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Online Colouring Overlap Graphs

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Contents

1 Abstract

This paper looks at systems of intervals and their associated overlap graphs. Specifically, it examines interval systems created by excluding interval combinations of size 3 and attempts to find what class of graphs the overlap graph creates. It successfully classifies each class as either perfect or not perfect.

2 Introduction

2.1 Motivation

One of the applications of online colouring of overlap graphs is in stacking problems. A stacking problem involves containers arriving and having to be stored for a duration on a stack, before eventually being removed from the top of the stack after a certain duration. One of the goals in this process is to minimise the number of times a container must be moved from somewhere other than the top of a stack, as this would require moving the above containers as well. We consider the perfect case where this is zero. Another goal of this process is to minimise the total number of stacks required. This can be modelled by minimising the number of colours in an overlap graph.

2.2 Colouring and perfect graphs

A graph is a set of vertices and a set of edges connecting those vertices, denoted $G(V, E)$. Colouring a graph involves assigning a colour to each vertex in the graph such that no two vertices that are linked by an edge are assigned the same colour, in the stacking problem this is equivalent to ensuring no two containers that overlap are in the same stack. The smallest number of colours that can be used is called the chromatic number or $\chi(G)$. A clique is a set of vertices where each vertex shares an edge with each other vertex. The size of the largest clique in the graph is called it's clique number or $\omega(G)$. An induced subgraph is a graph created by a subset of the vertices in G, with an edge between two vertices if they were adjacent in the original graph, denoted $G_A(A, E_A)$. A perfect graph is a graph where for every induced subgraph the chromatic number is equal to the clique number $\chi(G_A) = \omega(G_A)$. The strong perfect graph theorem states that a graph is perfect if and only if neither the graph nor it's complement contain an odd cycle of length at least 5. [1]

2.3 Overlap graphs

Overlap graphs correspond to a set of intervals where each vertex represents an interval, and two vertices are linked by an edge if their corresponding intervals overlap. An overlap occurs between two when the interval with the earlier start point, has a finishing point which is within the start and endpoint of the second interval. In the context of stacking problems, if the containers represented by the first and second intervals were on the same stack, when the first container is to be removed, it would be underneath the second container.

2.4 Partitioning an interval system

In their paper 'A Note on Online Colouring Problems in Overlap Graphs and Their Complements' [2], Marc Demange and Martin Olsen find an upper bound for colouring of overlap graphs when the intervals are presented from left to right, this is based around partitioning the system into subsets where each subset can be optimally coloured. This is done by ensuring that no combination of three intervals in a partition forms a certain configuration. Finding other configurations that when excluded the result can be perfectly coloured, may result in an improvement to this bound.

2.5 Results in this paper

This paper classifies interval configurations of size 3 as to whether or not excluding them from an interval system always results in a perfect overlap graph. For each interval configuration this paper either provides an example of an interval system that excludes them and is not perfect, or provides a proof that excluding them always produces a perfect graph.

2.6 Notation used

The positioning and overlap of intervals can be written as a series of numbers. Each interval is assigned a number, the first occurrence of a number represents the start of an interval, while the second occurrence of a number represents the end of the interval. The order the numbers are written is the same order as the corresponding points in the interval system. As a convention, intervals are labelled in order of their leftmost points. For example, the interval system represented by 1212, would consist of a pair of intervals that overlap. Each interval system of size 3 can be uniquely described by a combination of numbers containing exactly two copies of one, two and three, with the first occurrence of 1 before 2, which is before the first occurrence of 3.

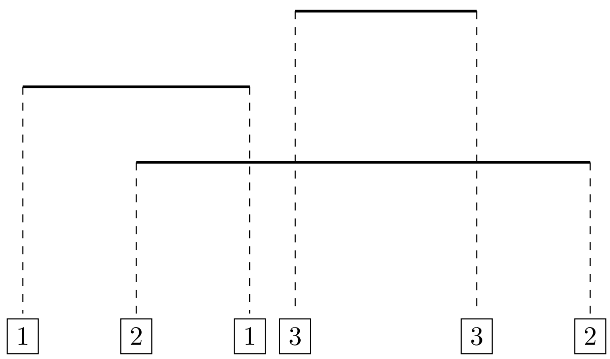
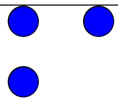
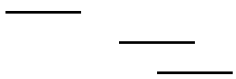
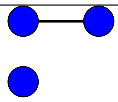
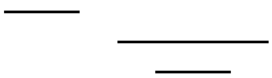
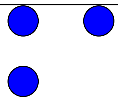
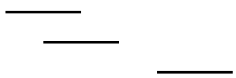
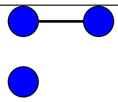
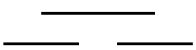
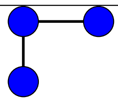
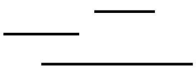
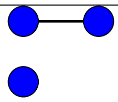
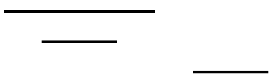
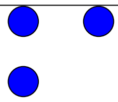
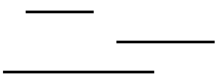
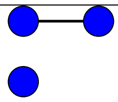
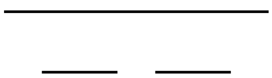
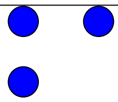
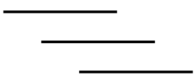
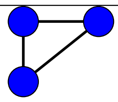
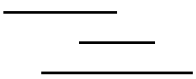
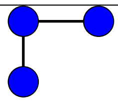
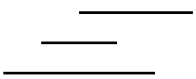
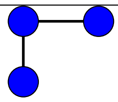
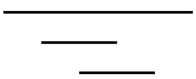
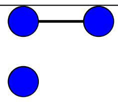
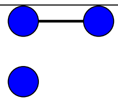
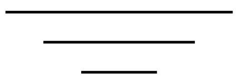
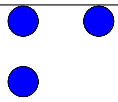


Figure 1: Diagram showing the conversion from interval system to notation

2.7 Statement of authorship

Under the supervision of Marc Demange I developed notation to work with intervals and found counter examples to the R free class being perfect for many intervals by finding possible constructions of C5 and the complement of C7. as well as finding a proof that the R free class of left and right tanks are perfect. Proofs that the R free class for Bridge, Contained Edge as well as Left and Right Cannon were found by Marc Demange and Martin Olsen.

3 Table of Results

Interval system numbers	Interval system lines	Overlap graph	Is the R free class perfect?
112233 or 3T1			no, contains C5
112323 or Vertex + Edge			no, contains complement of C7
112332 or T1 + T2			no, contains C5
121233 or Edge + Vertex			no, contains complement of C7
121323 or Bridge			yes
121332 or Left Tank			yes
122133 or T2+T1			no, contains C5
122313 or Right Tank			yes
122331 or Dolmen			no, contains c5
123123 or K3			no, contains c5
123132 or Left Cannon			yes
123213 or Right Cannon			yes
123123 or Contained Edge			yes
123312 or Containing Edge			no, contains c5
123321 or T3			no, contains c5

4 R free classes that are not perfect

4.1 Constructions of C5 and interval configurations not contained within

As the smallest odd cycle of size 5 or greater, it serves as a simple counterexample for many interval combinations.

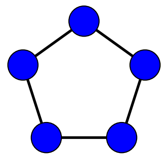


Figure 2: Graph of the cycle of size 5

R free classes that contained C5 were: 3T1, T1 + T2, T2 + T1, Dolmen, K3, Containing Edge, and T3 There are two distinct interval systems of which the overlap graph results in the cyclic graph of size 5 figure 3 gives the first of these.

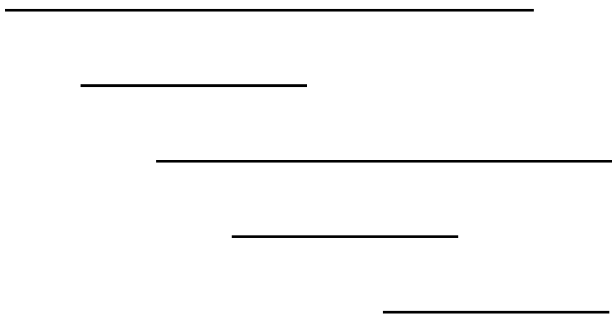


Figure 3: One interval representation of C5

Figure 3 does not contain the following interval configurations. 3T1, T1 + T2, T2 + T1, Dolmen, K3, T3

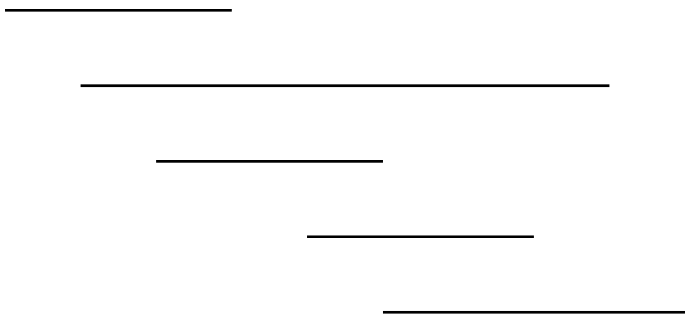


Figure 4: Another interval representation of C5

Figure 4 does not contain a Containing Edge.

4.2 Constructions of the complement of C7 and interval configurations not contained within

Since the complement of C5 is C5, the complement of C7, is the smallest complement of an odd cycle of size 5 or greater, not already covered, so serves as a simple counterexample for some others.

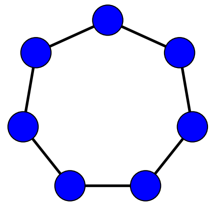


Figure 5: Graph of the complement of the cycle of size 7

R free classes that contained in the complement of C7 that are not contained in C5 were:
Edge + Vertex and Vertex + Edge
There are two distinct interval systems of which the overlap graph results in the complement of the cyclic graph of size 7 figures 6 and 7.

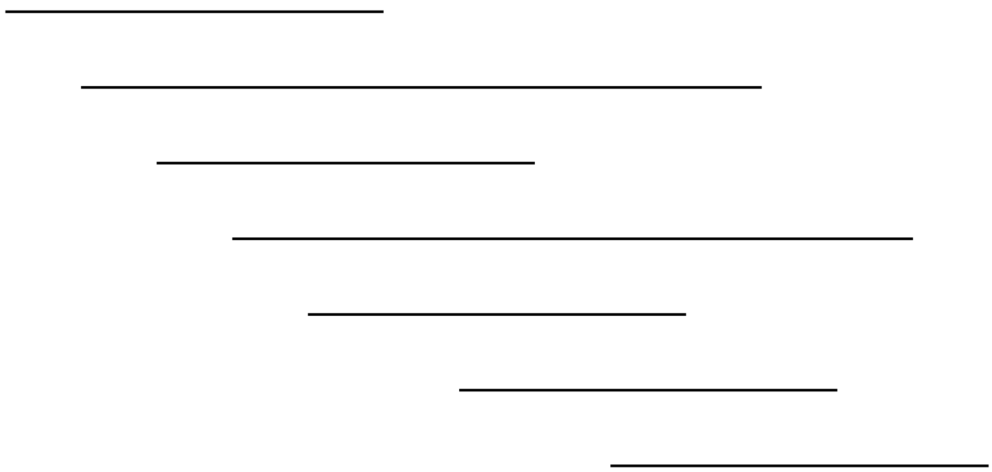


Figure 6: One interval representation of the complement of C7

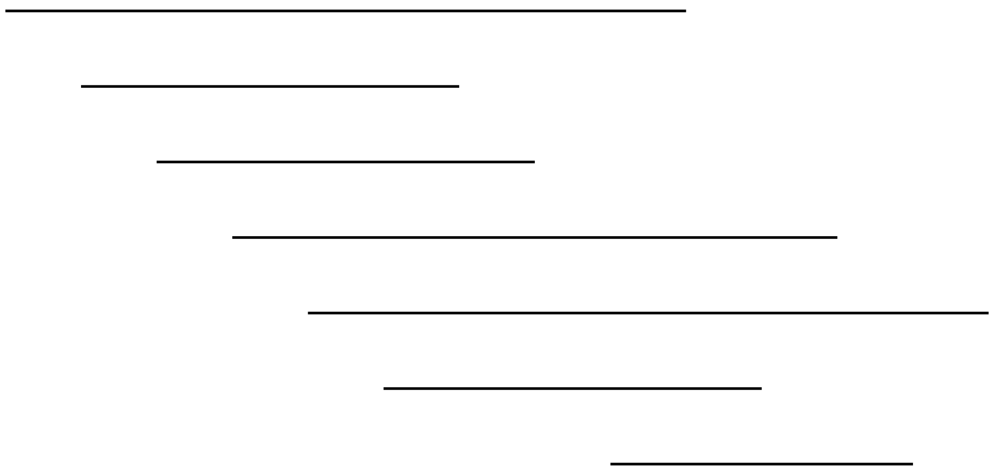


Figure 7: Another interval representation of C5

Neither Vertex + Edge or Edge + Vertex are contained in figure 7.

5 R free classes that are perfect

5.1 Bridge

See 'A Note on Online Colouring Problems in Overlap Graphs and Their Complements' by Marc Demange and Martin Olsen for proof.[2]

5.2 Left and Right Cannons

Proof again by Marc Demange and Martin Olsen
Consider first-fit and the first time colour t is used for an interval I. Then $t-1$ intervals coloured $1, \dots t-1$, before I in left to right order, overlap I. It constitutes a clique.

5.3 Contained Edge

Proof again by Marc Demange and Martin Olsen
Maintain a first class of intervals such that none are contained inside another, this can be coloured perfectly as it is equivalent to an interval graph. For every interval contained inside another colour contained interval the same colour as the containing interval. This will be a valid colouring as no two intervals contained in the same interval will overlap. As no additional colours will be used for this colouring, the colouring will also be perfect.

5.4 Left and Right Tank

In order to prove that excluding the Left Tank will result in a perfect overlap graph, it suffices to show that every odd cycle of size 5 or greater contains a Left Tank as well as the complement of every odd cycle of size

5 or greater contains a Left Tank. This is due to the strong perfect graph theorem which states that a graph is perfect if and only if an odd cycle of size 5 or greater is not an induced subgraph of either the graph or it's complement.

5.4.1 Proof any odd cycle of size 5 or greater has a left tank

There exists a leftmost interval, call this 1

11

Intervals 2 and n both overlap 1, they must both intersect at the rightmost point of interval 1. Since they intersect and do not overlap, one must be contained inside the other. Assume n is contained in 2, a similar argument applies if 2 is contained in n.

12n1n2

Interval 3, may either be contained inside 1, or to the right of 1. If it is contained inside 1, the configuration is as it must overlap 2 and cannot overlap n.

1323n1n2

Intervals 2,3 and n form a left tank.

In the second case where it is to the right of 1 the configuration formed is

12n1n323

An induction to show that n-1 is to the right of 1.

If interval k is entirely to the right of 1, Interval k+1 must at some point intersect interval k, as they overlap. So, interval k+1 has a point to the right of 1, interval k+1 cannot have a point left of 1, so, if interval k+1 does not overlap interval 1, interval k+1 must be entirely to the right of 1.

Base case

Interval 3 is entirely to the right of 1. Since the induction is valid for k+1 does not equal 2 or n. The property holds for n-1.

Since interval n-1 is to the right of 1, and does not overlap 2, and does overlap 1, we are left with the configuration

12n1(n-1)n(n-1)2.

Intervals 1,2 and n-1 form a left tank.

5.4.2 Proof any odd anti cycle of size 7 or greater has a left tank

There exists a leftmost interval, call this 1

11

Intervals 2 and n, overlap each other, so must both either be contained inside 1, or to the right of 1. Case contained

12n2n1

Interval 3 cannot be contained inside 2, as it must overlap 1, similarly it cannot be to the left of 2, as it must overlap n, It can also not contain 2, as in order to overlap 1, it would also need to contain n, so it must lie to the right.

12n23n13

Interval 4 cannot be either to the right of or contained with 3, as it must overlap 2. it cannot be to the left of 3 as it must overlap 1, and cannot overlap on the left. So interval 4 must contain 3, leaving us with

12n423n134

234 is a left tank. Case right

112n2n

Interval 3 must overlap 1, so must begin within 1, it must also overlap n, so must intersect 2, as it has points both to the left of and intersecting 2, but does not overlap, it must contain 2. Leaving us with

1312n23n

Intervals 123 form a left tank.

6 Discussion and conclusion

We have successfully categorised interval configurations of size 3, by whether or not excluding them from an interval system results in a perfect overlap graph. For cases where excluding them does not result in a perfect graph, an example of a non perfect graph has been provided. For cases where excluding them does result in a perfect graph, a proof has been provided. For most configurations whose exclusion results in a perfect overlap graph, a perfect colouring is known for the online case, however for the Left and Right Tanks, this is still unknown. The next step would be to determine if a perfect colouring exists for the Left and Right Tank is possible in the online case, either by providing an algorithm to colour it, or by forcing a non perfect colouring. If it were found to have a perfect colouring in the online case, it could lead to a reduction in the upper bound of online colouring of Overlap Graphs with potential applications in stacking problems.

References

[1] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, "The strong perfect graph theorem," *Annals of mathematics, ISSN 0003-486X, Vol. 164, Nº 1, 2006, pags. 51-229*, vol. 164, 01 2003.

[2] M. Demange and M. Olsen, *A Note on Online Colouring Problems in Overlap Graphs and Their Complements*, pp. 144–155. 01 2018.