))_{i\in\mathbb{N}}$ with rate 1, where $W_i(x, y)$ are the inter-arrival times. Define

$$T_x := \inf\{t : X_t = x\}. \tag{2.5}$$

We convey that $\inf \varnothing = +\infty$. We set $X_{T_x+1} = y$ if and only if y is the neighbour of x which minimises $a^{-1}_{x,y} W_1(x, y)$. Moreover, define recursively

$$T^{(j)}_x := \inf\{t > T^{(j-1)}_x : X_t = x\}$$

with $T^{(0)}_x = T_x$. Set $b(x, y, j)$ to be the number of times the process jumped from x to y before time $T^{(j)}_x$. We have that $\{X_{T^{(j)}_x+1} = y\}$ if and only if y is the neighbour of x which minimises

$$a^{-1}_{x,y} W_{b(x,y,j)}(x, y).$$

Definition 9 (See [CHS15] and [CHL17]). *For each N, let ψ_N be a function $\psi_N : \{-1, 1\}^N \mapsto \{-1, 1\}$. We say that $(\psi_N)_N$ is* **compatible** *if whenever $(\xi_i)_i$ is a sequence of i.i.d. mean-zero random variables on $\{-1, 1\}$, then the sequence $\big(\psi_N(\xi_1, \xi_2, \dots, \xi_N)\big)_N$ is also i.i.d. mean zero. In other words, if $(\xi_i)_i$ are the increments of simple symmetric random walk, so does $\big(\psi_N(\xi_1, \xi_2, \dots, \xi_N)\big)_N$.*

Example 10.

- *If $\psi_N(x_1, x_2, \dots, x_N) = \prod_{i=1}^N x_i$ then $(\psi_N)_N$ is compatible.*
- *If $(\psi_N)_N$ is compatible, and we define $\overline{\psi}_N = \prod_{i=1}^N \psi_i$, then $(\overline{\psi}_N)_N$ is compatible.*
- *If $\psi_1(x_1) = x_1$ and $\psi_N = x_{N-1}x_N$, for all $N \geq 2$, then $(\psi_N)_N$ is compatible.*

Definition 11. *For each vertex v of the tree $\mathcal{T}$, denote by p_v the vertices on the path connecting ρ to v, i.e. the shortest self-avoiding path connecting the two.*

Denote by $(\xi_v)_v$ a sequence of i.i.d. Bernoulli(1/2) indexed by the vertices of the tree. Let $\eta_v = 2\xi - 1 \in \{-1, 1\}$. Our main object is to study a random walk on a general tree that follows the rule of cellular automata. We assign to the root ρ the pair (η_ρ, η_ρ). Suppose that we assigned to each vertex at level less or equal to $N - 1$ a pair. Repeat the following reasoning for each vertex v at level N. Let $\Xi_v = (\eta_\mu)_{\mu\in p_v}$, and assign to v the pair $(\eta_v, \psi_N(\Xi_v))$.

We colour v blue if the coordinates of the pair assigned to v are the same, and v is coloured red otherwise. We define a random walk $\mathbf{X} = (X_n)_n$ on the tree as follows. It starts at ρ, i.e. $X_0 = \rho$ and it jumps to nearest neighbour. Whenever the random walk is at a blue (resp. red) vertex its next jump follows a fixed transition probability denoted by $P^{\bullet}$ (resp. $P^{\bullet}$). Let x be the probability, using the transition $P^{\bullet}$ that next step is further from the root. In other words,

$$x^{\bullet} = 1 - P^{\bullet}(\text{transition to its parent})$$

and define $x^{\bullet}$ similarly.

We are interested in studying this process when defined on Galton-Watson trees. Let $\mathcal{T}$ be a Galton-Watson tree and let $\mu > 1$ be its offspring mean. We define the process as a two step mechanism:

- We observe a realisation of the random tree $\mathcal{T}$, and
- we define the process on such realisation.

Theorem 12. *Suppose that*

$$\frac{\mu}{2}\left(x^{\bullet} + x^{\bullet}\right) > 1, \tag{2.6}$$

then the process is transient.

Definition 13 (see [Col06])**.** *A vertex at level 1 is good if it is visited by the process at time 1. A vertex v at level $n > 1$ is good if*

- *If its parent is good, and*
- *'It is visited by the process before it goes back to its grandparent v^{-2}', i.e.*

$$\frac{1}{\mathbb{P}^{\bullet}(v^{-1} \mapsto v)} W_1(v^{-1}, v) < \frac{1}{\mathbb{P}^{\bullet}(v^{-1} \mapsto v^{-2})} W_1(v^{-1}, v^{-2}),$$

  where $\bullet \in \{\bullet, \bullet\}$ is the color of v^{-1}.

Proposition 14. *The collection of good vertices evolve like a Galton-Watson tree.*

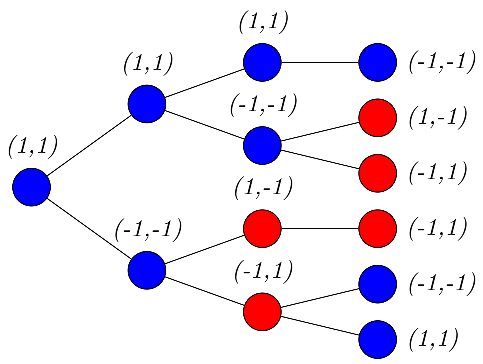
Proof. The first vertex at level one to be visited is good. Suppose that a vertex v, with $\|v\| \geq 2$, is good. Consider a child μ of v. Denote by A the event that μ is good. Let Ξ_v be the collection of Poisson processes

$$\left\{W_i(x, y) \colon i \in \mathbb{N}, \text{ both } x \text{ and } y \text{ are descendants of } v\right\}.$$

Denote by χ_v the collection of all the other Poisson processes used to generate the jumps of the random walk. We have that A is determined by Ξ_v, i.e. A is measurable with respect to the smallest σ-algebra generated by Ξ_v. If we pick any other vertex τ that is not a descendant of v, the event that τ is good is determined by χ_v, and this establishes the independence.

As the number of offspring for each vertex share the same distribution we have the following. For any pair of vertices v and μ, conditionally on both being good, the number of offspring of v that are good has the same distribution of the number of μ with the same property. ∎

Example 15. *We have a realisation of the Galton-Watson tree $\mathcal{T}$ with mean $\mu = 2$, and the process is defined on the tree as follows:*



Furthermore, we assign $x^{\color{blue}\bullet}$ *and* $x^{\color{red}\bullet}$ *as follows. If a vertex is blue, the probability of jumping towards its parent is* $\frac{1}{3}$*, thus* $x^{\color{blue}\bullet} = \frac{2}{3}$*; If a vertex is red, the probability of jumping towards its parent is* $\frac{1}{6}$*, thus* $x^{\color{red}\bullet} = \frac{5}{6}$*. This process is transient since* $\frac{\mu}{2}\left(x^{\color{red}\bullet} + x^{\color{blue}\bullet}\right) = \frac{2}{2}\left(\frac{5}{6} + \frac{2}{3}\right) > 1$.

Proof of Theorem 12. As we proved in Proposition 13, the collection of good vertices evolve like a Galton-Watson tree. Let ξ^g be the number of offspring of a good vertex v. Notice that the random variable is independent of the choice of v. Let

$$\mu^g = \mathbb{E}[\xi^g].$$

Next we prove that $\mu^g > 1$, which implies that with positive probability there is an infinite number of good vertices. This in turn implies that the process $\mathbf{X}$ will never revisit the starting point with a positive probability. Let $\mathbb{P}(red)$ (resp. $\mathbb{P}(blue)$) be the probability that the next child of the good process is coloured red (resp. blue). As the pair $(\eta_v, \psi_N(\Xi_v))$ has outcomes (1,1), (1,-1), (-1,1) and (-1,-1) with equal probability, $\mathbb{P}(red) = \frac{1}{2}$ and $\mathbb{P}(blue) = \frac{1}{2}$. Let $\mathbb{E}[\xi^g|red]$ (resp. $\mathbb{E}[\xi^g|blue]$) denote the conditional expectation that the expected number of offspring given the child of the good process is coloured red (resp. blue).

$$\begin{aligned}
\mu^g = \mathbb{E}[\xi^g] &= \mathbb{E}[\mathbb{E}[\xi^g|colour]] \\
&= \mathbb{E}[\xi^g|\text{red}] \cdot \mathbb{P}(\text{red}) + \mathbb{E}[\xi^g|\text{blue}] \cdot \mathbb{P}(\text{blue}) \\
&= \mu \cdot x^{\color{red}\bullet} \cdot \frac{1}{2} + \mu \cdot x^{\color{blue}\bullet} \cdot \frac{1}{2} \\
&= \frac{\mu}{2} \cdot \left(x^{\color{red}\bullet} + x^{\color{blue}\bullet}\right) > 1
\end{aligned} \tag{2.7}$$

∎

3 Conclusions and further studies

We provided a sufficient condition for transience of walks that are related to cellular automata. The example we considered could have been studied with different methods. The strategy we used can be generalized to cover a large class of processes. We aim to study, in future work, a more general class of walks related to general cellular automata rules, and provide sharp conditions for both transience and recurrence.

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