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# Naturally Reductive Metrics on Homogeneous Spaces


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Vacation Research Scholarships are funded jointly by the Department of Education, Skills and Employment and the Australian Mathematical Sciences Institute.



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# 1 Prelude

## 1.1 Acknowledgements

I would like to thank Dr Artem Pulemotov and Dr Timothy Buttsworth for their mentorship and insight throughout this project. I have thoroughly enjoyed our conversations and look forward to working with you in the future.
I would like to also thank the Australian Mathematical Sciences Institute (AMSI) for funding, the opportunity to experience pure mathematics research over the summer and for hosting AMSI Connect.
To my family, thank you for your unwavering support and encouragement. I am incredibly grateful to you all for empowering me to follow my curiosity and pursue my passions. I hope I have done the same for you.

## 1.2 Abstract

Naturally reductive homogeneous spaces are a useful example of homogeneous Riemannian manifolds due to their relatively simple algebraic and geometric properties. We explore the theory underlying differential geometry and homogeneous spaces to elucidate the properties of such metrics and why they provide such an interesting class to study. Further, we discuss recent work which uses the computation of the Ricci curvature of naturally reductive metrics to address the Prescribed Ricci Curvature problem on compact Lie groups. Finally, next steps in the computation of the Ricci curvature on more general classes and hence the Prescribed Ricci Curvature on these spaces are discussed.

## 1.3 Statement of Authorship

The information presented in this report is cited from many texts, though the structure, notation and presentation is varied from the sources.
The mathematical theory presented in this report is well-established and no new results are presented here. All results have been cited. This project has been an opportunity to understand the process of pure mathematical research and prepare for future work to be completed in an Honours thesis.

# 2 Introduction

A manifold is a Hausdorff topological space which locally resembles Euclidean space, in the sense that for every neighbourhood of the manifold there exists a homeomorphism to a Euclidean space. Surfaces and curves in $\mathbb{R}^3$ provide classical examples in geometry of manifold structures. Smooth manifolds allow for calculus to be done on the differential structure.

A Riemannian metric, which assigns each point on the manifold an inner product on the associated tangent space at that point, gives a manifold 'shape'. Amongst many other uses, it allows for determining the length of curves, area of surfaces and intrinsic curvature of a smooth manifold. Such a metric also provides insight as to the degree that a manifold deviates from Euclidean space.

A Lie group is a set which has a compatible smooth manifold and group structure. The geometry of such manifolds can then be studied through the properties of the group structure. The Lie algebra associated with this group is diffeomorphic to the tangent space at the identity of this group. Riemannian metrics can be defined on this real vector space and so computations of curvature can be reformulated into algebraic problems in this space.

This report aims to introduce key definitions and results in Riemannian geometry and homogeneous spaces. It will then provide motivation for and define a left-invariant naturally reductive Riemmanian metric. The classification theorems of naturally reductive metrics in the compact and non-compact case will be presented and some key curvature computations will be perfomed. Finally, naturally reductive homogeneous spaces on which the Ricci curvature has not been computed will be discussed. Such computations will lead to further understanding of the Prescribed Ricci Curvature problem on these spaces.

# 3 Differential Geometry

This section provides an introduction to differential geometry and is a summary of topics presented in Chapter 1 of [2] and [1].

## 3.1 Smooth Manifolds

Recall that a manifold, $M$, is a Hausdorff topological space which locally resembles a Euclidean space. Before proceeding into specific examples of manifolds, a formal mathematical definition of 'locally resembles' must be given.

An *open chart* on such an $M$ is a pair $(U, \varphi)$, where $U$ is an open subset of $M$ and $\varphi$ is a homeomorphism of $U$ onto an open subset of $\mathbb{R}^m$. Recall that a homeomorphism is a continuous function which has a continuous inverse. Such charts allow for the definition of a differentiable structure on $M$.

**Definition:** Differentiable Structure
Suppose $M$ is a Hausdorff topological space. A *differentiable structure* on $M$ of dimension $n$ is a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ on $M$ where $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^m$ such that the following conditions are satisfied:

(a) $M = \bigcup_{\alpha\in A} U_\alpha$

(b) For each pair $\alpha, \beta \in A$, the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable mapping of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$, in the sense of smooth funcions between subsets of Euclidean spaces.

(c) The collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is a maximal family of open charts for which the first two conditions hold.

A differentiable (smooth) manifold of dimension $m$ is a Hausdorff space with a differentiable structure of dimension $m$.

**Example:**
Consider the sphere $S^n = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$ in $\mathbb{R}^{n+1}$. This is a smooth manifold of dimension $n$. It can be covered by two charts, $U_+ = \{\mathbf{x} \in S^n : x_{n+1} > -1\}$ and $U_- = \{\mathbf{x} \in S^n : x_{n+1} < 1\}$. Then, $\phi_+ : U_+ \to \mathbb{R}^n$ defined by $\phi_+(\mathbf{x}) = \left(\frac{x_1}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}}\right)$ and $\phi_- : U_- \to \mathbb{R}^n$ defined by $\phi_-(\mathbf{x}) = \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right)$.

**Remarks:**

(a) A *local chart* on $M$ is a pair $(U_\alpha, \varphi_\alpha)$ for some $\alpha \in A$.

(b) If $p \in U_\alpha$ and $\varphi_\alpha(p) = (x_1(p), \ldots, x_m(p))$, the set $U_\alpha$ is called a *coordinate neighbourhood* of $p$ and the numbers $x_i(p)$ are called *local coordinates* of $p$.

Through the use of local charts, the differentiability of real valued functions on a manifold can be determined. Note that the differentiability of functions between manifolds can also be determined, see [1] for details.

**Definition:** Differentiable Function
Let $M$ be an $n$-dimensional smooth manifold and suppose $f$ is a real-valued function on $M$; i.e. $f : M \to \mathbb{R}$. Then, $f$ is a differentiable (smooth) function at a point $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that $f \circ \varphi_\alpha^{-1}$ is a differentiable function on $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. The function is called *differentiable* if it is differentiable at all $p \in M$.

Let $\mathcal{F}(M)$ denote the set of all smooth real-valued function on $M$. For smooth manifolds, such as the sphere, it is natural to consider vectors which are tangential to the manifold at a certain point. Such a vector can be defined as follows.

### 3.2 Tangent Vectors

**Definition:** Tangent Vector
Consider a point, $p$, of a manifold $M$. A *tangent vector* to $M$ at $p$ is a real valued function $v : \mathcal{F}(M) \to \mathbb{R}$ that, for $f, g \in \mathcal{F}(M)$ and $a, b \in \mathbb{R}$, satisfies:

(a) $v(af + bg) = av(f) + bv(g)$

(b) $v(fg) = v(f)g(p) + f(p)v(g)$ (Leibniz rule)

As the name suggests, tangent vectors form a vector space at every point of a smooth manifold.

**Definition:** Tangent Space
At each point $p \in M$, let $T_p(M)$ be the set of all tangent vectors to $M$ at $p$. Then, for $a \in \mathbb{R}$, $v, w \in T_p(M)$ and $f \in \mathcal{F}(M)$, the set $T_p(M)$ is made into a real vector space under the operations:

$$(v + w)(f) = v(f) + w(f), \qquad (av)(f) = av(f)$$

This real vector space is called the *tangent space* of $M$ at $p$ and is of the same dimension as $M$.

See [1] for a construction of the basis of the tangent space. From the tangent space, a new manifold can be constructed.

**Definition:** Tangent Bundle
Suppose $M$ is a smooth manifold of dimension $n$ and consider the set:

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} (\{p\} \times T_pM)$$

An element of this set is a pair $(p, v)$, where $p \in M$ and $v \in T_pM$. The set $TM$ can be made into a manifold of dimension $2n$, called the *tangent bundle* of $M$. The *canonical projection* $\pi : TM \to M$ is defined as $\pi(p, v) = p$ for $p \in M$ and $v \in T_pM$. The manifold structure of $TM$ is chosen such that this mapping is smooth.

### 3.3 Vector Fields

With vector fields now defined, it is natural to consider a function on a smooth manifold which maps every point to a tangent vector at that point. Such a function is called a vector field.

**Definition:** Vector Field
A *vector field* $X$ on a manifold $M$ is a function that assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$ to $M$ at $p$. Thus, $X : M \to TM$ where $X(p) = (p, X_p)$.

From [1], a vector field can be thought of as a collection of arrows (vectors) at each point of the manifold. Just as we defined a differentiable real-valued function on a manifold, we can define a smooth vector field. If $X$ is a vector field on $M$ and $f \in \mathcal{F}(M)$, let $Xf$ denote the real-valued function on $M$ given by $Xf(p) = X_p(f)$.

**Definition:** Smooth Vector Field
A vector field $X$ is called smooth if the function $Xf$ is smooth for all $f \in \mathcal{F}(M)$, in the sense of differentiable real-valued functions on $M$.

Let $\chi(M)$ denote the set of smooth vector fields on $M$. Viewing smooth vector fields as a map $X : \mathcal{F}(M) \to \mathcal{F}(M)$ for which $f \mapsto Xf$ leads to an important operation on vector fields.

**Definition:** Bracket Operation
Suppose $X, Y \in \chi(M)$. The *bracket* operation on $\chi(M)$ is defined as $[X, Y] : XY - YX$. Important properties of this operation are discussed in [1]. The operation will be of great importance in section 4.

### 3.4 Riemannian Manifold

**Definition:** Riemannian Metric
A Riemannian metric on a smooth manifold $M$ is a correspondence which associates to each point $p \in M$ an inner product $g_p = \langle \cdot, \cdot \rangle_p$ on the tangent space $T_pM$ such that for all $X, Y \in \chi(M)$, the map $p \mapsto \langle X_p, Y_p \rangle_p$ is smooth.

A Riemannian metric allows for the computation of various geometrical objects, including lengths of curves, area of surfaces and intrinsic curvature of the manifold.

**Definition:** Affine Connection
An affine connection $\nabla$ on a smooth manifold $M$ is a mapping $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ denoted by $(X, Y) \mapsto \nabla_X Y$ such that for all $X, Y, Z \in \chi(M)$ and $f, g \in \mathcal{F}(M)$:

(a) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(b) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(c) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

The connection is a way of taking covariant derivatives on a manifold, i.e. $\nabla_X Y$ representing the derivative of $Y$ in the direction of $X$.

**Theorem:** Given a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, there exists a unique connection (called the Levi-Civita or Riemannian connection) such that for all $X, Y, Z \in \chi(M)$:

(a) $[X, Y] = \nabla_X Y - \nabla_Y X$,

(b) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

This connection is characterised by the Koszul formula:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle$$

Now that Riemannian metrics and Levi-Civita connections have been defined, various curvature tensors can be defined. The definitions will be stated in this section and example computations will be given for special cases in later sections, namely for bi-invariant metrics and special cases of naturally reductive metrics.

**Definition:** Riemann Curvature Tensor
Let $M$ be a Riemannian manifold with Levi-Civita connection $\nabla$. The Riemann curvature tensor is the map:

$$R : \chi(M) \times \chi(M) \times \chi(M) \to \chi(M)$$

given by:

$$R(X, Y)Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$$

**Definition:** Ricci Curvature
The Ricci curvature $\mathrm{Ric}(X, Y)$ of a Riemannian manifold $M$ is the trace of the map $Z \mapsto R(X, Z)Y$. If $\{e_i\}$ is an orthonormal basis of the tangent space $T_pM$ at point $p \in M$, then the Ricci curvature is given by

$$\mathrm{Ric}(X, Y) = \sum_i \langle R(X, e_i)Y, e_i\rangle$$

The Ricci curvature can be seen as a measurement of how a shape is deformed along a geodesic. Excellent conceptual explanations for this definition is provided by [12]. Other examples of curvature properties include sectional curvature tensors and the scalar curvature, for more details see [1, 2].

# 4 Lie Groups and Lie Algebras

This section gives an introduction to Lie groups, Lie algebras, left-invariant metrics and bi-invariant metrics. It summarises relevant definitions and results from Chapter 1 of [1] and Chapter 2 of [2]. Computations of curvature of bi-invariant metrics are then provided.

## 4.1 Lie Groups and Associated Lie Algebra

**Definition:** Lie Group
Suppose $G$ is a smooth manifold. Then, $G$ is called a Lie group if:

(a) $G$ is a group

(b) The group operations $G \times G \to G$, $(x, y) \mapsto xy$ (multiplication) and $G \to G$, $x \mapsto x^{-1}$ (inversion) are smooth functions.

**Examples:**

(a) The general linear group, $\mathrm{GL}(n, \mathbb{R})$, under matrix multiplication forms a Lie group. For a specific example, $\mathrm{GL}(2, \mathbb{R})$ is a 4-dimensional, noncompact Lie group.

(b) The special orthogonal group, $\mathrm{SO}(n, \mathbb{R})$, containing real $n \times n$ orthogonal matrices with determinant one.

(c) The special unitary group, $\mathrm{SU}(n)$, containing complex $n \times n$ unitary matrices with determinant one.

**Definition:** Left-Invariant Vector Fields
A vector field $X$ on a Lie group $G$ is left-invariant if $dL_p X = X$ for all $p \in G$. More explicitly, $X_{ag} = (dL_a)_g(X_g)$ for all $a, g \in G$.

Left-invariant vector fields can be identified with the tangent vectors in $T_eG$. In fact, given a tangent vector $\bar{X} \in T_eG$, there exists a unique left-invariant vector field $X$ on $G$ such that $X_e = \bar{X}$. A detailed proof of this fact is provided in [2].

Consider the tangent vectors $\bar{X}, \bar{Y} \in T_eG$ and the corresponding left-invariant vector fields $X, Y$. Then, the vector field given by $[X, Y]$ is also left-invariant and the corresponding tangent vector $[X, Y]_e$ is denoted $[\bar{X}, \bar{Y}]$. Then, the vector space $T_eG$, combined with the bracket operation, is called the Lie algebra of $G$, denoted $\mathfrak{g}$. Note that the dimension of the Lie algebra is equal to the dimension of the Lie group. Properties of the Lie algebra have been given below:

**Definition:** Lie Algebra
A real vector space $\mathfrak{g}$ is a *Lie algebra* if it has a bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties:

(a) $[X, Y] = -[Y, X]$, (skew-symmetry)

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
$[Z, aX + bY] = a[Z, X] + b[Z, Y]$ ($\mathbb{R}$-bilinearity)

(c) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, (Jacobi Identity)

for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{R}$.

### 4.2 Basic Lie Theory

Before considering Riemannian metrics on Lie groups, it will be important to understand some important definitions and theorems in Lie theory. This section will provide insight into the correspondence of Lie groups and Lie algebras, adjoint representations and the Killing form.

**Definition:** Lie Subalgebras and Ideals
Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$.

(a) $\mathfrak{h}$ is called a Lie subalgebra of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.

(b) $\mathfrak{h}$ is called an ideal in $\mathfrak{g}$ if $[A, X] \in \mathfrak{h}$ for all $A \in \mathfrak{g}$ and $X \in \mathfrak{h}$.

**Theorem:**

(a) **(Lie's Third Theorem)** For any real Lie algebra $\mathfrak{g}$ there is a simply connected Lie group $G$ (not necessarily unique) whose Lie algebra is $\mathfrak{g}$.

(b) **(Subgroup-Subalgebra Correspondence)** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. If $H$ is a Lie subgroup of $G$ with Lie algebra $\mathfrak{h}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, for each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup $H$ of $G$ which has $\mathfrak{h}$ as its Lie algebra.

If $G$ is a matrix Lie group, then the corresponding Lie algebra $\mathfrak{g}$ is given by all matrices whose matrix exponential lies in $G$. Explicitly, $\text{Lie}(G) = \{X \in M_n(\mathbb{C}) \, : \, \exp(tX) \in G, \forall t \in \mathbb{R}\}$.

**Example:** Lie algebra of $G = \text{SO}(2, \mathbb{R})$, $\mathfrak{g} = \mathfrak{so}(2)$

Consider the Lie group $G = \text{SO}(2, \mathbb{R})$, with the definition:

$$\text{SO}(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

The Lie algebra associated with this Lie group is $\mathfrak{g} = \mathfrak{so}(2)$, defined as the vector space of $2 \times 2$ skew-symmetric matrices:

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in \mathbb{R} \right\} \cong T_e(\text{SO}(2, \mathbb{R}))$$

This can be seen by considering the matrix exponential. For $A \in \mathfrak{so}(2)$:

$$\begin{aligned} \exp(tA) &= \exp\left( \begin{pmatrix} 0 & ta \\ -ta & 0 \end{pmatrix} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & ta \\ -ta & 0 \end{pmatrix}^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \begin{pmatrix} 0 & (ta)^{2k+1} \\ -(ta)^{2k+1} & 0 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \begin{pmatrix} (ta)^{2k} & 0 \\ 0 & (ta)^{2k} \end{pmatrix} \\ &= \begin{pmatrix} \cos(ta) & -\sin(ta) \\ \sin(ta) & \cos(ta) \end{pmatrix} \in \text{SO}(2, \mathbb{R}), \forall t \in \mathbb{R} \end{aligned}$$

Before considering decompositions of Lie algebras, some important representations of a Lie group and Lie algebra will be introduced. First, an automorphism of a Lie group is a map $\phi : G \to G$ that is a diffeomorphism and a group isomorphism.

Consider a Lie group $G$ and some $p \in G$. The map $I_p : G \to G$ which sends $g \in G$ to $pgp^{-1}$ is both a group homomorphism and a diffeomorphism. It is a diffeomorphism as it is the composition of left and right translations.

**Definition:** Adjoint Representation of $G$
The *adjoint representation* of $G$ is the homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ given by $\mathrm{Ad}(g) = (dI_g)_e$.

**Definition:** Adjoint Representation of $\mathfrak{g}$
The *adjoint representation* of $\mathfrak{g}$ is the homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(X) = (d\mathrm{Ad})_e(X)$.

**Remarks:**

(a) The adjoint representation of $\mathfrak{g}$ satisfies the equality $\mathrm{ad}(X)Y = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

(b) If $G$ is a matrix Lie group, then $\mathrm{Ad}(g)X = gXg^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$.

The adjoint representation of $\mathfrak{g}$ allows for an inner product to be defined on the Lie algebra $\mathfrak{g}$ which is useful for classifying different classes of Lie groups.

**Definition:** Killing Form
The *Killing form* of a Lie algebra $\mathfrak{g}$ is the function $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(X, Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$.

From now on, the Killing form of Lie group $G$ is referring to the Killing form of the associated Lie algebra $\mathfrak{g}$. Several key results will be presented which relate the properties of the Killing form to the structure of the Lie algebra which it is associated with (see [1] for proofs of result).

**Definition:** Semisimple lie Group
A Lie group $G$ is semisimple if its Killing form is non-degenerate. Thus, a semisimple Lie algebra $\mathfrak{g}$ can be thought of as one that has no proper subspaces $\mathfrak{h}$ with $[X, Y] = 0$ if $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

The following theorem relates the structure of $G$ to the value of the Killing form on $\mathfrak{g}$. The proof is quite long and so won't be presented here, see [2] for details.

**Theorem:**
Suppose $G$ is a connected Lie group. Then, $G$ is compact and semisimple if and only if the Killing form is negative definite on $\mathfrak{g}$.

Returning to Riemannian metrics on Lie groups $G$, recall that the corresponding Lie algebra is isomorphic to the tangent space of $G$ at the identity. As such, a Riemannian metric of particular interest to study is one which is entirely determined by its value at $T_eG \cong \mathfrak{g}$.

## 4.3 Left-Invariant Riemannian Metrics

**Definition:** Left-Invariant Metrics
A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Lie group $G$ is called *left-invariant* if for all $g, h \in G$ and $u, v \in T_gM$;

$$\langle u, v \rangle_g = \langle (dL_h)_g u, (dL_h)_g v \rangle_{hg}$$

An important property of left-invariant Riemannian metrics is that they are entirely determined by their value at the identity. Suppose $\langle \cdot, \cdot \rangle$ is a left-invariant Riemmanian metric, $g \in G$ and $u, v \in T_gM$. Then:

$$\begin{aligned} \langle u, v \rangle_g &= \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle_{g^{-1}g} \\ &= \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle_e \end{aligned}$$

In fact, there is a one-to-one correspondence between left-invariant metrics on a Lie group $G$ and scalar products on its Lie algebra $\mathfrak{g}$.

**Remarks:**

(a) The space of left-invariant metrics on an $n$-dimensional Lie group is $\frac{1}{2}n(n+1)$-dimensional [4].

(b) See [4] for a survey on curvatures of left invariant metrics on Lie groups, including the sectional curvature and Ricci curvature.

After considering Riemannin metrics in which left translations are isometries, it is natural to consider metrics in which both left and right translations are isometries. These metrics are called bi-invariant.

### 4.4 Bi-Invariant Riemannian Metrics

**Definition:** Bi-Invariant Metrics
A Riemannian metric that is both left-invariant and right-invariant is called a *bi-invariant* Riemannian metric.

**Proposition:** Consider a Lie group $G$ with corresponding Lie algebra $\mathfrak{g}$ and suppose that $\langle \cdot, \cdot \rangle$ is a bi-invariant Riemannian metric on $G$. Then, for all $X, Y, Z \in \mathfrak{g}$:

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

This proposition allows for many curvature computations of bi-invariant metrics to be simplified.

**Proposition:**
Let $G$ be a Lie group with a bi-invariant metric. Then, for all $X, Y, Z \in \mathfrak{g}$:

(a) The Levi-Civita connection is given by:

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

(b) The Riemann curvature tensor is given by

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$$

(c) The Ricci curvature is given by

$$\text{Ric}(X, Y) = \frac{1}{4} \sum_i \langle [X, E_i], [Y, E_i] \rangle$$

where $\{E_i\}$ is an orthonormal basis for $\mathfrak{g}$.

**Proof:**

(a) As $\langle X, Y \rangle$ is constant (from left-invariance), $Z\langle X, Y \rangle = 0$ and so the Koszul formula simplifies to:

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle$$

As $\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$, this expression can be simplified to:

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \langle [X, Y], Z \rangle$$

From this, it follows that $\nabla_X Y = \frac{1}{2}[X, Y]$.

(b) Recall that the Riemann curvature tensor is given by:

$$\begin{aligned} R(X,Y)Z &= \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z \\ &= \frac{1}{2}[[X,Y],Z] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] \\ &= -\frac{1}{2}[Z,[X,Y]] - \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[Z,X]] \\ &= -\frac{1}{4}[Z,[X,Y]] \\ R(X,Y)Z &= \frac{1}{4}[[X,Y],Z] \end{aligned}$$

(c) Let $\{E_i\}$ be an orthonormal basis for $\mathfrak{g}$. Then:

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \mathrm{tr}\{Z \mapsto R(X,Z)Y\} \\ &= \sum_i \langle R(X,E_i)Y, E_i\rangle \\ &= \frac{1}{4}\sum_i \langle [[X,E_i],Y],E_i\rangle \\ \mathrm{Ric}(X,Y) &= \frac{1}{4}\sum_i \langle [X,E_i],[Y,E_i]\rangle \end{aligned}$$

□

Whilst these curvature computations are greatly simplified, the space of bi-invariant metrics is relatively sparse. For example, the space of bi-invariant metrics on a compact Lie group with simple Lie algebra is 1-dimensional. An important result from [4] states that bi-invariant metrics exists only on groups isomorphic to a direct product of a compact Lie group and a vector space, viewed as an abelian group.

# 5 Homogeneous Spaces

This section gives an introduction to coset manifolds, homogeneous spaces and reductive homogeneous spaces, with reference to [1, 2, 3]. The geometry of reductive homogeneous spaces will then be explored.

## 5.1 Homogeneous Spaces

Let $G$ be a Lie group and $K$ a closed subgroup of $G$. The set of all left cosest of $K$ in $G$, $G/K = \{gK : g \in G\}$, can be endowed with the structure of a smooth manifold. Let $\pi : G \to G/K$ denote the projection that maps $g \in G$ to the coset $gK$. Henceforth, $e$ will be used to denote the identity element in $G$ and $o$ will be used to denote the identity coset $eK = K$.

**Lemma:** Let $G$ be a Lie group and $K$ a closed subgroup of $G$. Then, there is a unique way to make $G/K$ a manifold so that the projection $\pi$ is a submersion. See [10] for a detailed proof of this result. A manifold $G/K$ constructed in this way is called a coset manifold.

**Definition:** Left Action
A *left action* of a group $G$ on a manifold $M$ is a smooth map $\lambda : G \times M \to M$ such that for all $a, b \in G$ and $m \in M$:

(a) $\lambda(e,m) = m$, where $e$ is the identity element of $G$,

(b) $\lambda(ab,m) = \lambda(a, \lambda(b,m))$.

A right action is defined analogously.

From this point, $\lambda(a, m)$ will be denoted $a \cdot m$. Also, $G$ will be a Lie group and $M$ will be a smooth manifold. Suppose $\lambda$ is an action of $G$ on $M$ and denote $\lambda_a : M \to M$ by $\lambda_a(m) = \lambda(a, m)$. For all $a \in G$, $\lambda_a$ is a diffeomorphism of $M$. In this way, $G$ can be conceptualised as a group of 'transformations' (diffeomorphisms) of $M$. A transitive action is an action such that for all $m, n \in M$, there exists $g \in G$ such that $g \cdot m = n$. This property allows an action to map any two points on a manifold to each other.

**Definition:** Isotropy Group and Orbit
Suppose $m \in M$. The set $G_m = \{g \in G \,|\, g \cdot m = m\} \subseteq G$ is called the isotropy group at $m$. The set $G \cdot m = \{g \cdot m \,|\, g \in G\} \subseteq M$ is called the orbit of a point $m \in M$.

**Proposition:** Let $G \times M \to M$ be a transitive action of a Lie group $G$ on a manifold $M$. Let $K = G_m$ be the isotropy subgroup of a point $m$. Then:

(a) The subgroup $K$ is a closed subgroup of $G$,

(b) The natural map $n : G/K \to M$ given by $n(gK) = g \cdot m$ is a diffeomorphism,

(c) The dimension of $G/K$ is $\dim G - \dim K$.

**Definition:** Homogeneous Space
A homogeneous space is a manifold, $M$, with a transitive action of a Lie group $G$. Equivalently, it is a manifold of the form $G/K$, where $G$ is a Lie group and $K$ is a closed subgroup.

Multiple Lie groups may act transitively on a homogeneous space. As such, a manifold may be a homogeneous space under different groups, as can be seen in the examples below.

**Examples:**

(a) All Lie groups $G$ are homogeneous spaces. This can be seen in multiple ways, including $G = G/\{e\}$ and $G = G \times G/G$.

(b) The $n$-sphere, $S^n$, is a homogeneous space as $S^n$ is diffeomorphic to $\mathrm{SO}(n+1)/\mathrm{SO}(n)$.

## 5.2 Reductive Homogeneous Spaces

Suppose $G/K$ is a homogeneous space and recall the projection $\pi : G \to G/K$ given by $\pi(g) = gK$. Consider the differential $d\pi_e : \mathfrak{g} \to T_o(G/K)$. Let $X \in \mathfrak{g}$ and $\exp tX$ be the corresponding one-parameter subgroup. It then follows that:

$$d\pi_e(X) = \frac{d}{dt}(\pi \circ \exp tX)\bigg|_{t=0} = \frac{d}{dt}\left((\exp tX)K\right)\bigg|_{t=0}$$

Notice that $d\pi_e(X) = 0$ iff $X \in \mathfrak{k}$, and so $\ker(d\pi_e) = \mathfrak{k}$. Since $d\pi$ is onto, the following canonical isomorphism is obtained:

$$\mathfrak{g}/\mathfrak{k} \cong T_o(G/K)$$

For any $X \in \mathfrak{g}$, a vector field $X^*$ on $G/K$ can be defined by:

$$X^*_{gK} = \frac{d}{dt}(\exp tX)gK\bigg|_{t=0}$$

From now on, let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$ respectively.

**Definition:** Reductive Homogeneous Space
A homogeneous space $G/K$ is *reductive* if there exists a suspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$ ($\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant).

**Remarks:**

(a) The subspace $\mathfrak{m}$ does not need to be closed under the Lie bracket, as $\mathfrak{k}$ does.

(b) $\mathrm{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ implies that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. The converse is true if $K$ is connected.

(c) If $G$ is a compact Lie group, then $G/K$ is reductive as the subspace $\mathfrak{m}$ can be taken as $\mathfrak{m} = \mathfrak{k}^{\perp}$ with respect to an Ad-invariant inner product on $\mathfrak{g}$.

As such, if $G/K$ is reductive then the following isomorphism is obtained:

$$\mathfrak{m} \cong T_o(G/K)$$

**Example:** $G/K = \mathrm{SO}(3)/\mathrm{SO}(2)$

Then, the Lie algebras for $G$ and $K$ are then $\mathfrak{g} = \mathfrak{so}(3)$ and $\mathfrak{k} = \mathfrak{so}(2)$ respectively. Note that:

$$\mathfrak{k} = \mathfrak{so}(2) \cong \begin{pmatrix} 0 & 0_2^T \\ 0_2 & \mathfrak{so}(2) \end{pmatrix}, \qquad 0_2 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then, $\mathfrak{g}$ can be expressed as the direct sum of $\mathfrak{k}$ and the subspace $\mathfrak{m}$ defined as:

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & b & c \\ -b & 0 & 0 \\ -c & 0 & 0 \end{pmatrix} : b, c \in \mathbb{R} \right\}$$

$G = \mathrm{SO}(3)$ is a compact Lie group and so the subspace $\mathfrak{m}$ can be defined as $\mathfrak{k}^{\perp}$ with respect to an Ad-invariant inner product on $\mathfrak{g} = \mathfrak{so}(3)$, such as the Killing form. The Killing form of $\mathfrak{so}(3)$ is $B(X, Y) = \mathrm{tr}(XY)$ and therefore $\mathfrak{m} = \mathfrak{k}^{\perp} = \{A \in \mathfrak{g} = \mathfrak{so}(3) : \mathrm{tr}(AB) = 0, B \in \mathfrak{so}(2)\}$.

Quick computations verify that $\mathfrak{m}$ as defined above satisfies $\mathfrak{m} = \mathfrak{k}^{\perp}$. It can then be seen that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Note that in this example $\mathfrak{m}$ is not closed under the Lie bracket, an example is given below. As such, $\mathfrak{m}$ is not a Lie subalgebra.

$$\begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ab \\ 0 & ab & 0 \end{pmatrix} \notin \mathfrak{m}$$

To show that $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant, recognise that:

$$K = \mathrm{SO}(2) \cong \begin{pmatrix} 1 & 0_2^T \\ 0_2 & \mathrm{SO}(2) \end{pmatrix}$$

To demonstrate that $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$, suppose that $m \in \mathfrak{m}$ and $k \in K$:

$$\begin{aligned} kmk^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & a\cos(\theta) - b\sin(\theta) & a\sin(\theta) + b\cos(\theta) \\ b\sin(\theta) - a\cos(\theta) & 0 & 0 \\ -a\sin(\theta) - b\cos(\theta) & 0 & 0 \end{pmatrix} \in \mathfrak{m} \end{aligned}$$

Therefore, $G/K = \mathrm{SO}(3)/\mathrm{SO}(2)$ is a reductive homogeneous space.

## 6 Naturally Reductive Riemannian Metrics

This section provides an introduction to naturally reductive metrics on homogeneous spaces and explores some key theorems of Kostant, D'Atri, Ziller and Gordon which provide insight into the properties of such metrics. Finally, the computation of the Ricci curvature for naturally reductive metrics is discussed.

### 6.1 Definition

Suppose $(M = G/K, g)$ is a reductive homogeneous Riemannanian manifold with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Recall that $\mathfrak{m}$ is identified with the tangent space at the identity $T_o(G/K)$.

**Definition:** Naturally Reductive Homogeneous Manifold
A reductive homogeneous Riemannian manifold, $(M = G/K, \langle\cdot,\cdot\rangle)$, is said to be *naturally reductive* (with respect to $G$ and the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$) if

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y,Z]_{\mathfrak{m}}\rangle$$

for all $X, Y, Z \in \mathfrak{m}$, where $U_{\mathfrak{k}}$ and $U_{\mathfrak{m}}$ denote the $\mathfrak{k}$ and $\mathfrak{m}$ components of $U \in \mathfrak{g}$.

Recall that $\mathfrak{m}$ is not necessarily a Lie subalgebra. If $\mathfrak{m}$ is not a Lie algebra, then $[\mathfrak{m},\mathfrak{m}] \not\subset \mathfrak{m}$. Informally, a naturally reductive metric can be seen as bi-invariant on the projection onto the subspace $\mathfrak{m}$. In this way, we can view naturally reductive metrics as an intermediary between left-invariant and bi-invariant metrics.

**Examples:**
Classical examples of naturally reductive homogeneous spaces include irreducible symmetric spaces, isotropy-irreducible homogeneous manifolds, Lie groups with a bi-invariant metric, and Riemannian 3-symmetric spaces. [5] gives a classification of naturally reductive metrics in dimension less than or equal to 6 and [11] gives a classification of 7 and 8 dimensional naturally reductive spaces.

### 6.2 Kostant's Theorem

A key theorem involving naturally reductive metrics is Kostant's theorem. This result was first proven for $M$ compact by Kostant and D'Atri/Ziller demonstrated that compactness was not required. See [6] for a proof of this theorem.

**Kostant's Theorem:**
If $M$ is naturally reductive with respect to $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, then $\bar{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m},\mathfrak{m}]$ is a $\mathfrak{g}$-ideal, the corresponding connected subgroup $\bar{G} \subset G$ is transitive on $M$, and there exists a unique $\mathrm{Ad}(G)$-invariant symmetric non-degenerate bilinear form $Q$ on $\mathfrak{g}$ such that $Q(\bar{\mathfrak{g}} \cap \mathfrak{k}, \mathfrak{m}) = 0$ and $Q_{|\mathfrak{m}}$ is the inner product induced by the Riemannian metric.
Conversely, if $M = G/K$ with $G$ connected, then for an $\mathrm{Ad}(G)$-invariant, symmetric, bilinear form $Q$ on $\mathfrak{g}$, which is non-degenerate on both $\mathfrak{g}$ and $\mathfrak{k}$ and positive-definite on $\mathfrak{m} = \mathfrak{k}^{\perp}$, the metric on $M$ defined by $\langle\cdot,\cdot\rangle = Q_{|\mathfrak{m}}$ is naturally reductive.

Kostant's theorem implies that all naturally reductive metrics on a homogeneous space $M$ can be found by finding all transitive groups $G$ and, for all $\mathrm{Ad}(G)$-invariant forms $Q$ on $\mathfrak{g}$, determining $\mathfrak{m} = \mathfrak{k}^{\perp}$ and $Q_{|\mathfrak{m}}$ on $T_eM$. Unfortunately, finding all such transitive groups is generally quite a difficult problem.

Kostant's theorem has allowed for the form of all naturally reductive metrics to be classified on both compact [6] and non-compact groups [7]. First, we consider the compact case.

## 6.3 Compact Case

**Theorem (D'Atri/Ziller):**
For any compact Lie group $G$ with bi-invariant metric $g$ and connected subgroup $K$, the following left-invariant metrics:

$$\langle\cdot,\cdot\rangle = \alpha g_{|\mathfrak{a}} + h_{|\mathfrak{k}_0} + \alpha_1 g_{|\mathfrak{k}_1} + \cdots + \alpha_r g_{|\mathfrak{k}_r}$$

are naturally reductive with respect to $G \times K$. Note that $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, $\mathfrak{k}_0$ is the centre of $\mathfrak{k}$, $\mathfrak{k}_i$ simple ideals, $h$ an arbitrary inner product on $\mathfrak{k}_0$ and $\mathfrak{a} = \mathfrak{k}^\perp$. Further, a left-invariant metric on a compact simple Lie group which is naturally reductive is of the above form.

**Remarks:**

(a) As $G$ is assumed to be a compact Lie group, a bi-invariant metric $g$ is guaranteed to exist.

(b) $h$ can always be diagonalized in a $g$-orthonormal basis of $\mathfrak{k}_0 = z(\mathfrak{k})$, giving the further splitting $\mathfrak{k}_0 = \mathfrak{k}_{r+1} \oplus \cdots \mathfrak{k}_{r+s}$. The metric $\langle\cdot,\cdot\rangle$ can then be written in the form:

$$\langle\cdot,\cdot\rangle = \alpha g_{|\mathfrak{a}} + \alpha_1 g_{|\mathfrak{k}_1} + \cdots + \alpha_{r+s} g_{|\mathfrak{k}_{r+s}}$$

D'Atri and Ziller also computed the Ricci curvature for naturally reductive metrics on compact Lie groups [6].

**Theorem:**
Suppose $G$ is a compact Lie group with bi-invariant metric $g$ and connected subgroup $K$. Let $\langle\cdot,\cdot\rangle$ be of the form given above. Let $B$ denote the Killing form of $\mathfrak{g}$ and $B_i$ denote the Killing form of $\mathfrak{k}_i$. As $\mathfrak{k}_i$ is simple for $i = 1, \ldots, r$, $B_i = c_i B$ with $c_i > 0$. Also, note that $c_i = 0$ for $i = 0$ or $i > r$. Then, the Ricci curvature is given by:

$$\begin{aligned}
\mathrm{Ric}(\mathfrak{a}, \mathfrak{k}_i) &= \mathrm{Ric}(\mathfrak{k}_i, \mathfrak{k}_j) = 0 && 1 \le i, j \le r+s \\
\mathrm{Ric}_{|\mathfrak{k}_j} &= -\frac{4}{\alpha^2}\left((\alpha^2 - \alpha_j^2)c_j + \alpha_j^2\right) B_{|\mathfrak{k}_j} && 1 \le j \le r+s \\
\mathrm{Ric}_{|\mathfrak{a}} &= \left(-\frac{1}{2}\sum_{i=1}^{r+s}\left(\frac{\alpha_i}{\alpha} - 1\right)\frac{d_i(1-c_i)}{n} + \frac{1}{4}\right) B_{|\mathfrak{a}}
\end{aligned}$$

where $d_i = \dim \mathfrak{k}_i$ for $i = 1, \ldots, r+s$ and $n = \dim \mathfrak{a}$.

This computation for the Ricci curvature of compact Lie groups $G$ has been used to compute the scalar curvature [8], find Einstein metrics [6] and address the Prescribed Ricci Curvature problem [8]. With such results being determined in the compact case, it is natural to then consider naturally reductive metrics in the non-compact case.

## 6.4 Non-compact Case

As in the compact case, Gordon utilised Kostant's theorem to classify all naturally reductive metrics on non-compact Lie groups. However, two results must be stated to obtain the form of all such metrics. The proofs can be seen in [7].

**Theorem (Gordon):**
A left-invariant Riemannian metric on a homogeneous space $G/L$ of a connected semisimple Lie group $G$ of non-compact type is naturally reductive if and only if the following conditions are satisfied:

(a) Let $\mathfrak{k}$ be any maximal compactly embedded subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$. Then $L$ is a normal subgroup of the corresponding subgroup $K$ of $G$.

(b) The Riemannian metric is $\mathrm{Ad}(K)$-invariant for $K$ as in (a).

(c) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition and let $\mathfrak{f}$ be a $\mathfrak{k}$-ideal complementary to $\mathfrak{l}$. Relative to the inner product $\langle,\rangle$ on $\mathfrak{f} + \mathfrak{p}$ induced by the Riemannian metric, $\mathfrak{f} \perp \mathfrak{p}$.

Note that $\mathfrak{f} + \mathfrak{p}$ is naturally identified with the tangent space of $G/L$ at the base point. When these conditions hold, the metric is naturally reductive with respect to $I_0(M) = G \times F/D$, where $F$ is the connected subgroup of $G$ with Lie algebra $\mathfrak{f}$ and $D$ is discrete. The metric is not naturally reductive with respect to any proper subgroup of $I_0(M)$.

Through an equivalence theorem, Gordon classified the form of all naturally reductive metrics on non-compact groups with respect to decompositions of the subspaces $\mathfrak{p}$ and $\mathfrak{f}$.

**Theorem (Gordon):**
Assume the notation from the above theorem and suppose that $L$ satisfies condition (a). Let

$$g = g_{(1)} \oplus \cdots \oplus g_{(n)}$$

be the decomposition of $\mathfrak{g}$ into simple ideals, let

$$\mathfrak{p}_{(j)} = \mathfrak{p} \cap \mathfrak{g}_{(j)}, \qquad j = 1, \ldots, n$$

and let

$$\mathfrak{f} = \mathfrak{f}_{(1)} \oplus \cdots \oplus \mathfrak{f}_{(r)} \oplus z(\mathfrak{f})$$

be the decomposition of $\mathfrak{f}$ into simple ideals and center $z(\mathfrak{f})$. A left-invariant Riemannian metric on $G/L$ satisfies conditions (b) and (c) if and only if $\mathfrak{f}_{(i)}$, $(1 \leq i \leq r)$, $\mathfrak{p}_{(j)}$, $(1 \leq j \leq n)$ and $z(\mathfrak{f})$ are pairwise orthogonal and

$$\langle \cdot, \cdot \rangle = -\alpha_1 B_{|_{\mathfrak{f}_{(1)}}} - \cdots - \alpha_r B_{|_{\mathfrak{f}_{(r)}}} + \beta_1 B_{|_{\mathfrak{p}_{(1)}}} + \cdots + \beta_n B_{|_{\mathfrak{p}_{(n)}}} + A_{z(\mathfrak{f})}$$

where $B$ is the Killing form of $\mathfrak{g}$, $\alpha_i$ and $\beta_j$ are positive constants and $A$ is any inner product on $z(\mathfrak{f})$.

In [7], the sectional curvature of such a metric is given for orthonormal vectors, however the Ricci curvature was not computed. Recently, [9] computes the Ricci curvature in the case where the Lie group $G$ is simple however the computation has not been done when the simplicity assumption is dropped. [9] then uses this result to address the Prescribed Ricci Curvature problem on such spaces.

## 7 Conclusion

This report has aimed to introduce relevant theory in differential geometry and homogeneous spaces to elucidate why naturally reductive metrics are an interesting class of metrics to study. Specific results about such metrics, including the classification results of D'Atri, Ziller and Gordon, were then presented.

In particular, recent work involving the computation of the Ricci curvature of naturally reductive metrics on simple, non-compact Lie groups was briefly mentioned. Such computations have not been made where the simplicity assumption is dropped. I aim to determine the computation of Ricci curvature and consider the Prescribed Ricci Curvature problem for the more general case in an Honours thesis.

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