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# The Price of Anarchy, the Price of Stability, and the Price of Communication in Interacting Intensive Care Units

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# 1 Prelude

## 1.1 Acknowledgements

I would like to thank my supervisor, Dr Mark Fackrell, for the commitment he has shown towards my project. Mark's guidance provided a valuable insight into how research is conducted and presented. I always learnt something new during our regular discussions. Thank you to the AMSI Vacation Research Scholarships team for their continual support. To the coordinators of the Vacation Scholarships program at the University of Melbourne, your efforts to organise lunchtime events for scholars and to ensure that everyone gained the most out of the program as possible despite the challenges imposed by COVID-19 is much appreciated.

## 1.2 Abstract

Maximising patient throughput in a system of multiple interacting intensive care units (ICUs) is important for an efficient healthcare system. The price of stability (PoS) and the price of anarchy (PoA) are measures of how inefficient the system is if interacting ICUs do and do not communicate, respectively. The price of communication (PoC) is the ratio of PoA to PoS. Considering both non-cooperative and semi-cooperative frameworks, this work models the interaction between multiple ICUs as a normal form game, where each ICU chooses a bed occupancy threshold. Diversion of patient arrivals to another ICU becomes possible on or above this threshold. For each configuration of thresholds, we model the system of interacting ICUs as a continuous-time Markov chain (CTMC). By establishing the stationary distribution of the CTMC, we then calculate the throughput and bed utilisation rate for each ICU. A bed utilisation target helps to align the interests of individual ICUs and those of society. The PoA, PoS and PoC are calculated for a range of demand levels and utilisation targets to explore the optimal target that aligns these interests. By performing multiple experiments with different parameter sets, the merits of having more ICUs in the system as well as the value of communication and cooperation is investigated.

## 1.3 Statement of Authorship

Many of the ideas, models and figures for this project were generalised from those in the literature, particularly from the paper by Knight *et al* (2017). My supervisor, Dr Mark Fackrell, proposed the original project and prepared some initial code which reproduced the results for two interacting ICUs in the paper by Knight *et al* (2017). This existing initial code was very useful so that I could write code to generalise the findings of Knight *et al* (2017) to a system of any number of interacting ICUs. Many of the help guides available on the MATLAB website and forums enabled me to understand and implement some of the functions in MATLAB. Forums and help guides for presenting work in LaTeXwere also of great assistance. Unless otherwise indicated, I produced the models, results and findings for the system of $N$ interacting ICUs in this report, with guidance from Dr Mark Fackrell.

# 2 Introduction

Ensuring that the maximum number of patients can be accommodated across all hospitals is important for an efficient healthcare system. This becomes pivotal for people who are critically ill or in a condition which is unstable. According to Knight *et al* (2017), intensive care units (ICUs) are special wards in hospitals which provide intensive treatment and monitoring for such patients.

ICUs rarely operate in isolation to other ICUs in the healthcare system. If the bed occupancy is high in one ICU at a given time, it can sometimes be possible to divert these patients to an ICU that has a lower bed occupancy. This helps to reduce the number of patients being turned away from the system or having to wait

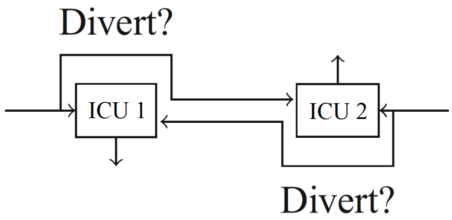



Figure 2.1: Diagrammatic representation of ICU interaction through patient diversion for a system of 2 ICUs. Patients arrive at one of the ICUs, spend some time in that ICU, and then leave. But in particular circumstances, the ICU can divert patient arrivals to the other ICU. (Adapted from Knight *et al* (2017).)

until a bed becomes available at the ICU they originally present to. A diagrammatic representation of this is shown in Figure 2.1.

Knight *et al* (2017) studied in detail the interaction between two ICUs. The interaction was studied in a game theoretic setting, where each ICU chooses a diversion (or bed occupancy) threshold. It is on or above this threshold that the ICU declares being of a high occupancy status and diversion of patient arrivals to the other ICU becomes possible.

For every possible configuration of thresholds (or strategy profile), Knight *et al* (2017) implemented a continuous time Markov chain to establish the long term distribution of bed occupancies at each ICU in the system. This allowed the throughput (or number of patients passing through the system) and utilisation rate for each ICU to be determined for each configuration of thresholds. A useful stochastic modelling reference is Karlin and Taylor (1998). Two useful game theory references are Morris (1994) and Maschler *et al* (2013).

To align the interests of individual ICUs and those of society, a bed utilisation target can be helpful. Knight *et al* (2017) described an optimal target to be one which is high enough to align these interests, but low enough to have beds available for the care of emergency patient arrivals. The best response of an ICU is found at the diversion threshold which has its bed utilisation rate as close as possible to the target.

This project aims to extend the work of Knight *et al* (2017) to investigate the interaction between multiple ICUs, possibly all in close proximity to one another to care for patients in a particular geographic region. We implement a more general game theoretic setting to that developed for a system of two ICUs. We consider intersections of best responses for the system of multiple ICUs and use throughput calculations to define three prices - the price of anarchy (PoA), the price of stability (PoS) and the price of communication (PoC). The definitions of these three prices are based on those implemented by Knight *et al* (2017), Anshelevich *et al* (2004) and Li (2017), respectively. The PoS and PoA are measures of how inefficient the system is if interacting ICUs do and do not communicate, respectively. The PoC is the ratio of PoA to PoS.

We explore two different models for the system of multiple interacting ICUs - *strict diversion* and *soft diversion*, which were first introduced by Knight *et al* (2017). The strict diversion model is non-cooperative in the sense that when all ICUs are of a high occupancy status, patients are lost from the system of interacting ICUs. This model allows for more beds to be reserved for emergencies, but often leads to increased uncoordinated behaviour. In reality, patients are not lost from the healthcare system. They just have to wait in the recovery room or emergency department in one of the hospitals until a bed becomes available in one of the ICUs.

The soft diversion model is considered to be semi-cooperative because when all ICUs are of a high occupancy status, each ICU still has to admit the patients that turn up to that ICU, until it has reached its full capacity. The ICUs are working less selfishly in this model, but each ICU still sets its threshold independently of the other ICUs. Numerous sets of parameters are considered for both models in this project.

For each set of parameters in these models, the PoA, PoS and PoC are calculated for a range of patient demand levels and utilisation targets. From these results, the optimal target that aligns the interests of the individual ICUs and those of society is established. Considering all such demand levels and utilisation targets, we determine the maximum value of PoA, PoS and PoC for each set of parameters. This leads to an investigation of the merits of having more ICUs in the system as well as the value of communication and cooperation for the interacting ICUs, in our more generalised setting.

# 3 Game Theoretic Models

We consider a system of $N$ interacting intensive care units (ICUs). Let $n$ refer to ICU $n$, where $n \in \{1, 2, ..., N\}$. Each ICU has a bed capacity $c_n$, patient arrivals which follow a Poisson process with rate $\lambda_n$, and patient length of stay times exponentially distributed with mean $1/\mu_n$.

Each ICU chooses a diversion threshold $K_n$, where $0 \le K_n \le c_n$ and $K_n \in \mathbb{Z}$. ICU $n$ can set its threshold at any number of beds, up to its full capacity $c_n$. Let $v_n(\tau)$ be the number of patients in ICU $n$ at time $\tau$, such that $0 \le v_n(\tau) \le c_n$ and $v_n(\tau) \in \mathbb{Z}$ for all $\tau$. For every possible configuration of diversion thresholds $(K_1, K_2, ..., K_N)$, a continuous-time Markov chain (CTMC) can be set-up. The state space of the CTMC is the same for all configurations of thresholds and is given by:

$$\begin{aligned} S &= S(c_1, c_2, ..., c_N) \\ &= \{(v_1(\tau), v_2(\tau), ..., v_N(\tau)) \in \mathbb{Z}^N \mid 0 \le v_1(\tau) \le c_1, 0 \le v_2(\tau) \le c_2, ..., 0 \le v_N(\tau) \le c_N\}. \end{aligned}$$

In total, the number of states is $C = \prod_{n=1}^{N}(c_n + 1)$. We can view the CTMC for each configuration of thresholds as being $N$-dimensional because each state in the CTMC is represented as an $N$-tuple. The $n$-th component of the $N$-tuple is the number of occupied beds in ICU $n$.

For each CTMC, we define a stochastic transition rate matrix (or an infinitesimal generator matrix), denoted by $Q = Q(c_1, c_2, ..., c_N)$. This matrix as well as all other calculations and plots for the PoA, PoS and PoC (that are introduced later) were created in MATLAB. The code can be accessed here.

To understand the entries of the stochastic transition rate matrix, we must first categorise the bed occupancy states of each individual ICU below, on and above the chosen diversion threshold for that ICU. If $v_n(\tau) < K_n$, then ICU $n$ is of a *low occupancy* status at time $\tau$, which we denote by $l$. If $v_n(\tau) \ge K_n$, then ICU $n$ is of a *high occupancy* status at time $\tau$, which we denote by $h$.

Let $r_n(\tau)$ be the status of ICU $n$ at time $\tau$, where $r_n(\tau) \in \{l, h\}$ is a binary variable. We let $\mathbf{r}(\tau) = (r_1(\tau), r_2(\tau), ..., r_N(\tau))$ represent the status of the $N$ ICUs at time $\tau$. At time $\tau$, there are $2^N$ possibilities for $\mathbf{r}(\tau)$. For example, if we had a system of $N = 5$ interacting ICUs, a possible status vector at an arbitrary time $\tau$ could be $\mathbf{r}(\tau) = (h, h, l, l, l)$. This would mean at time $\tau$, ICUs 1 and 2 would be of a high occupancy status, whilst ICUs 3, 4 and 5 would be of a low occupancy status.

Given a status vector $\mathbf{r}(\tau)$, we define $\lambda_n^{\mathbf{r}(\tau)}$ to be the arrival rate of patients that ICU $n$ admits to their ward. Note that this is different to $\lambda_n$, which is the arrival rate of patients to ICU $n$. Since the ICUs are interacting and patients can be diverted or rejected under certain circumstances, not all patients that arrive to ICU $n$ will be admitted to ICU $n$. A diagrammatic representation of $\lambda_n^{\mathbf{r}(\tau)}$ for the case where the system consists of 2 interacting ICUs is shown in Figure 3.1. In this example, $n \in \{1, 2\}$.

At a given time $\tau$, let $L(\tau)$ $(H(\tau))$ be the set of ICUs which are of a low (high) occupancy status. Since each ICU can only have a status of either low or high occupancy at time $\tau$, $L(\tau)$ and $H(\tau)$ form a partition of the set of all $N$ ICUs in the system. This means $L(\tau) \cup H(\tau) = \{1, 2, ..., N\}$ and $L(\tau) \cap H(\tau) = \emptyset$.

For a specific configuration of thresholds $(K_1, K_2, ..., K_N)$, we can now define the non-diagonal entries $q_{i,j}$ (for $j \ne i$) of the stochastic transition rate matrix $Q = Q(c_1, c_2, ..., c_N)$. Let $i$ represent the current state of the CTMC at some time $\tau$. Suppose that the CTMC transitions into a new state after time $\tau$. Let $j$ represent this new state, which is reached by time $\tau + \epsilon$, where $\epsilon \in \mathbb{R}$ is small and $j \ne i$.

In the current state $i$, the number of patients in each ICU is given by $(v_1(\tau), v_2(\tau), ..., v_N(\tau))$. This means $\mathbf{r}(\tau) = (r_1(\tau), r_2(\tau), ..., r_N(\tau))$ is the status vector in the current state $i$ of the CTMC. In state $j$ which is being transitioned into, the number of patients in each ICU is given by $(v_1(\tau + \epsilon), v_2(\tau + \epsilon), ..., v_N(\tau + \epsilon))$. If we let $R$ be the set of all possible $\mathbf{r}(\tau)$, then the transition rates (non-diagonal entries of $Q$) are:

$$q_{i,j} = q_{(v_1(\tau),v_2(\tau),...,v_N(\tau)),(v_1(\tau+\epsilon),v_2(\tau+\epsilon),...,v_N(\tau+\epsilon))}$$

$$= \begin{cases} v_n(\tau)\mu_n, & \text{if } (v_1(\tau), v_2(\tau), ..., v_N(\tau)) - (v_1(\tau+\epsilon), v_2(\tau+\epsilon), ..., v_N(\tau+\epsilon)) = (0, ..., 0, \overbrace{1}^{n}, 0, ..., 0), \\ & \quad \forall n \in \{1, 2, ..., N\} \\ \lambda_n^{\mathbf{r}(\tau)}, & \text{if } (v_1(\tau), v_2(\tau), ..., v_N(\tau)) - (v_1(\tau+\epsilon), v_2(\tau+\epsilon), ..., v_N(\tau+\epsilon)) = (0, ..., 0, \overbrace{-1}^{n}, 0, ..., 0), \\ & \quad v_y(\tau) < K_y \;\; \forall y \in L(\tau), \quad v_z(\tau) \geq K_z \;\; \forall z \in H(\tau), \quad \forall n \in \{1, 2, ..., N\}, \quad \forall \mathbf{r}(\tau) \in R \\ 0, & \text{otherwise} \end{cases}.$$

The diagonal entries $q_{i,i}$ of the matrix $Q = Q(c_1, c_2, ..., c_N)$ are given by $q_{i,i} = -\sum_{j \neq i} q_{i,j}$ for all states $i$ in the CTMC. This completes the definition of the stochastic transition rate matrix.

Starting from an arbitrary time $\tau$, the length of stay for the first patient that leaves ICU $n$ is the minimum of the length of stays of all $v_n(\tau)$ patients currently in ICU $n$. Patient length of stay times are exponentially distributed with rate $\mu_n$ (or mean $1/\mu_n$). This means that the length of stay for the first patient that leaves ICU $n$ is also exponentially distributed, but with rate $v_n(\tau)\mu_n$.

We can interpret the CTMC for each configuration of thresholds as a type of birth and death process. A 'death' ('birth') occurs when the bed occupancy decreases (increases) by 1 in one of the ICUs, which means a patient has been discharged (admitted) from (to) that ICU. The rate for a 'death' in ICU $n$ is $v_n(\tau)\mu_n$. The rate for a 'birth' in ICU $n$ is $\lambda_n^{\mathbf{r}(\tau)}$, which depends partially on the status vector $\mathbf{r}(\tau)$.

The relevant $\mathbf{r}(\tau)$ is found by establishing $L(\tau)$ and $H(\tau)$ - the set of ICUs which are operating with a low and high occupancy status in the current state of the CTMC, respectively. This is done using the method introduced earlier - consider the configuration of thresholds $(K_1, K_2, ..., K_N)$ as well as $(v_1(\tau), v_2(\tau), ..., v_N(\tau))$.

Once $\mathbf{r}(\tau)$ is known, the 'birth' rate $\lambda_n^{\mathbf{r}(\tau)}$ can be calculated using two different models, which will be discussed next. Both models are generalisations of those from Knight *et al* (2017), which were influenced by discussions with a local health board that oversees the ICUs in two real hospitals.

## 3.1 Strict Diversion (non-cooperative framework)

In the strict diversion model, patients continue to be accepted to the system of $N$ interacting ICUs until all ICUs are of a high occupancy status. When all ICUs are of a low occupancy status, all ICUs admit their own patients. Patient diversion arises when some ICUs have low occupancy and some have high occupancy. For all $n \in \{1, 2, ..., N\}$ and $\mathbf{r}(\tau) \in R$, the arrival rates (at some time $\tau$) with this framework are:

$$\lambda_n^{\mathbf{r}(\tau)} = \begin{cases} \lambda_n + \dfrac{1}{|L(\tau)|} \displaystyle\sum_{d \in H(\tau)} \lambda_d, & \text{if } r_n(\tau) = l \\ 0, & \text{if } r_n(\tau) = h \end{cases}.$$

If a given ICU has a high occupancy status, it will not accept any more patients to its ward until its status changes to low occupancy. Once its status is low occupancy, the ICU will accept its own patients as well as a fraction of patients from each of the ICUs that have a high occupancy status (if there are any). Note that if $r_n(\tau) = l$ for ICU $n$, then $n \notin H(\tau)$. In this model, we assume that each of the ICUs with a low occupancy status will share a uniform proportion of patient arrivals from the high occupancy status ICUs.

Consider again the example where $N = 5$ and, for some time $\tau$, $\mathbf{r}(\tau) = (h, h, l, l, l)$. In this example, not all ICUs have a high occupancy status. Therefore, all patients arriving to the system of ICUs will be admitted to one of the 5 ICUs. This means that $\sum_{n=1}^{5} \lambda_n^{(h,h,l,l,l)} = \sum_{n=1}^{5} \lambda_n$ in this example, since:

$$\lambda_1^{(h,h,l,l,l)} = \lambda_2^{(h,h,l,l,l)} = 0, \quad \lambda_3^{(h,h,l,l,l)} = \lambda_3 + \frac{\lambda_1 + \lambda_2}{3}, \quad \lambda_4^{(h,h,l,l,l)} = \lambda_4 + \frac{\lambda_1 + \lambda_2}{3}, \quad \lambda_5^{(h,h,l,l,l)} = \lambda_5 + \frac{\lambda_1 + \lambda_2}{3}.$$

The strict diversion model implies that once ICU $n$ chooses its diversion threshold $K_n$, and the first $K_n$ beds are occupied, the remaining $c_n - K_n$ beds will never be occupied. In reality, this means that the system of ICUs is prepared for emergency patient arrivals. (An extension of this model could consider separate arrival rates for emergency patients and elective patients.) Leaving beds empty like this may be viewed as non-cooperative. However, this model will be a useful benchmark for the other model we propose next.

### 3.2 Soft Diversion (semi-cooperative framework)

The soft diversion model is similar to the set-up for strict diversion, except that ICUs must accept their own patients when all ICUs are of a high occupancy status. This can be viewed as a form of cooperation. For all $n \in \{1, 2, ..., N\}$ and $\mathbf{r}(\tau) \in R$, the arrival rates (at some time $\tau$) with this framework are:

$$\lambda_n^{\mathbf{r}(\tau)} = \begin{cases} \lambda_n + \dfrac{1}{|L(\tau)|} \displaystyle\sum_{d \in H(\tau)} \lambda_d, & \text{if } r_n(\tau) = l \\ 0, & \text{if } r_n(\tau) = h, H(\tau) \neq \{1, 2, ..., N\} \\ \lambda_n, & \text{if } r_n(\tau) = h, H(\tau) = \{1, 2, ..., N\} \end{cases}.$$

It should be acknowledged that it is still possible for patients to be lost from the system in the soft diversion framework, which is why the model is only semi-cooperative. This occurs when all ICUs are of a high occupancy status, and when the ICU that the patient arrives to has already reached its full bed capacity. In this situation, patients cannot be diverted to another ICU, even if there exists an ICU that has not yet reached its full bed capacity. This is because when all ICUs have a high occupancy status, each ICU will only accept its own patients, irrespective of the actual number of beds that are occupied.

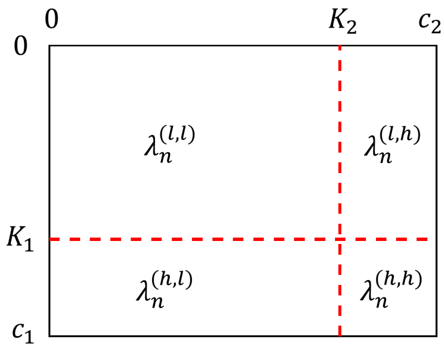



Figure 3.1: General arrival rates at each region for a system of 2 ICUs and given thresholds $K_1$ and $K_2$. (Adapted from Knight *et al* (2017).)

### 3.3 Calculation of the Price of Anarchy, Stability, and Communication

In this section, we work towards defining the three prices that are used to analyse the models in this project. The first step in this direction is to find the stationary (or long term) distribution of the bed occupancy at each ICU, for each configuration of thresholds. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_C)$ be the stationary distribution for a specific CTMC. As defined earlier, $C$ is the number of states in the CTMC. Assume that the components of $\boldsymbol{\pi}$ are listed in the same order as the rows/columns of the stochastic transition rate matrix $Q = Q(c_1, c_2, ..., c_N)$.

The ordering of the CTMC states is arbitrary, but we will list first the bed occupancies for ICU 1 (from 0 to $c_1$), followed by those for ICU 2, and so on up to ICU $N$. To calculate $\boldsymbol{\pi}$ for a specific CTMC, we solve the system of $C$ linear equations $\boldsymbol{\pi} Q = \mathbf{0}$, where $\mathbf{0}$ is a row vector (length $C$) of zeros. These $C$ equations are linearly dependent. Therefore, to find a unique solution for $\boldsymbol{\pi}$, we replace the final equation in this system with the normalising equation $\pi_1 + \pi_2 + ... + \pi_C = 1$.

The final system of $C$ linear equations is of the form $\boldsymbol{\pi}A = \mathbf{b}$, where $A$ is a $C \times C$ matrix, and $\mathbf{b}$ is a row vector (length $C$). By inverting $A$, we use MATLAB to calculate the stationary distribution $\boldsymbol{\pi} = \mathbf{b}A^{-1}$.

Let $\boldsymbol{\omega}_n = (\omega_{0,n}, \omega_{1,n}, ..., \omega_{c_n,n})$ be the stationary distribution of bed occupancies at ICU $n$. For $m_n \in \{0, 1, ..., c_n\}$, we can find the component $\omega_{m_n,n}$ of $\boldsymbol{\omega}_n$ by finding the sum of all $\pi_i$ for which $i$ corresponds to a state of the CTMC where the bed occupancy in ICU $n$ is equal to $m_n$.

This leads us to find the expected number of occupied beds in each ICU $n$, for a given configuration of thresholds. From this, we can define the utilisation rate ($U_n$) and throughput ($T_n$) in ICU $n$:

$$U_n = \frac{\mathbb{E}(\text{occupancy in ICU } n)}{c_n} = \frac{1}{c_n} \sum_{m_n=0}^{c_n} m_n \omega_{m_n,n}, \quad T_n = \mu_n \times \mathbb{E}(\text{occupancy in ICU } n) = \mu_n \sum_{m_n=0}^{c_n} m_n \omega_{m_n,n}.$$

These two related formulas will each be useful in their own right. The utilisation rate is needed when considering best responses. Note that $0 \leq U_n \leq 1$ always holds. Each ICU calculates its optimal threshold (best response) to the thresholds of the other ICUs. Consider ICU $n$ and suppose that this ICU knows $K_1, K_2, ..., K_{n-1}, K_{n+1}, ..., K_N$ (the thresholds chosen by the other $N-1$ ICUs). Based on the thresholds of the other ICUs, the best response for ICU $n$ is found by solving the optimisation problem:

$$\begin{aligned} &\min\ (U_n - t)^2 \\ &\text{such that} \quad 0 \leq K_n \leq c_n, \quad K_n \in \mathbb{Z}. \end{aligned}$$

To solve this problem, we calculate $U_n$ for each possible threshold $K_n$ available to ICU $n$, whilst holding the thresholds of the other $N-1$ ICUs fixed at the values stated above. We let $t$ be the bed utilisation target, set by central control which oversees all $N$ ICUs. $t$ is the same for all ICUs and each ICU maximises its utility when its bed utilisation rate $U_n$ is as close as possible to this bed utilisation target.

In the case where there are multiple thresholds that minimise $(U_n - t)^2$, it is assumed that the best reply for ICU $n$ will be the lowest such threshold $K_n$ that minimises $(U_n - t)^2$. An ICU would prefer to set its threshold lower (if its utility will be the same) as this allows the ICU to divert patients even when it is less busy. If an ICU sets its diversion threshold lower, this increases the workload for some of the other ICUs. The convention adopted here is also consistent with the definition of the Price of Anarchy (PoA), which can be viewed as measuring the maximum possible extent of uncoordinated behaviour.

Note that the only strategies available to ICU $n$ are pure strategies. This means that once ICU $n$ chooses its diversion threshold, it will not change it (at least for a fixed or agreed period of time). According to an analysis conducted by Knight *et al* (2017), interacting ICUs generally keep the same diversion thresholds in the long term, so this model is reasonable. Of course, in reality, ICU $n$ would not know the thresholds of the other ICUs when it makes its decision. Hence, the above optimisation problem needs to be solved for all possible configurations of $(K_1, K_2, ..., K_{n-1}, K_{n+1}, ..., K_N)$. We take a brute force approach here.

An example of this process for a system of 2 interacting ICUs is shown in Figure 3.2. In this example, if ICU 2 chooses to set threshold at 10 beds, the best response of ICU 1 is to set its threshold at 6 beds (a diamond). Recall that on an ICUs threshold, the status of the ICU changes from low to high occupancy. Similarly, if ICU 1 chooses to set its threshold at 6 beds, the best response of ICU 2 is to set its threshold at 8 beds (a cross). A similar explanation also applies to the other thresholds in the diagram.

Once all $N$ ICUs have solved their relevant optimisation problems, each ICU will have a list of their best response to each of the configurations of thresholds of the other ICUs. A Nash equilibrium is found if all $N$ ICUs have a best response on their list for which the $N$ thresholds all match. We refer to this as an intersection of best responses. For the example in Figure 3.2, a Nash equilibrium occurs at the blue circle, where ICU 1 and 2 set their thresholds at 5 and 8 beds, respectively.

At a Nash equilibrium, unilateral deviation does not benefit any ICU. This means that if ICU 1 in the diagram decides to change its threshold to 4 or 6 beds, say, whilst ICU 2 keeps its threshold at 8 beds, then

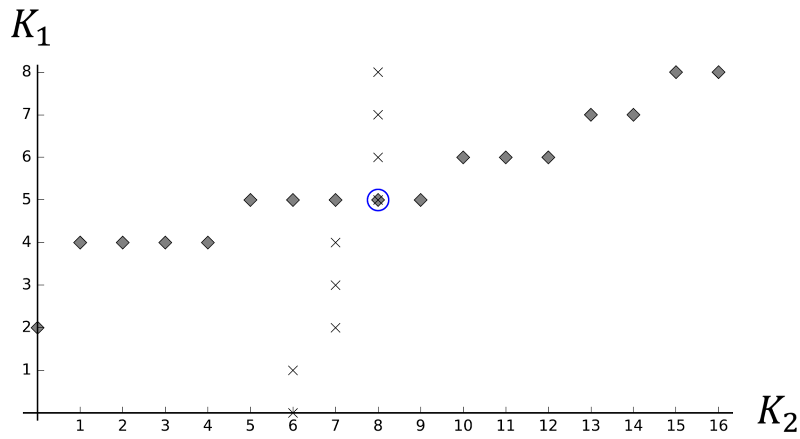



Figure 3.2: Example of best responses for a system of 2 ICUs. ICU 1 (2) can set their diversion threshold anywhere between 0 beds and their bed capacity of 8 (16) beds. Diamonds (crosses) are the best responses for ICU 1 (2). (Adapted from Knight *et al* (2017).)

ICU 1 cannot do any better (that is, achieving a bed utilisation closer to the target $t$). We will now consider the throughputs to define the three prices in this project.

For every possible configuration of thresholds $(K_1, K_2, ..., K_N)$, we determine the sum of throughputs $T_1 + T_2 + ... + T_N$, using the formula introduced earlier. Let $T^*$ be the maximum sum of throughputs across all configurations of thresholds. Since the sum of throughputs is a measure of how many patients pass through the system of these $N$ interacting ICUs, it is in society's best interests for $T^*$ to be as large as possible.

$T^*$ is independent of the bed utilisation target $t$. Intuitively, one may expect $T^*$ to be obtained for $(K_1, K_2, ..., K_N) = (c_1, c_2, ..., c_N)$. However, Knight *et al* (2017) advised that this is not always the case, according to numerical experiments.

Let $T^+$ ($T^-$) be the largest (smallest) sum of throughputs when considering only threshold configurations which correspond to Nash equilibria. In some cases, there can be more than one Nash equilibrium. But if there is only one, then $T^+ = T^-$. Using this notation, we can define the Price of Anarchy (PoA), the Price of Stability (PoS), and the Price of Communication (PoC):

$$\text{PoA} = \frac{T^*}{T^-}, \qquad \text{PoS} = \frac{T^*}{T^+}, \qquad \text{PoC} = \frac{T^+}{T^-}.$$

The PoA (PoS) is the ratio of the socially optimum sum of throughputs to that sum for the 'worst' ('best') Nash equilibrium. The PoA is relevant when communication is not possible, since this price considers the worst case scenario. The PoS is relevant when the $N$ ICUs can communicate by coming together to determine what the 'best' Nash equilibrium would be. The PoA and PoS have similar scales, and so comparisons between them are meaningful. The PoC scale is quite different due to its definition, and so is usually analysed separately.

A single ICU cannot do any better by deviating from a Nash equilibrium. Hence, an individual ICU should be indifferent between each of the Nash equilibria, if there is more than one. The PoC is the ratio of the sum of throughputs for the 'best' Nash equilibrium to that sum for the 'worst' Nash equilibrium. The PoC can equivalently be defined as the ratio PoA/PoS. When there is a single Nash equilibrium, PoC = 1. In the more interesting cases where there are multiple Nash equilibria, the PoC could be greater than 1.

## 4 Results

To analyse the strict and soft diversion models, we considered a variety of different sets of parameters. The parameters $N$, $\mathbf{c} = (c_1, c_2, ..., c_N)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_N)$ introduced in the previous section distinguish each parameter set. We produced separate 'checkerboard' plots of the PoA, PoS and PoC for each parameter set, for both the strict diversion and soft diversion models.

In producing these plots, we considered the variables $t$ and $x$. $t$ is the bed utilisation target and is indicated on the vertical axis. We increased $t$ in steps of 0.1 from 0 to 1, which means $t \in \{0, 0.1, 0.2, ..., 0.9, 1\}$. $x$ refers to a modified arrival rate and is indicated on the horizontal axis. For all $n \in \{1, 2, ..., N\}$, we consider the original arrival rate $\lambda_n$ of ICU $n$ and transform this value into a modified arrival rate given by $\lambda_n(1 + x)$. We can interpret $x$ as the relative increase in demand, which is applied to all $N$ ICUs. We increased $x$ in steps of 0.3 from $-0.9$ to 2.1, which means $x \in \{-0.9, -0.6, -0.3, ..., 1.8, 2.1\}$.

Note that $x = 0$ is included in this set and corresponds to the 'original' arrival rates. In total, we considered 11 values for $t$ and 11 values for $x$. Hence, there are 121 'checkerboard' squares when we consider every possible $t$ and $x$ combination. For each combination, the PoA, PoS and PoC were calculated for both diversion models. Plots for an example set of parameters are shown in Figure 4.1, where there are $N = 4$ ICUs.

In Figure 4.1, consider the squares corresponding to $t = 0.2$ and $x = 0$ in the strict diversion model. For this particular combination of $x$ and $t$, two Nash equilibria exist. The 'best' Nash equilibrium occurs for diversion thresholds $(K_1, K_2, K_3, K_4) = (1, 1, 1, 1)$, and the sum of throughputs is $T^+ = 0.7342$. The 'worst' Nash equilibrium occurs for diversion thresholds $(K_1, K_2, K_3, K_4) = (0, 0, 1, 1)$, and the sum of throughputs is $T^- = 0.3572$. The socially optimum sum of throughputs is $T^* = 1.3315$, and this occurs for $(K_1, K_2, K_3, K_4) = (2, 2, 3, 3)$, which happens to also be the bed capacities of the 4 ICUs in this particular example.

From these calculations, we can determine the PoA, PoS and PoC for $t = 0.2$ and $x = 0$ in the strict diversion model, for the parameter set in Figure 4.1. We find that PoA $= T^*/T^- = 1.3315/0.3572 = 3.7277$, PoS $= T^*/T^+ = 1.3315/0.7342 = 1.8136$, and PoC $= T^+/T^- = 0.7342/0.3572 = 2.0554$. By definition, we always have PoA $\geq 1$, PoS $\geq 1$, and PoC $\geq 1$. The yellow square which corresponds to PoC $= 2.0554$ in Figure 4.1 is the only combination of $x$ and $t$ (of those tested) which gives a PoC value greater than 1, in the strict diversion model. This means the remaining navy blue squares in the PoC plot refer to $x$ and $t$ values for which there is only one Nash equilibrium, unless two or more Nash equilibria have an identical sum of throughputs.

For each set of parameters, we can identify the 'smallest' (denoted by $s$) and 'biggest' (denoted by $b$) ICU in the system. We identify these two ICUs by ranking the $N$ ICUs according to their bed capacity such that $c_1 \leq c_2 \leq ... \leq c_N$. It follows that ICU 1 and ICU $N$ are ICUs $s$ and $b$, respectively. In the results that follow, ICU $s$ (or ICU 1) is assumed to have the lowest arrival rate and highest service rate, whilst ICU $b$ (or ICU $N$) is assumed to have the highest arrival rate and lowest service rate.

Whilst the values $\lambda_s, \lambda_b, \mu_s$ and $\mu_b$ for ICUs $s$ and $b$ used in this project were based on real data, the arrival and service rates for the remaining ICUs were derived through a procedure which we now outline. The remaining ICUs were made to have arrival and service rates between the two extremes such that $\lambda_s = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N = \lambda_b$ and $\mu_s = \mu_1 \geq \mu_2 \geq ... \geq \mu_N = \mu_b$. In our tests, we spaced the arrival and service rates uniformly between the two extreme values (when there were more than 2 ICUs). For all $n \in \{1, 2, ..., N\}$, we have:

$$\lambda_n = \frac{2}{N} \times \left\{ \lambda_s + \left[ \frac{n-1}{N-1} \times (\lambda_b - \lambda_s) \right] \right\}, \qquad \mu_n = \mu_s - \left[ \frac{n-1}{N-1} \times (\mu_s - \mu_b) \right].$$

In our results, we analysed the effect of varying the number of ICUs in the system. In order to make fair comparisons in this type of analysis, we wanted the total arrival rate in the system of ICUs to remain fixed. When there are 2 ICUs in the system, the total arrival rate is $\lambda_s + \lambda_b$. To ensure this is still the case when there are more than 2 ICUs, the arrival rates in the formula above are weighted by the factor $2/N$. Note that this is not relevant for the service rates, which is why there is no factor of $2/N$ in the formula for the service rates.

As part of their analysis for a system of 2 ICUs, Knight *et al* (2017) analysed data from two real hospitals (with one ICU in each hospital). From this data, the parameters $c_1 = 8$, $c_2 = 16$, $\lambda_1 = \lambda_s = 1.50$, $\lambda_2 = \lambda_b = 2.24$, $\mu_1 = \mu_s = 0.262$, and $\mu_2 = \mu_b = 0.198$ were obtained. The arrival and service rates both had units of 'per day'. This agrees with our logic that the bigger the ICU, the higher (lower) the arrival (service) rate should be. For

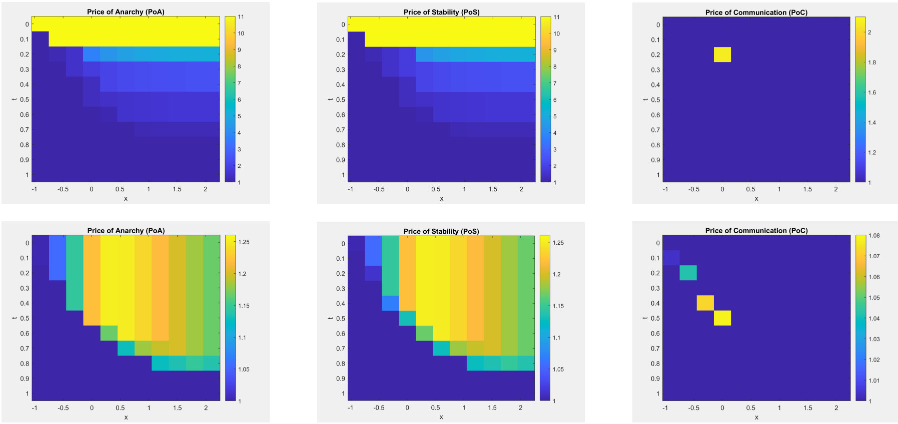



Figure 4.1: PoA, PoS and PoC plots for strict (top row) and soft (bottom row) diversion models with $N = 4$ ICUs and bed capacities $\mathbf{c} = (2, 2, 3, 3)$. Arrival and service rate (per day) parameters are $\lambda_s = 0.5625$, $\lambda_b = 0.84$, $\mu_s = 0.262$, and $\mu_b = 0.198$. $x$ refers to a modified arrival rate found by the transformation $\lambda_n \to \lambda_n(1 + x)$ for all $n \in \{1, 2, ..., N\}$. $t$ is the bed utilisation target.

our system of $N$ ICUs, we always set $\mu_s = 0.262$ (per day) and $\mu_b = 0.198$ (per day) in this project.

Whilst the numerical values of the service rates for a given number of ICUs $N$ remain fixed, the arrival rates do not. The system studied by Knight *et al* (2017) had $8 + 16 = 24$ beds in total. In the example in Figure 4.1, there are $N = 4$ ICUs, and $2 + 2 + 3 + 3 = 10$ beds in total. Since there are fewer beds in this system of 4 ICUs, we adjust the (extreme) arrival rates. We start with the values $\lambda_s = 1.50$ (per day) and $\lambda_b = 2.24$ (per day) used by Knight *et al* (2017), but scale both values down by an appropriate factor (in this case 3/8).

The final (extreme) arrival rates used to produce the plots in Figure 4.1 are $\lambda_s = (3/8) \times 1.50 = 0.5625$ (per day) and $\lambda_b = (3/8) \times 2.24 = 0.84$ (per day). A similar approach was taken for other sets of parameters.

Note that the values $\lambda_s, \lambda_b, \mu_s$ and $\mu_b$ are sufficient to derive the arrival and service rates for every ICU in the system of $N$ ICUs. For the example in Figure 4.1, using the formulae described above, we have $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.2813, 0.3275, 0.3738, 0.42)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4) = (0.262, 0.2407, 0.2193, 0.198)$. As expected, these arrival rates keep the total arrival rate fixed at $\lambda_s + \lambda_b$, since we find that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0.2813 + 0.3275 + 0.3738 + 0.42 = 0.4025 = 0.5625 + 0.84 = \lambda_s + \lambda_b$.

For some combinations of $x$ and $t$, it is possible that at one or more of the Nash equilibria, the sum of throughputs will be equal to zero. This often occurs when a Nash equilibrium corresponds to all $N$ ICUs setting their diversion thresholds at 0 beds. In situations where $T^-$ and/or $T^+$ are equal to 0, the PoA, PoS and PoC can take infinite values, if $T^*$ is non-zero. In Figure 4.1, we sometimes observe this in the strict diversion case when $t = 0$ or $t = 0.1$. We consider later the possibility where $T^*$ is also zero.

So that a reasonable plot can still be produced in these instances, we represent 'infinity' on these plots to be approximately twice the highest non-infinite value of PoA/PoS/PoC in the plot. In both the PoA and PoS plots for strict diversion in Figure 4.1, 5.2221 is the highest non-infinite value (which corresponds to $t = 0.2$ and $x = 2.1$). The numerical values of all other 'checkerboard' squares in Figure 4.1 can be accessed in tables here.

In cases where two or more of $T^*$, $T^-$ and $T^+$ are zero, it is possible that PoA, PoS or PoC can take the value 0/0. Clearly, 0/0 is undefined, and MATLAB would indicate that the value is not a number ('NaN'). In these cases, we choose to set PoA, PoS or PoC to be 1. This choice is quite natural given that we have identical sums of throughputs in the numerator and denominator of the PoA, PoS or PoC in these cases.

If the calculation of PoC is performed using the ratio PoA/PoS (instead of $T^+/T^-$), and PoA and PoS are both infinite, we would find that PoC takes the value $\infty/\infty$. In this case, we also set PoC equal to 1, since the PoA and PoS take the same value. This would usually indicate that there is a single Nash equilibrium.

From the tests we conducted, infinite values for PoA, PoS and PoC were only observed in the strict diversion model. In this model, patients are lost from the system when all ICUs have a high occupancy status. In the soft diversion model, we do not observe Nash equilibria where the sum of throughputs is equal to zero. The parameter set in Figure 4.1 is just one of the sets we considered in this project. Plots of the PoA, PoS and PoC for 20 of the other parameter sets considered are displayed in Appendix B.

### 4.1 Computational Time

For a given parameter set and a given combination of $x$ and $t$, the PoA, PoS and PoC were calculated simultaneously in MATLAB. We first include the details of the computer used for our simulations. All tests were performed on a Windows 10 Pro (PC), 2019. Processor: Intel(R) Core (TM) i5-9400F CPU @ 2.90 GHz. Installed Memory (RAM): 16.0 GB. System Type: 64-bit Operating System, x64 based processor.

To understand the time it takes to produce all 121 calculations in a single set of PoA, PoS and PoC plots, it is useful to determine the time it takes to run the calculations for a single $x$ and $t$ combination (which we refer to as a trial). If one trial takes too long, it is not worthwhile to perform calculations for all 121 trials.

For the parameter sets considered in this project, most trials took 5 minutes or less to run, with the majority taking only a few seconds. The parameter set in Figure 4.1 took the longest, with the trial $x = 0$ and $t = 1$ taking 17 minutes. As discussed later, there is an immediate alignment of interests between the individual ICUs and those of society whenever $t = 1$. We have PoA = PoS = PoC = 1 for this target. In this project, we chose not to consider parameter sets which took over 20 minutes to run a single trial. 121 such trials would take $20 \times 121 = 2420$ minutes (40.3 hours) to run, which is not viable here. We include below the computational time for the trial $x = 0$ and $t = 1$ for a few of the parameter sets we chose not to include in the project results.

- $N = 3$, $\mathbf{c} = (5, 8, 11)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$ took time 6 hours, 27 minutes.
- $N = 4$, $\mathbf{c} = (4, 5, 7, 8)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$ took time over 7 hours (exact time not determined).
- $N = 5$, $\mathbf{c} = (1, 2, 2, 2, 2)$, $\lambda_s = (3/8) \times 1.50 = 0.5625$ and $\lambda_b = (3/8) \times 2.24 = 0.84$ took time 41 minutes.
- $N = 6$, $\mathbf{c} = (1, 1, 1, 2, 2, 2)$, $\lambda_s = (3/8) \times 1.50 = 0.5625$ and $\lambda_b = (3/8) \times 2.24 = 0.84$ took time 2 hours, 23 minutes.

In general, the number of rows (or columns) in the stochastic transition rate matrix dictates the time a trial takes to run. The relationship between the number of rows and computational time is not linear (closer to exponential). The difference in time between the strict and soft diversion model calculations is negligible.

### 4.2 Analysis of Parameter Sets

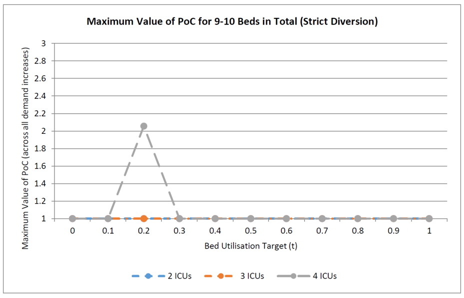
A series of line graphs were produced to analyse the key aspects of the PoA, PoS and PoC plots for each parameter set. Some graphs are displayed below, but others are included in Appendix A. The data (or the numerical values of the points) that produce these line graphs can be accessed in tables here. The findings from these line graphs can be divided into three categories - the maximum value of PoC, the maximum value of PoA (or PoS) and the optimal bed utilisation target. In some cases, the maximum is infinite when finding the maximum value of PoA, PoS or PoC. This makes it difficult to achieve a reasonable scale on the line graph.

So that the line graphs show an accurate trend, we represent any infinite values as approximately twice that of the highest non-infinite point on the graph. In the scale of the line graphs shown, these infinite points are not visible. However, it is clear from the lines in these graphs that infinite points exist. For examples, see the strict diversion model line graphs in Figures A.11 to A.14. This is only relevant for the figures themselves. In the tables that contain the data for these figures, we do not 'modify' infinite values, unless we indicate otherwise.

The maximum value of PoC is analysed in one of three ways - maximising across all $x$ and $t$ values in a given plot, maximising across all $x$ values for each value of $t$ in the plot, or maximising across all $t$ values for each value of $x$ in the plot. Some key findings (based on Figure 4.2 and Figures A.1 to A.6) are given below:

- When the PoC is greater than 1, it is higher for the strict diversion model compared to the soft diversion model. But the frequency of multiple Nash equilibria in soft diversion is much greater, which is why we argue that communication is more valuable in the soft diversion model.

- When there are more ICUs in the system, there are more instances of multiple Nash equilibria (particularly for soft diversion). The value of the PoC also seems to increase, but there are exceptions to this.
- Our PoC plots are fairly coarse. We considered only 11 values for $x$ and 11 values for $t$. It is possible that some instances of Nash equilibria could be missed, particularly when a plot shows only a single instance of multiple Nash equilibria. The values of $x$ and $t$ for which multiple Nash equilibria are observed is more or less random. However, we typically observe it for low demand and low to medium targets. It is less common to see multiple Nash equilibria for high targets, and rare to see it for high demands.
- It is difficult to establish the relationship between the bed capacity of the ICU and when we observe multiple Nash equilibria. But when the system consists of 2 ICUs, a larger bed capacity tends to lead to more instances of multiple Nash equilibria. These are mostly observed for low demands and for high targets. However, the value of the PoC is not always higher just because there is a larger bed capacity.



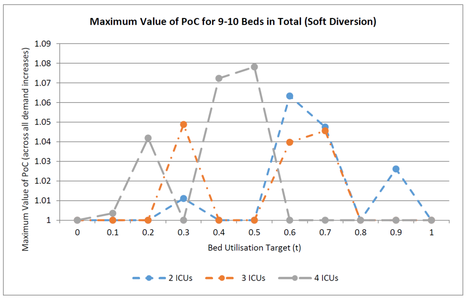




Figure 4.2: Maximum value of PoC for each bed utilisation target ($t$), with 9-10 beds in total across the entire ICU system and varying parameter $N$ (number of ICUs in the system). Arrival and service rate parameters (per day) are $\lambda_s = 0.5625$, $\lambda_b = 0.84$, $\mu_s = 0.262$, and $\mu_b = 0.198$, for all parameter sets.

The maximum value of PoA (or PoS) is analysed in a similar way to PoC, except that we only maximise over one variable at a time (rather than over $x$ and $t$ simultaneously). In the strict diversion model, for each $x$ value, we always obtain infinite values of PoA (or PoS) when $t = 0$. This is logical since the bed utilisation target is to have no beds occupied in this case. So, when maximising over all $t$ values for each $x$ in the strict diversion model, we choose to only maximise over non-infinite values of the PoA (or PoS).

We take this approach so that we can see some variation in the strict diversion model line graphs in Figure 4.3 and Figures A.7 to A.10. Some key findings (based on Figure 4.3 and Figures A.7 to A.14) are given below:

- For each demand level ($x$) in the soft diversion model, the maximum value of PoA across all $t$ values is generally lower when there are fewer ICUs in the system. The severity of the uncoordinated behaviour is not as bad with fewer ICUs. The maximum PoA for soft diversion is usually larger for the medium demand levels, which explains the ‘peak’ in Figures 4.3, A.7 and A.8. In soft diversion, patient diversion is possible when the system is not too ‘quiet’/‘busy’. When there is a fixed total number of beds across the system of ICUs, the $x$ value at which the ‘peak’ occurs does not vary by adding more ICUs. But when there are the same number of beds at each ICU, the ‘peak’ shifts to higher $x$ values when more ICUs are added. In these cases, higher system demand is needed before uncoordinated behaviour starts to emerge.
- For strict diversion, the association between number of ICUs and the maximum PoA for each $x$ value is not always as strong. But having fewer ICUs seems to slightly raise the maximum PoA for this model. The maximum PoA for strict diversion is usually larger for higher demand levels. As $x$ increases, the maximum PoA appears to approach a constant value. In Figure A.8, where each ICU has the same number of beds, we find that the maximum PoA for 2, 3 and 4 ICUs all converge to the same constant value.
- When there are medium arrival rates, the ICUs seem to make the most of their opportunity to act in a selfish manner when the framework is soft diversion. As a result, the Nash equilibrium sum of throughputs

shows a greater divergence from the socially optimum sum. In all cases though, both the PoA and PoS are much lower in the soft diversion model compared to strict diversion (where there is no cooperation). The trends for PoA and PoS are similar for a given model, as can be seen in Figures A.10 and A.14.

- For each target ($t$), the maximum value of PoA across all $x$ values is usually larger for lower targets. As $t$ increases to 1, the maximum PoA decreases until a PoA value of 1 is reached. For soft (strict) diversion, this decrease is concave (convex). The target needs to be set a lot higher in the soft diversion case before there is a noticeable drop in the maximum PoA. As $t$ approaches 0, the maximum PoA appears to approach a constant value for soft diversion, but diverges to infinity for strict diversion.
- When there is a low bed capacity across the system of ICUs, the number of possible configurations of thresholds is also low. Hence, for a fixed demand, it is common for the PoA to be the same across multiple targets $t$. For a fixed number of ICUs, lower bed capacities tend to make the maximum PoA higher in the soft diversion case (whether we maximise over $t$ values or over $x$ values). This is because ICUs become busier if beds are removed from the system. The maximum PoA 'peak' mentioned earlier for soft diversion seems to shift to lower $x$ values when there is a lower bed capacity. So, when there are fewer beds in the system, individual ICUs have less scope to behave in an uncoordinated way when demand is higher.
- In the strict diversion model, a lower bed capacity tends to result in a lower maximum PoA. This trend is mainly clear when maximising over $t$ values for each $x$. As $x$ increases, the maximum PoA appears to approach a constant value for each set of bed capacities, but this value is not the same for each set.
- Even though most of our parameter sets consist of a low number of beds (sometimes just one or two beds at each ICU), the previous observation is somewhat unexpected. However, a similar observation can be made from the plots produced by Knight *et al* (2017) for the system of 2 ICUs. Having an extra bed in an ICU means that ICU has an extra choice to set its diversion threshold at. When the ICU system is small, this can sometimes outweigh the benefits of having a greater bed capacity, and uncoordinated behaviour increases. We can refer to this as the 'small system' effect.

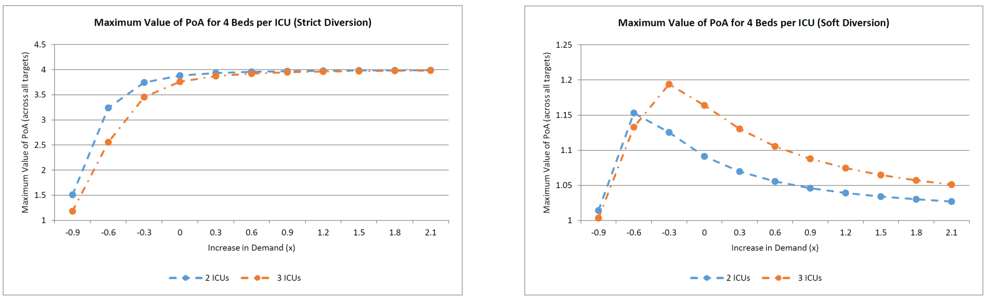
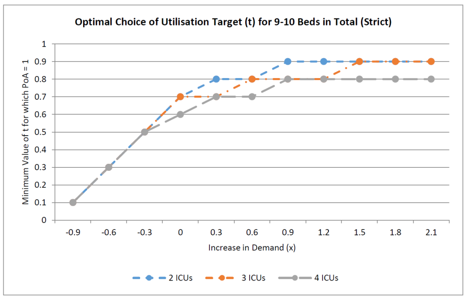



Figure 4.3: Maximum value of PoA for each increase in demand ($x$), with 4 beds at each ICU in the system and varying parameter $N$ (number of ICUs in the system). Arrival and service rate parameters (per day) are $\lambda_s = 1.50$, $\lambda_b = 2.24$, $\mu_s = 0.262$, and $\mu_b = 0.198$, for all parameter sets.

When infinity is excluded from a maximisation calculation, it can sometimes result in undesirable modifications to the raw data. In addition, sometimes the values for which we are calculating the maximum are all infinite. This would occur if we maximised over all $x$ values when $t = 0$ in the strict diversion PoA (or PoS) plots. Hence, in general, care must be taken when excluding infinity from calculations. The context in which we exclude infinity in this project is treated cautiously to retain the general trends in the data.

For each value of $x$, when communication is (is not) possible in the system of interacting ICUs, we define the optimal bed utilisation target to be the minimum value of $t$ such that the PoS (PoA) is equal to 1. Setting $t = 1$ would ensure an immediate alignment of interests, since rather than diverting patients to other ICUs, each ICU would be striving to completely fill its own ward. However, a target of $t = 1$ is not always optimal, since this leaves no room for emergency patient arrivals. The optimal target aims to align the interests of the individual ICUs and those of society and to also ensure that the ICU system is not unnecessarily overcrowded.



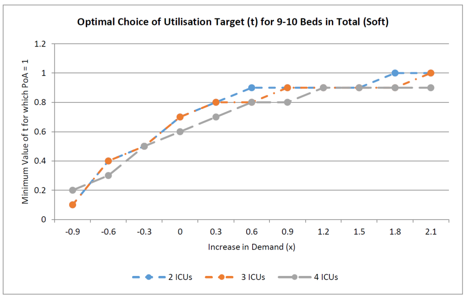




Figure 4.4: Optimal bed utilisation target for each increase in demand ($x$), with 9-10 beds in total across the entire ICU system and varying parameter $N$ (number of ICUs in the system). Arrival and service rate parameters (per day) are $\lambda_s = 0.5625$, $\lambda_b = 0.84$, $\mu_s = 0.262$, and $\mu_b = 0.198$, for all parameter sets.

In general, we can consider the system of interacting ICUs to be performing better when the optimal target can be set lower. Note that we never observe an optimal target of $t = 0$ for strict diversion. In this case, the PoA (or PoS) is infinite, which emphasises that such a target is not a good choice from society's perspective. Some key findings (based on Figure 4.4 and Figures A.15 to A.20) are given below:

- For a fixed total number of beds across the system of ICUs, adding more ICUs usually leads to a lower optimal target. This is because more ICUs means each ICU has fewer beds (as the beds are dispersed across a greater number of ICUs), so each ICU has fewer thresholds to choose from. This is the case for both the strict and soft diversion models. Whilst it may be beneficial to have more ICUs in the system, the cost of operating more ICUs also needs to be considered.
- Adding more ICUs when each ICU in the system has the same number of beds also leads to a lower optimal target. Naively, one might predict that with a larger number of ICUs, there is greater scope for uncoordinated behaviour, since the system is becoming larger and ICUs can get away with behaving in a more selfish way. But as more ICUs have to work with one another, the ability of the ICUs to behave in an uncoordinated manner is actually reduced and a single ICU has less influence on the system. The ICUs cannot divert patients as easily if every ICU is trying to do the same thing. Hence, each ICU is forced to take more of its own patients compared to when there were fewer ICUs, and diversion occurs less often.
- It is consistently observed in the line graphs that a higher demand leads to a higher optimal target, since ICUs behave most selfishly when they become busy. We expect the optimal target to also be higher for lower bed capacities. This is not always the case in our graphs, which may be a consequence of the 'small system' effect mentioned earlier. Setting a target may not be so reliable when the system of ICUs is small.
- Above some $x$ value, the optimal target often does not seem to change. This indicates that once central control chooses its bed utilisation target in a high demand situation, the target will be valid over a wide variation of $x$ in this high demand regime. When communication is (is not) possible between the ICUs, we use the PoS (PoA) to find the optimal target. As can be seen from figure A.18, whether or not there is communication has negligible impact on the optimal target (usually only a difference of 0.1).
- With medium to high demand, the optimal target is slightly higher for soft diversion than for strict diversion. The disparity is more evident when there are a greater number of ICUs and more beds across the system. This may be a result of ICUs taking advantage of their limited scope to divert patients whenever they can in the soft diversion model. As discussed earlier, the PoA and PoS values for soft diversion are much smaller than for strict diversion, particularly when demand is high and the target is low. Although the higher optimal target may indicate a greater misalignment of interests with soft diversion, the severity is small, which is why soft diversion is still the better choice of model.

## 5 Discussion and Conclusion

This work modelled the interaction between a system of $N$ ICUs as a normal form game, where each ICU chooses a bed occupancy threshold. On or above this threshold, it is possible for patients to be diverted to another ICU. For each configuration of thresholds, the system was modelled as a continuous-time Markov chain (CTMC), using two different frameworks. The strict diversion model is non-cooperative and allows patients to be lost from the system when all ICUs are busy. The soft diversion model is semi-cooperative and forces ICUs to still accept their own patients when they are busy.

By establishing the stationary distribution of the CTMC, we calculated the throughput and bed utilisation rate for each ICU. The utilisation rate led us to find the best response for each individual ICU, for each possible configuration of thresholds of the other ICUs. An ICU's best response corresponds to the threshold which has the bed utilisation as close as possible to some target set by central control. From these best responses, the Nash equilibria (or intersections of best responses for all $N$ ICUs) were determined. The maximum possible throughput for the system and the throughput for each Nash equilibria were calculated.

We defined three different prices in this project - the Price of Anarchy, Stability, and Communication. By considering many different sets of parameters, we used these prices and throughput calculations to establish the circumstances in which the most uncoordinated behaviour is observed in the system. Through an investigation of different targets and demand levels, we also introduced and found an optimal target for each parameter set, which aimed to align the interests of the individual ICUs and those of society.

We considered the variation of multiple parameters in this project, including the diversion model, the number of ICUs in the system, the bed capacity of each ICU, and the patient arrival rates. We indicate below some of the limitations of the models and methods we used that may have influenced some of our results. Several possible future directions for this work are also discussed:

- In our models, only pure strategies were considered when each ICU chose its diversion threshold. Although ICUs generally keep the same diversion thresholds in the long term, it may be useful to consider mixed strategies, where the decision made by each ICU can change over time. According to Knight *et al* (2017), this could allow more flexibility for ICU managers since they could assess their resources (including staff availability) periodically, and make decisions accordingly. This would allow ICUs to accept more/fewer patients when demand is high based on their individual ability to cope with the increased system load at the time. Alternatively, the ICUs could pick their diversion thresholds based on a probability distribution.
- An ICU can only have a status of low occupancy or high occupancy. Whilst this simplifies the model, a real system of ICUs may use more than two categories to assess their bed occupancy. For example, an ICU could be of a medium occupancy status, where the ICU accepts some of its own patients but diverts its other patients to one or more of the other ICUs. The strict and soft diversion models could be adjusted to include additional bed occupancy categories and make the decision making of the ICUs less restrictive.
- In our models, the patient length of stay in the system is dependent upon the ICU in which the patient receives their treatment. According to Knight *et al* (2017), it could be meaningful to also take into account the service rate in the ICU which the patient originally presented to (before any diversion took place). In reality, the patient is likely to have still spent some time in the original ICU, and in some cases, they may have spent more time in the original ICU than the ICU they are diverted to. The patient could even have been diverted more than once. These additional considerations would lead to a more complicated CTMC (due to the multiple service rates), but would make the model more realistic. It would also be interesting to consider the effect of varying service rates, since this project focuses mainly on varying arrival rates.
- The soft diversion model rejects a patient when they turn up to an ICU that has reached its full bed capacity, and all ICUs are of a high occupancy status. It may be worth considering a more cooperative model, where patients can only be lost from the system when nearly all ICUs have reached their full bed capacity. We say 'nearly all' because there must be some beds available for emergency patient arrivals.

- The models in this project did not distinguish between emergency and elective patients. However, it was assumed that setting a lower bed utilisation target would leave room for emergency patient arrivals. Elective patients undergo operations scheduled in advance, which are not deemed to be medical emergencies. An elective patient may require intensive care in their recovery from an operation, and be admitted to an ICU. An extension of the model could consider separate arrival rates for the two types of patients.
- In a real healthcare setting, if patients cannot be accepted in any of the ICUs immediately, they could wait for treatment in the emergency department or recovery room in one of the hospitals. An extension of the CTMC in this project could incorporate a queueing system for these patients. Examples could include a first come first served model or ranking patients based on the urgency of their need for treatment.
- It was assumed in our models that each ICU would have the same bed utilisation target. Since not every ICU has the same bed capacity, arrival rate and service rate, a further study could consider central control setting a different target for each ICU based on the individual parameters of the ICUs.
- For each parameter set tested in this project, at least one Nash equilibrium was always found. Indeed, Knight *et al* (2017) proved the existence of a Nash equilibrium for a system of 2 ICUs when the strict and soft diversion models are used. This proof relied on showing that the best response curve for each ICU is non-decreasing as the diversion threshold of the other ICU increases. Although we are confident that a Nash equilibrium will always exist for our system of $N$ ICUs when the strict and soft diversion models are used, this should be formally proved using ideas from the proof by Knight *et al* (2017).
- We considered the utility of each ICU to be maximised when the bed utilisation is as close as possible to the target. Knight *et al* (2017) suggested that it may also be important to consider a model that maximises patient survival, rather than just the number of patients passing through the ICU system.
- To find the best response for an individual ICU, we considered every possible configuration of thresholds of the other $N-1$ ICUs. By an exhaustive consideration of best responses, this led to finding the Nash equilibria. It may be useful to explore a less computationally draining approach to finding Nash equilibria than the brute force approach taken in this project. Due to the long computation times, it may also be worth exploring other programs to implement our models. A supercomputer program, such as Spartan, would be a better choice than MATLAB for computational efficiency. However, the user interface for coding in Spartan may be more advanced than that for MATLAB, and more difficult to access.
- The soft diversion model is constructed so that Nash equilibria can be found, which is another reason why this model is only semi-cooperative. A fully cooperative model would not consider Nash equilibria, but instead make use of an $N$-dimensional *cooperative payoff set* for the system of $N$ ICUs. If we only consider pure strategies, this set can be defined to be a finite collection of discrete points in $\mathbb{R}^n$. Each configuration of thresholds has an associated *payoff vector*, and this vector corresponds to a point in the cooperative payoff set. The $n$-th component of this payoff vector is the *payoff* to ICU $n$, which is derived from the absolute value of the difference between the bed utilisation for ICU $n$ and the bed utilisation target. The smaller the absolute value, the larger the payoff to ICU $n$. The point in the cooperative payoff set deemed 'best', known as *Nash's solution*, can be found using a (generalised version of a) bargaining procedure proposed by John F. Nash. Nash's solution is fair to all $N$ ICUs, and the corresponding thresholds are 'selected' for the system of $N$ ICUs to implement. The bargaining is initiated by a *status quo point*, which is a configuration of thresholds deemed acceptable, but not necessarily optimal, to all $N$ ICUs. Each player's payoff in Nash's solution is no less than the payoff they would receive in the status quo point.

Whilst adjustments can be made to improve the models and methods we implemented, the results obtained in this project are still informative enough to understand the general trends in the optimal target and the severity of uncoordinated behaviour in the system of ICUs. We demonstrated that having more ICUs in the system can be beneficial in reducing the extent of uncoordinated behaviour. We also discovered that communication between individual ICUs in the system is most valuable when there is a significant amount of cooperation in the model (the soft diversion framework in our case).

## References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# Appendix A

This section includes the line graphs which were referred to, but not displayed, in the results section. Each figure refers to a pair of side-by-side line graphs. In the figure caption, the parameters used to produce the graphs (including diversion model type (strict or soft), number of ICUs in the system, bed capacities, and arrival rate parameters $\lambda_s$ and $\lambda_b$) are outlined, unless otherwise stated. Note that the service rate parameters $\mu_s = 0.262$ and $\mu_b = 0.198$ are fixed for all parameter sets. Hence, we do not need to refer to these values explicitly in the figure captions. The data for these line graphs can be accessed in tables here.

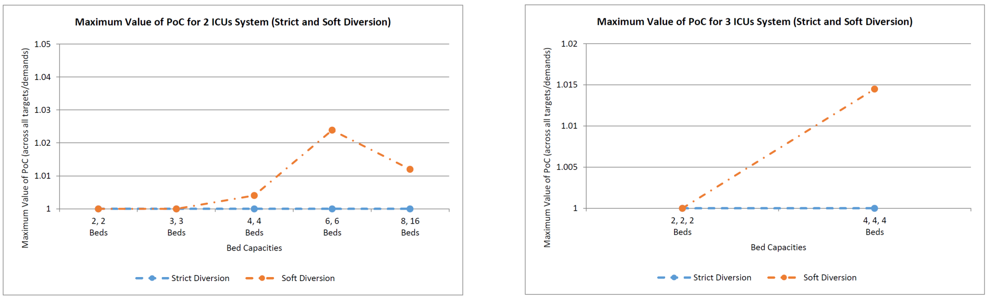



Figure A.1: Maximum value of PoC across all $x$ and $t$ values, for each bed capacity. Both the strict and soft diversion models are considered. Left graph has fixed parameters $N = 2$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$. Right graph has fixed parameters $N = 3$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.

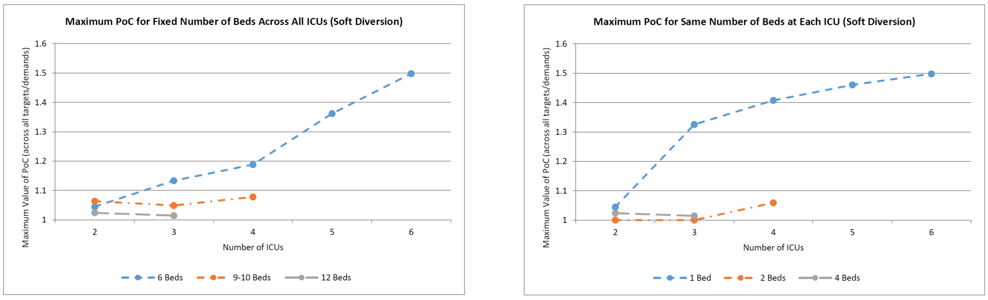



Figure A.2: Maximum value of PoC across all $x$ and $t$ values, for each parameter $N$ (number of ICUs in the system). Only the soft diversion model is considered. We change the fixed number of beds across the entire system in the left graph and the number of beds at each ICU in the system in the right graph. The values of $\lambda_s$ and $\lambda_b$ are fixed for the parameter sets which have the same line colour. Refer to Appendix B for details.

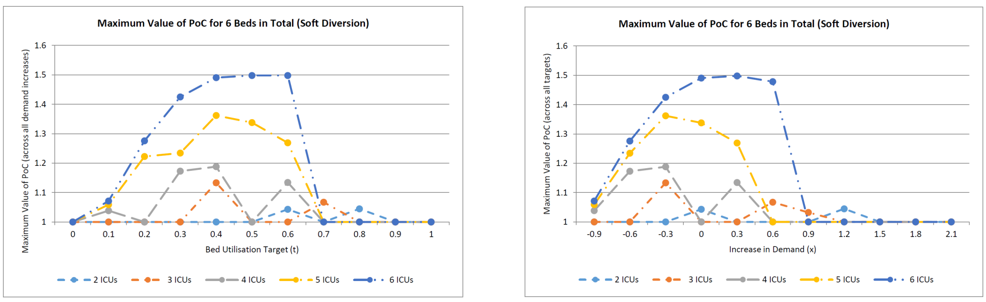



Figure A.3: Maximum value of PoC for each value of $t$ (left graph) and for each value of $x$ (right graph), with 6 beds in total across the entire ICU system and varying parameter $N$. Only the soft diversion model is considered. We fix $\lambda_s = 0.375$ and $\lambda_b = 0.56$ for all parameter sets.

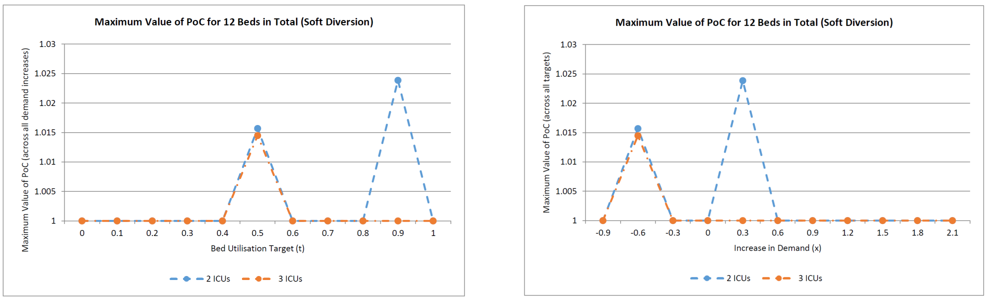



Figure A.4: Maximum value of PoC for each value of $t$ (left graph) and for each value of $x$ (right graph), with 12 beds in total across the entire ICU system and varying parameter $N$. Only the soft diversion model is considered. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

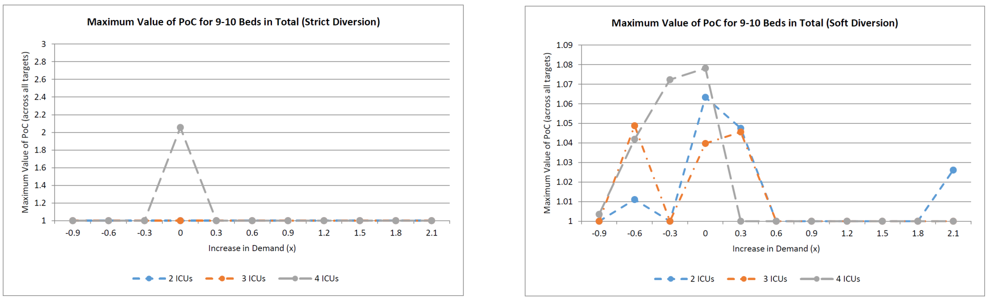



Figure A.5: Maximum value of PoC for each increase in demand ($x$), with 9-10 beds in total across the entire ICU system and varying parameter $N$. We fix $\lambda_s = 0.5625$ and $\lambda_b = 0.84$ for all parameter sets.

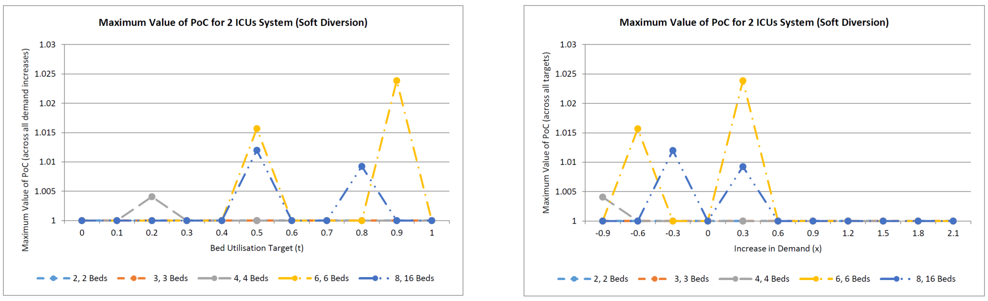



Figure A.6: Maximum value of PoC for each value of $t$ (left graph) and for each value of $x$ (right graph), with varying bed capacities, but with a fixed parameter $N = 2$. Only the soft diversion model is considered. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

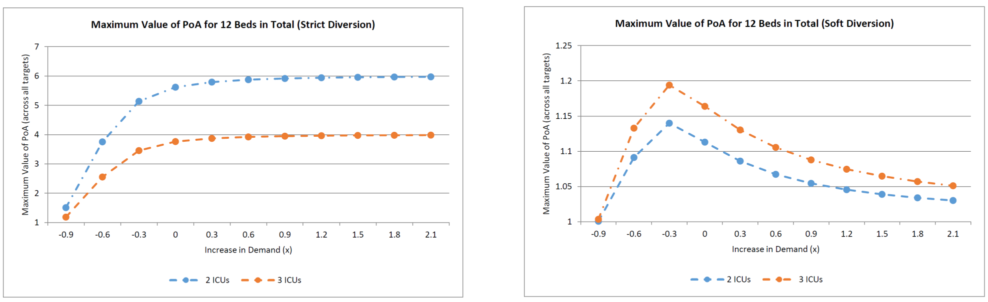



Figure A.7: Maximum value of PoA for each increase in demand ($x$), with 12 beds in total across the entire ICU system and varying parameter $N$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

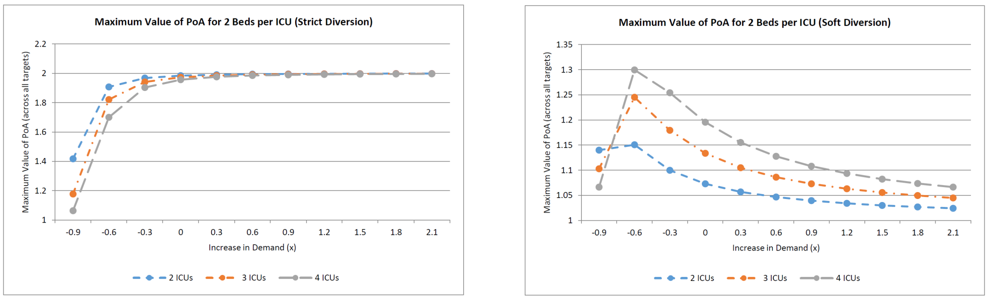



Figure A.8: Maximum value of PoA for each increase in demand ($x$), with 2 beds at each ICU in the system and varying parameter $N$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

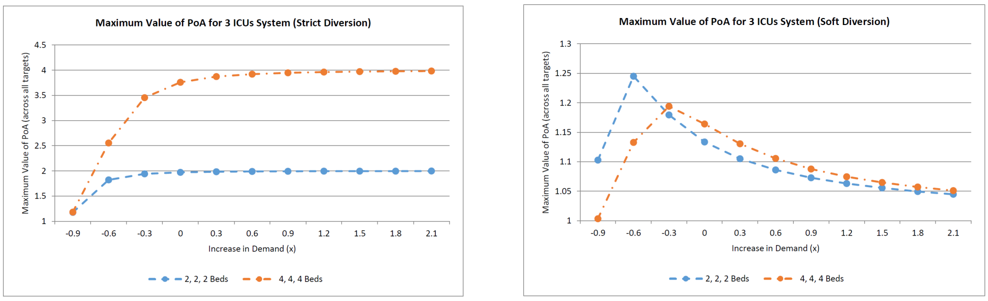



Figure A.9: Maximum value of PoA for each increase in demand ($x$), with varying bed capacities, but with a fixed parameter $N = 3$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

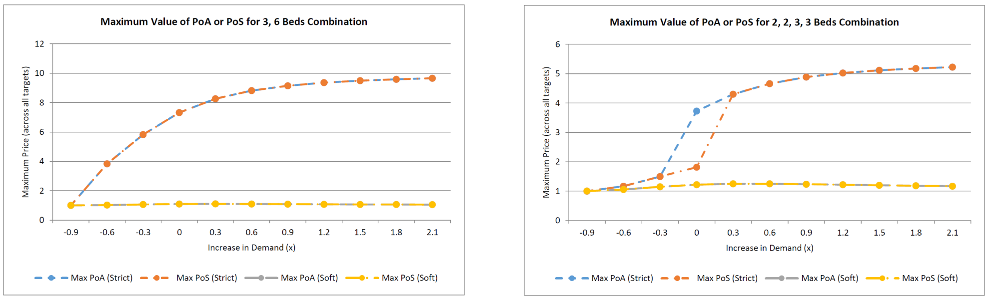



Figure A.10: Maximum value of PoA and maximum value of PoS for each increase in demand ($x$), for strict/soft diversion. Left graph has parameters $N = 2$, $\mathbf{c} = (3, 6)$, $\lambda_s = 0.5625$ and $\lambda_b = 0.84$. Right graph has parameters $N = 4$, $\mathbf{c} = (2, 2, 3, 3)$, $\lambda_s = 0.5625$ and $\lambda_b = 0.84$.

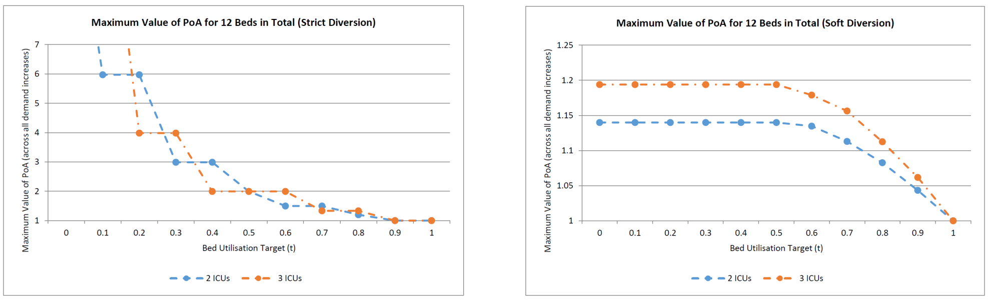



Figure A.11: Maximum value of PoA for each bed utilisation target ($t$), with 12 beds in total across the entire ICU system and varying parameter $N$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

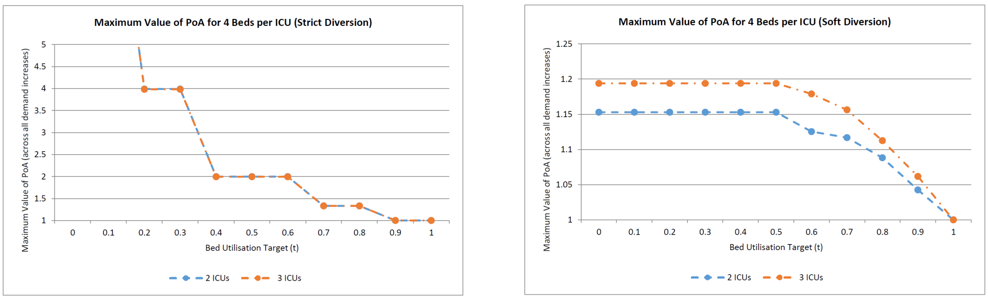



Figure A.12: Maximum value of PoA for each bed utilisation target ($t$), with 4 beds at each ICU in the system and varying parameter $N$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

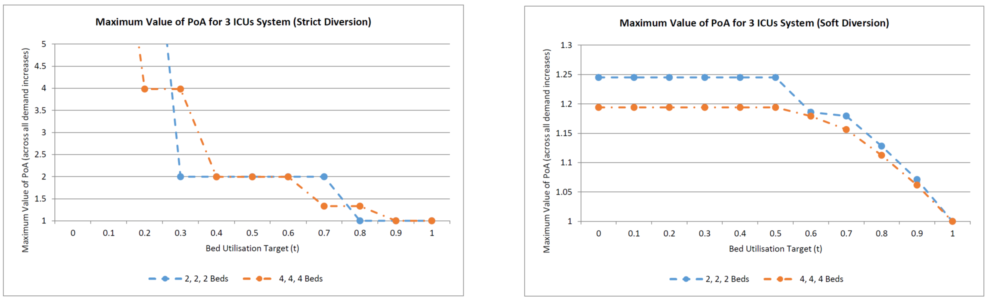



Figure A.13: Maximum value of PoA for each bed utilisation target ($t$), with varying bed capacities, but with a fixed parameter $N = 3$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

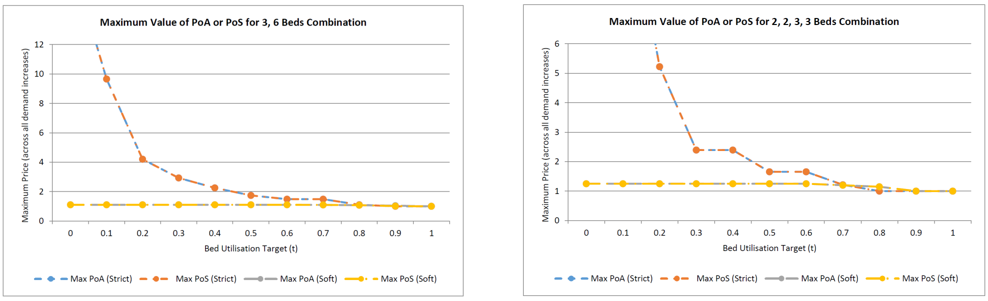



Figure A.14: Maximum value of PoA and maximum value of PoS for each bed utilisation target ($t$), for strict/soft diversion. Left graph has parameters $N = 2$, $\mathbf{c} = (3, 6)$, $\lambda_s = 0.5625$ and $\lambda_b = 0.84$. Right graph has parameters $N = 4$, $\mathbf{c} = (2, 2, 3, 3)$, $\lambda_s = 0.5625$ and $\lambda_b = 0.84$.

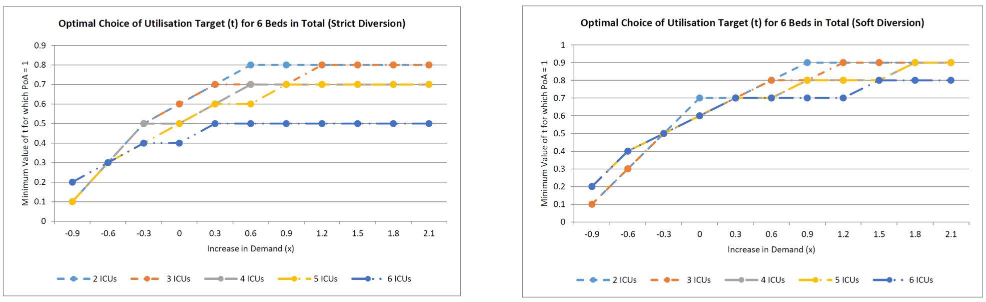



Figure A.15: Optimal bed utilisation target for each increase in demand ($x$), with 6 beds in total across the entire ICU system and varying parameter $N$. We fix $\lambda_s = 0.375$ and $\lambda_b = 0.56$ for all parameter sets.

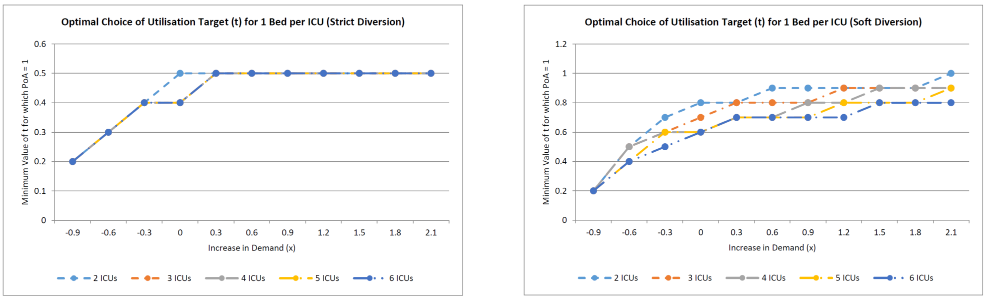



Figure A.16: Optimal bed utilisation target for each increase in demand ($x$), with 1 bed at each ICU in the system and varying parameter $N$. We fix $\lambda_s = 0.375$ and $\lambda_b = 0.56$ for all parameter sets.

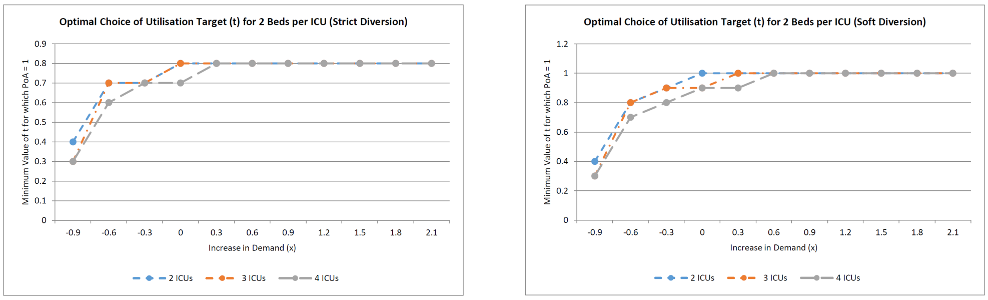



Figure A.17: Optimal bed utilisation target for each increase in demand ($x$), with 2 beds at each ICU in the system and varying parameter $N$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

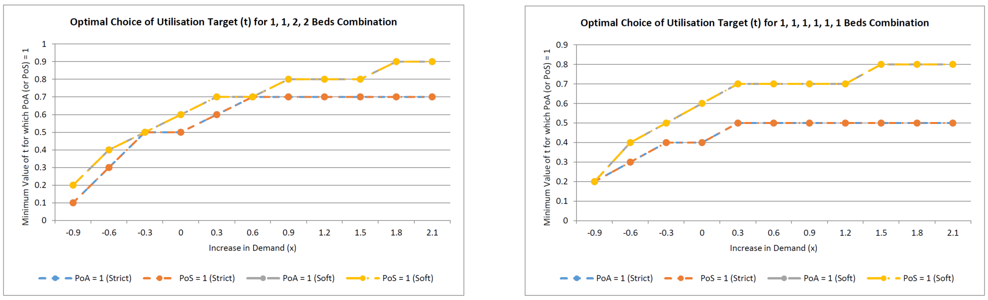



Figure A.18: Optimal bed utilisation target for each increase in demand ($x$), for strict/soft diversion and when communication is/is not allowed. Left graph has parameters $N = 4$, $\mathbf{c} = (1, 1, 2, 2)$, $\lambda_s = 0.375$ and $\lambda_b = 0.56$. Right graph has parameters $N = 6$, $\mathbf{c} = (1, 1, 1, 1, 1, 1)$, $\lambda_s = 0.375$ and $\lambda_b = 0.56$.

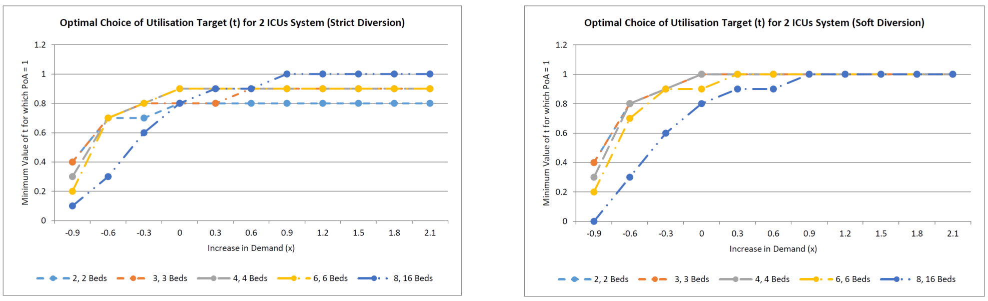



Figure A.19: Optimal bed utilisation target for each increase in demand ($x$), with varying bed capacities, but with a fixed parameter $N = 2$. We fix $\lambda_s = 1.50$ and $\lambda_b = 2.24$ for all parameter sets.

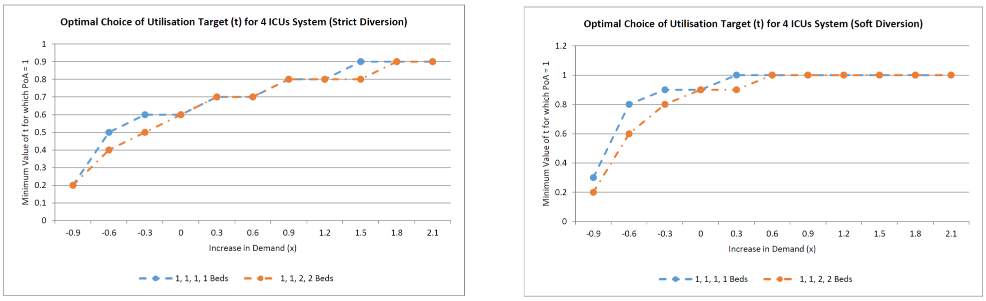



Figure A.20: Optimal bed utilisation target for each increase in demand ($x$), with varying bed capacities, but with a fixed parameter $N = 4$. We fix $\lambda_s = 0.375$ and $\lambda_b = 0.56$ for all parameter sets.

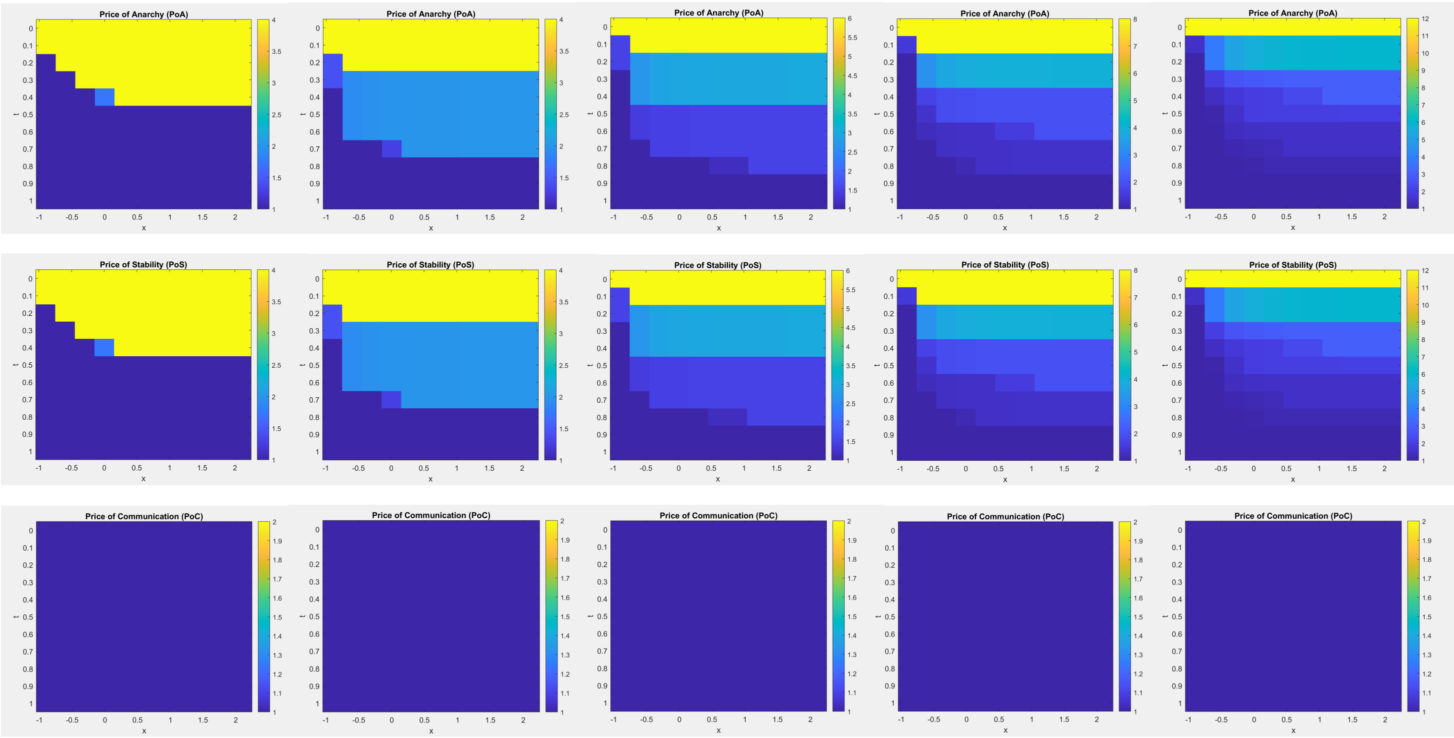
# Appendix B

This section includes the 'checkerboard' plots of the PoA, PoS and PoC (for both the strict and soft diversion models) for 20 parameter sets (not including the set already considered in Figure 4.1). The numerical values of all 121 'checkerboard' squares in each of the plots can be accessed in tables here.

We list below the parameters which distinguish each of the 20 different parameter sets. As mentioned in the results section, the service rate parameters $\mu_s = 0.262$ and $\mu_b = 0.198$ are fixed for all parameter sets. However, at least one of the number of ICUs in the system ($N$), bed capacities ($\mathbf{c}$), and arrival rate parameters $\lambda_s$ and $\lambda_b$ will be different between any two different parameter sets in the list.

1. $N = 2$, $\mathbf{c} = (1, 1)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
2. $N = 2$, $\mathbf{c} = (2, 2)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
3. $N = 2$, $\mathbf{c} = (3, 3)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
4. $N = 2$, $\mathbf{c} = (4, 4)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
5. $N = 2$, $\mathbf{c} = (6, 6)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
6. $N = 2$, $\mathbf{c} = (2, 4)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
7. $N = 2$, $\mathbf{c} = (3, 6)$, $\lambda_s = (3/8) \times 1.50 = 0.5625$ and $\lambda_b = (3/8) \times 2.24 = 0.84$.
8. $N = 2$, $\mathbf{c} = (8, 16)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
9. $N = 3$, $\mathbf{c} = (1, 1, 1)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
10. $N = 3$, $\mathbf{c} = (1, 2, 3)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
11. $N = 3$, $\mathbf{c} = (2, 2, 2)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
12. $N = 3$, $\mathbf{c} = (2, 2, 2)$, $\lambda_s = (5/2) \times 1.50 = 3.75$ and $\lambda_b = (5/2) \times 2.24 = 5.60$.
13. $N = 3$, $\mathbf{c} = (2, 3, 4)$, $\lambda_s = (3/8) \times 1.50 = 0.5625$ and $\lambda_b = (3/8) \times 2.24 = 0.84$.
14. $N = 3$, $\mathbf{c} = (4, 4, 4)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
15. $N = 4$, $\mathbf{c} = (2, 2, 2, 2)$, $\lambda_s = 1.50$ and $\lambda_b = 2.24$.
16. $N = 4$, $\mathbf{c} = (1, 1, 2, 2)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
17. $N = 4$, $\mathbf{c} = (1, 1, 1, 1)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
18. $N = 5$, $\mathbf{c} = (1, 1, 1, 1, 2)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
19. $N = 5$, $\mathbf{c} = (1, 1, 1, 1, 1)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.
20. $N = 6$, $\mathbf{c} = (1, 1, 1, 1, 1, 1)$, $\lambda_s = (1/4) \times 1.50 = 0.375$ and $\lambda_b = (1/4) \times 2.24 = 0.56$.

Strict Diversion Plots




Soft Diversion Plots

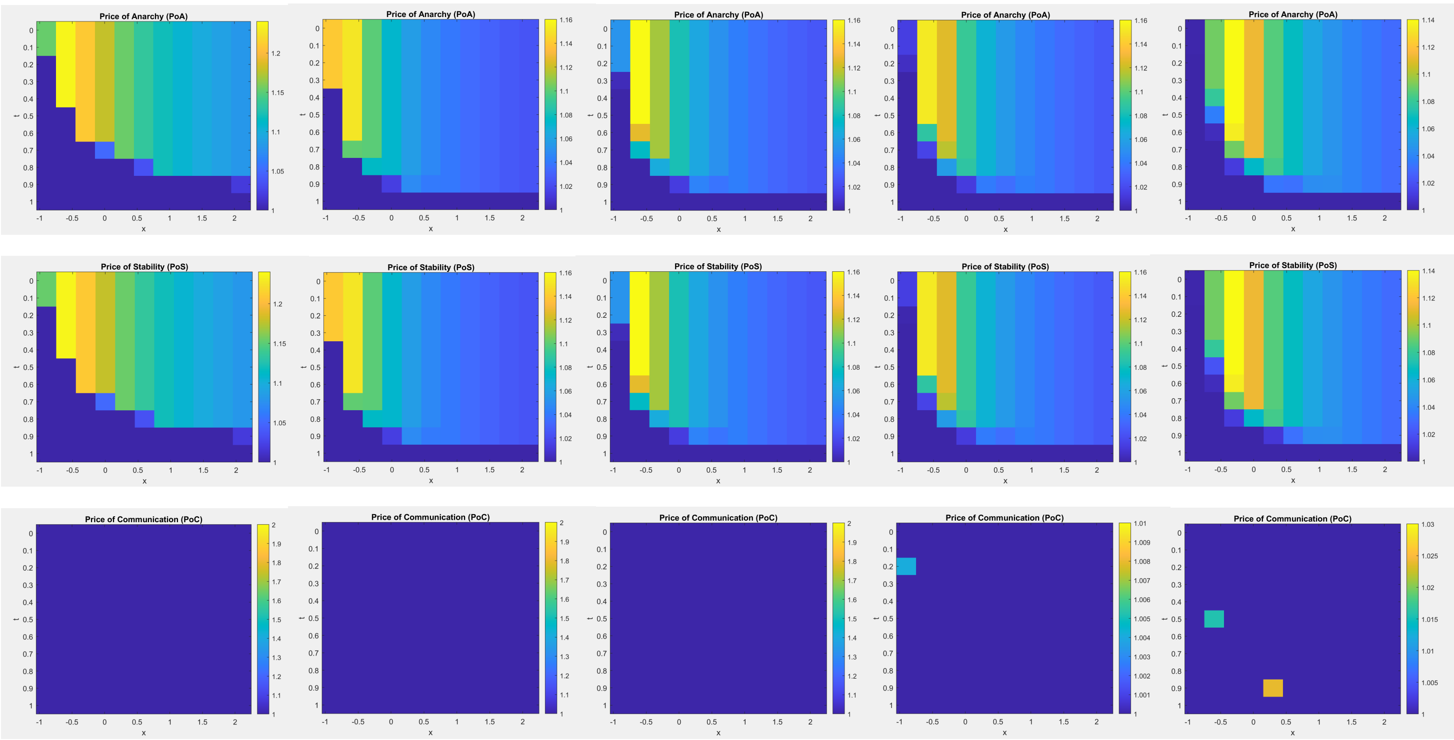
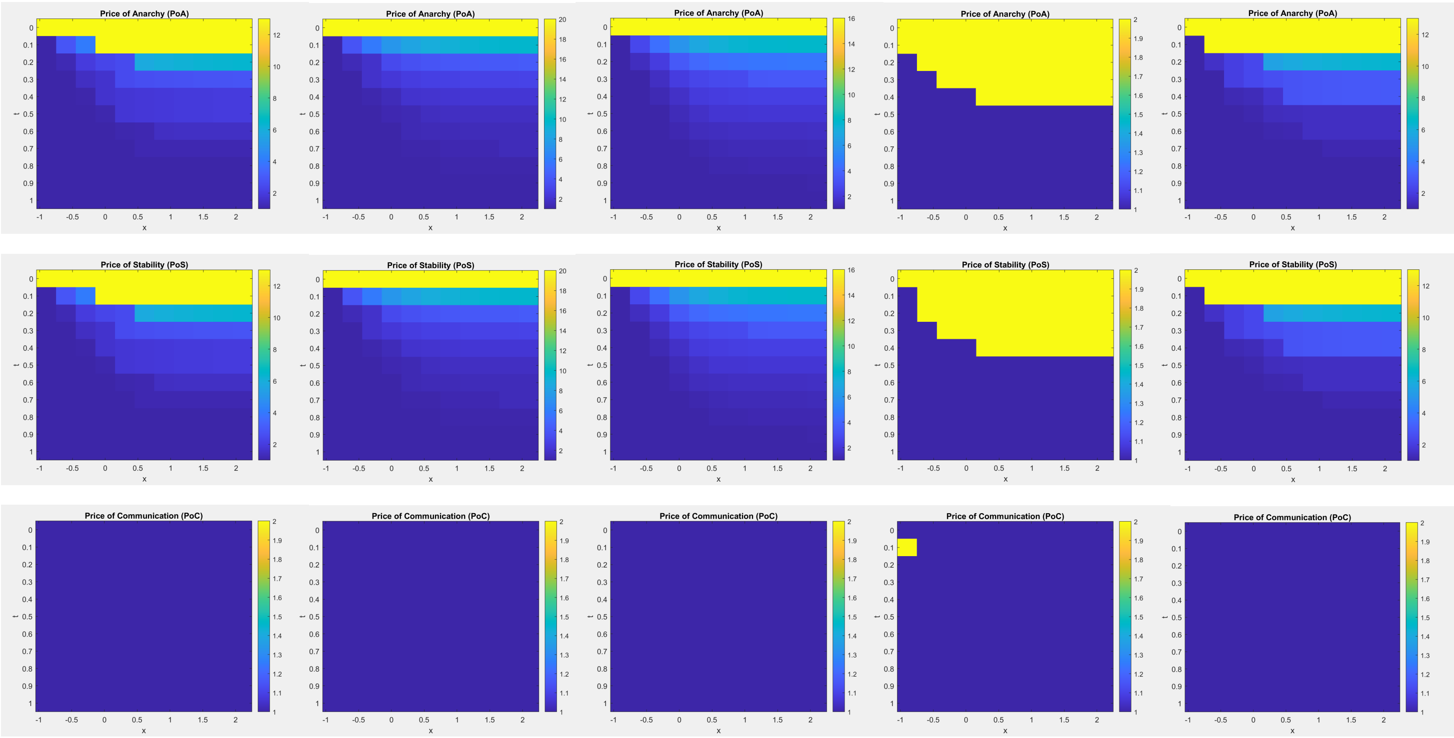



Figure B.1: PoA, PoS and PoC plots for the strict and soft diversion models. Columns 1, 2, 3, 4 and 5 refer to parameter sets 1, 2, 3, 4 and 5, respectively.

Strict Diversion Plots




Soft Diversion Plots

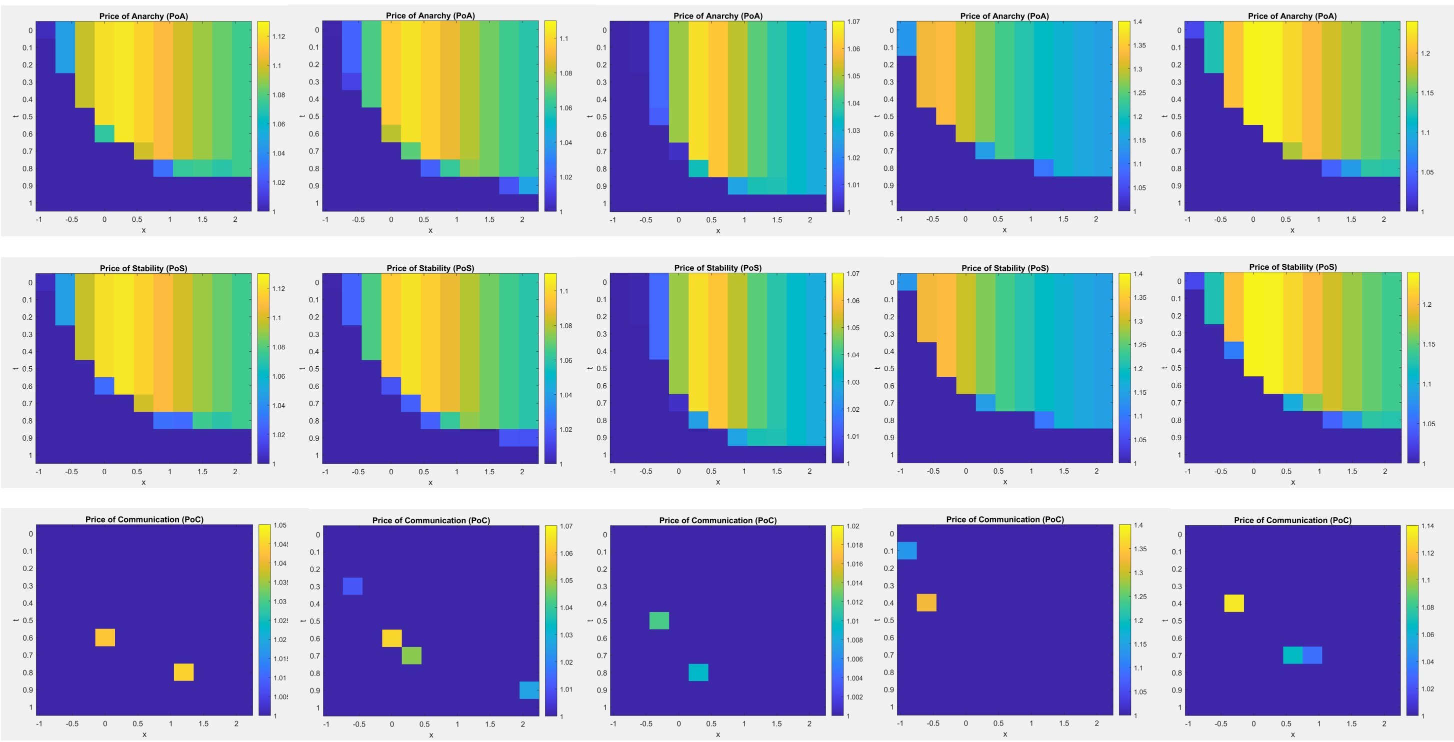
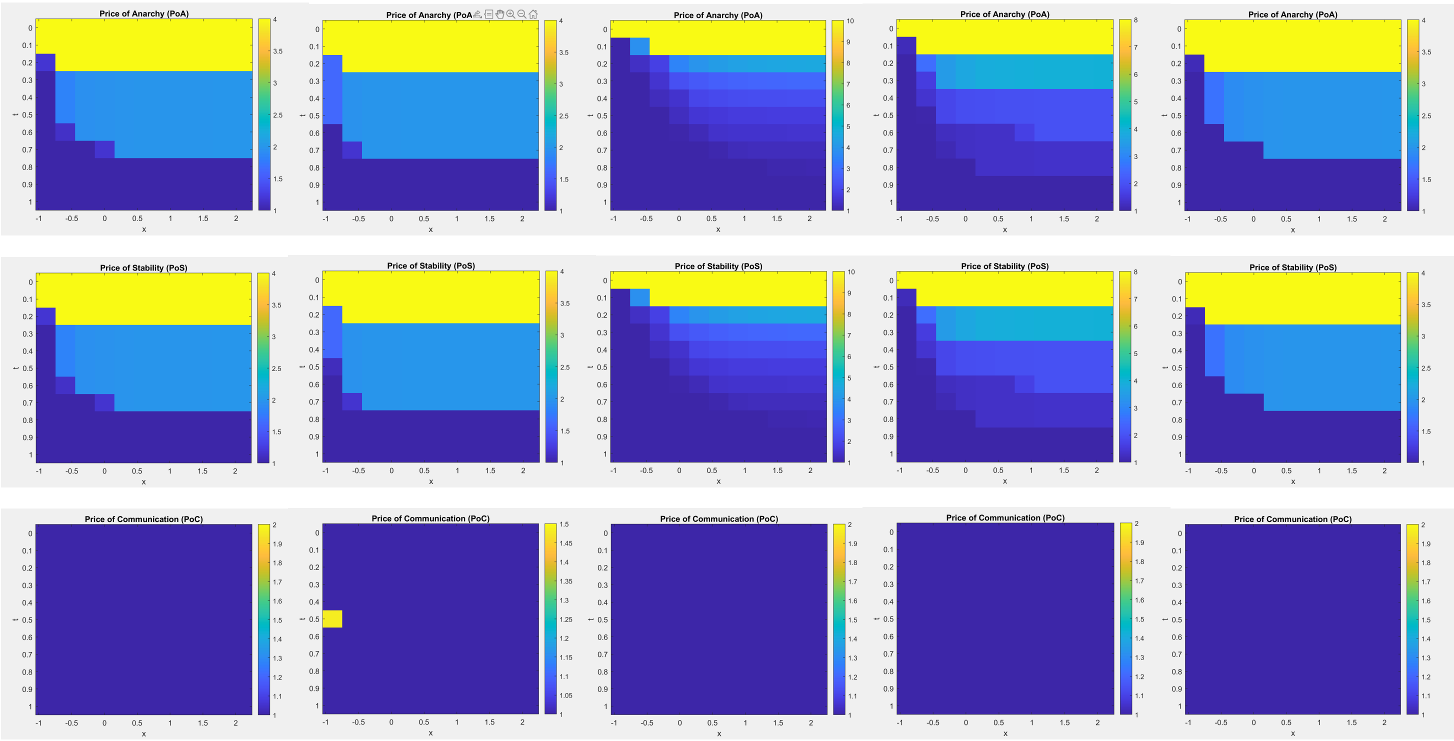



Figure B.2: PoA, PoS and PoC plots for the strict and soft diversion models. Columns 1, 2, 3, 4 and 5 refer to parameter sets 6, 7, 8, 9 and 10, respectively.

Strict Diversion Plots




Soft Diversion Plots

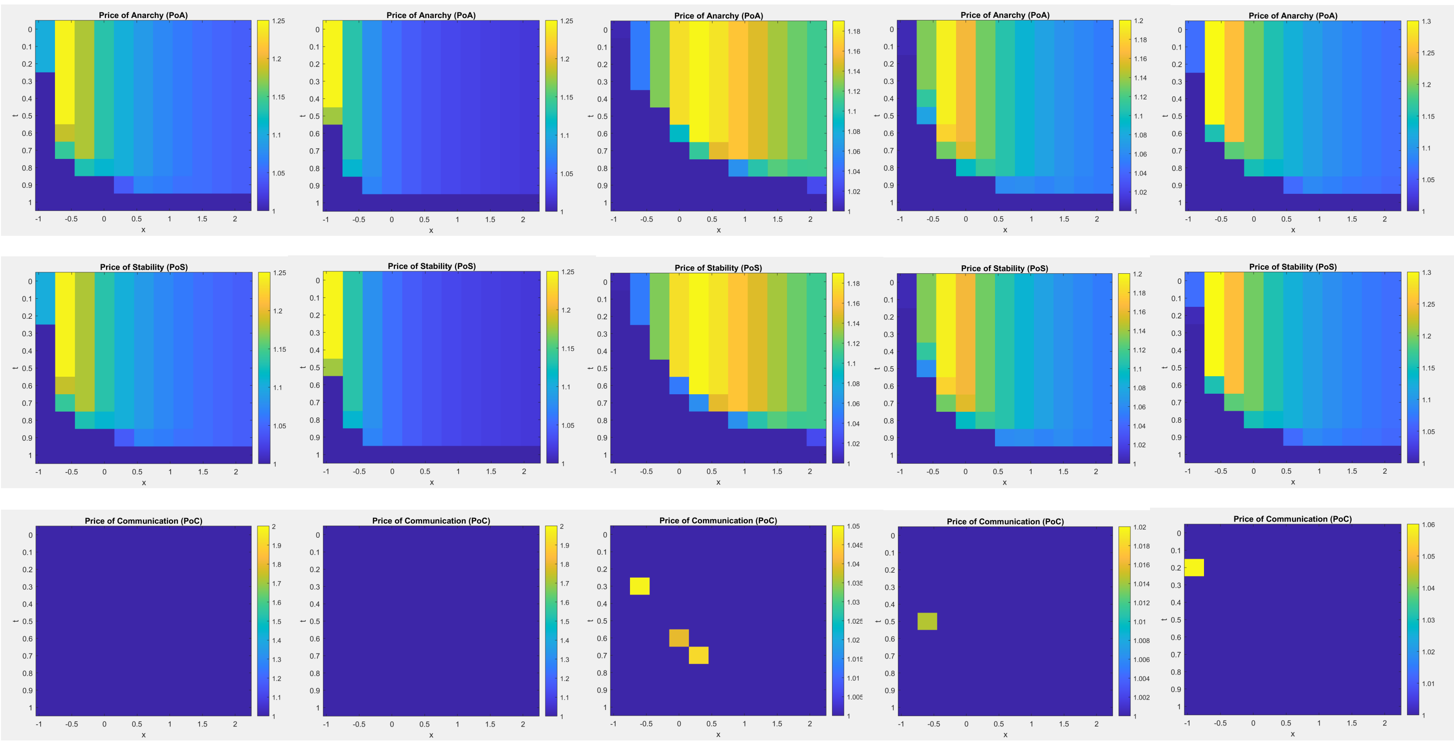
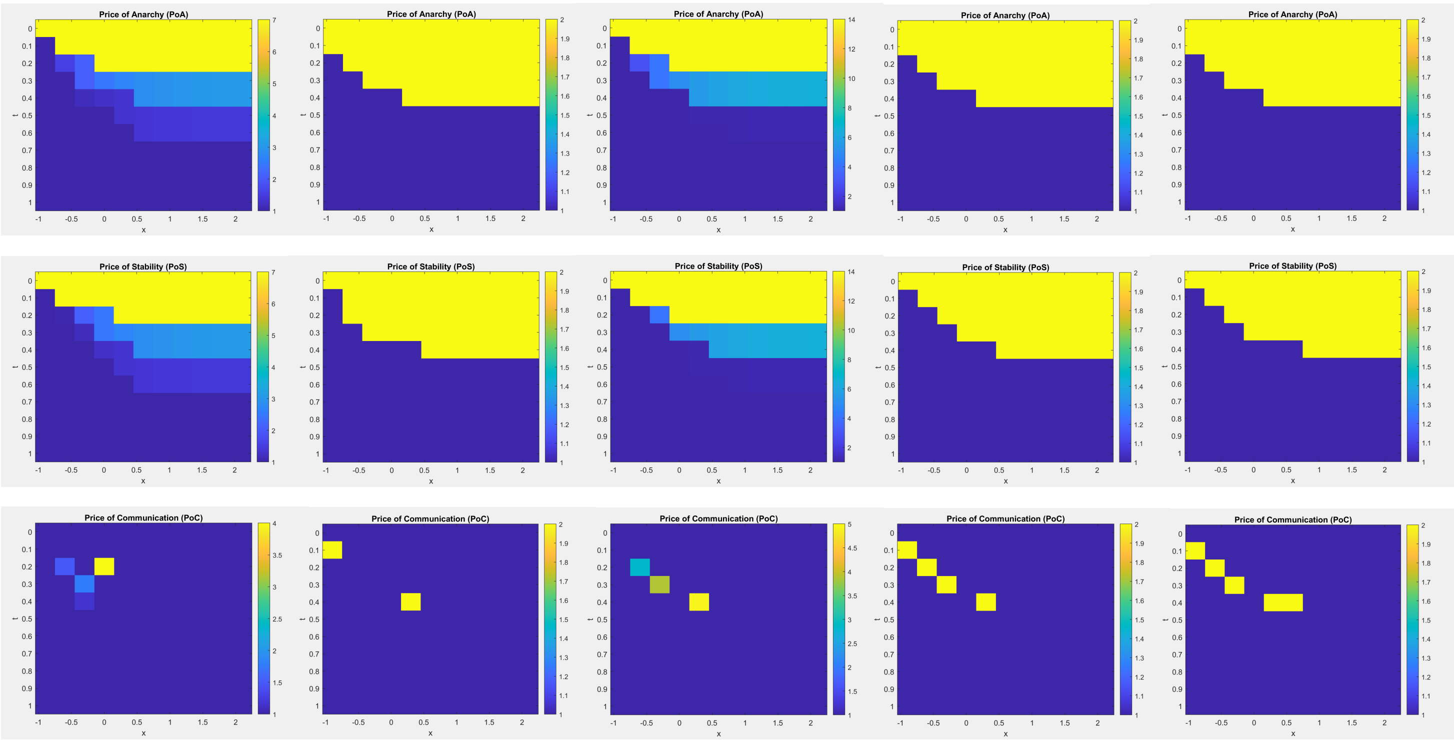



Figure B.3: PoA, PoS and PoC plots for the strict and soft diversion models. Columns 1, 2, 3, 4 and 5 refer to parameter sets 11, 12, 13, 14 and 15, respectively.

Strict Diversion Plots




Soft Diversion Plots

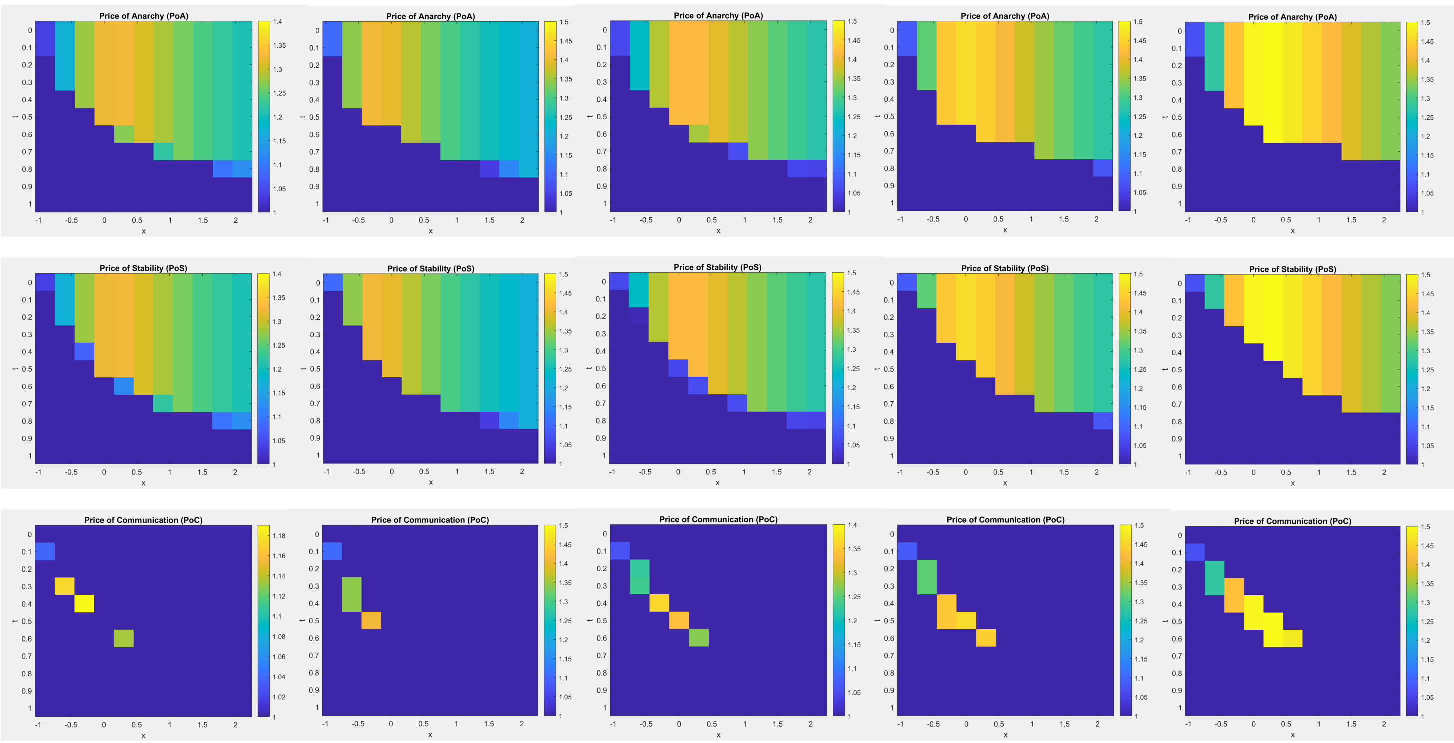



Figure B.4: PoA, PoS and PoC plots for the strict and soft diversion models. Columns 1, 2, 3, 4 and 5 refer to parameter sets 16, 17, 18, 19 and 20, respectively.