Length of Embedded Circuits in Geodetic Graphs

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Abstract

We attempt to extend a result in graph theory concerning geodetic graphs, recently proven by Elder and Piggott, to prove that if a geodetic graph has an embedded circuit of diameter at least 4 then a minimal length example of such an embedded circuit contains a geodesic subpath of length at least 4. Also, we present a program to enumerate all geodetic graphs and geodetic blocks up to size n.

1 Introduction

Since the 1980s, mathematicians have tried to characterise algebraically the families of groups that may be presented by various families of rewriting systems. This includes characterising the groups that may be presented by *finite length-reducing rewriting systems* [1]. If a group is presented by a finite convergent length-reducing rewriting system, then the corresponding Cayley graph is *geodetic*. So there is a strong connection between these rewriting systems and the problem of understanding and classifying geodetic graphs. (See [2] and Appendix A for all definitions used in this section).

The problem of constructing a general classification of all finite geodetic graphs, originally presented by Ore [3] nearly six decades ago, has proven very challenging. Although various advances have been made, a solution is still elusive.

Of relevance to the above problems is a new result in graph theory recently proved by Elder and Piggott [2] about locally-finite undirected simple geodetic graphs: if all isometrically embedded circuits of such a graph have length at most 5, then the diameter of any embedded circuit in it is at most 2. The first major part of this result is the following: if a geodetic graph has an embedded circuit of diameter at least 3 then a minimal length example of such an embedded circuit contains a geodesic subpath of length at least 3. This is Lemma 6 in [2]. Here we consider the next case up: if a geodetic graph has an embedded circuit of diameter at least 4 then a minimal length example of such an embedded circuit contains a geodesic subpath of length at least 4. Our approach is to start by making various assumptions about the embedded circuit without loss of generality, then going through each case of positions of vertices on the circuit and arriving at a contradiction. In contrast to [2, Lemma 6], the number of cases explodes, and further assumptions without loss of generality were introduced in an attempt to bring the number of cases under control.

While we were not able to complete this proof due to the unexpected large number of cases that arose, we present an almost complete proof which contains just a few outstanding cases. We believe with more work these cases can be eliminated and the result will be proved. No counterexample has been uncovered.

Also, we present a program to enumerate all geodetic graphs up to n vertices, as well as all biconnected geodetic graphs, by first generating non-isomorphic (biconnected) graphs using Brendan McKay's nauty program, then checking the geodetic property on this output.

Statement of Authorship

The workload of this project was divided as follows:

- John Cu wrote the enumeration program and produced this report.
- Professor Elder devised the main ideas and outline of the project, supervised the project, and proof-read this report.
- Both worked on the cases, continually cross-checking each other's progress, and discussing improvements and new strategies to cut down the number of cases.

2 Length of Embedded Circuits in Geodetic Graphs

Once again we refer the reader to Appendix A for all definitions used here. We make use of the following fact due to Stemple.

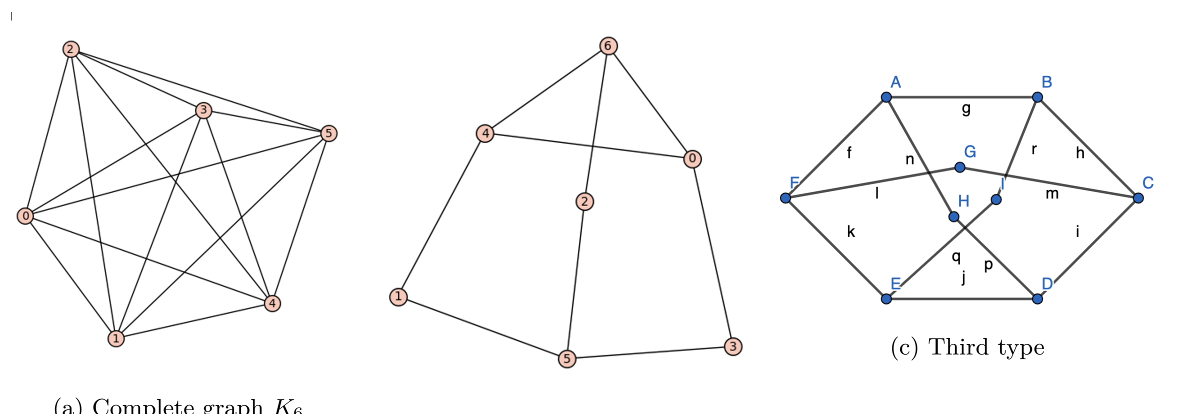
Lemma 1 ([4, Theorems 3.3 and 3.4])**.** *Let Γ be a simple, undirected, geodetic graph. If Γ contains an embedded circuit*

$$w_0, w_1, w_2, w_3, w_0$$

of length four, then these vertices lie in a complete subgraph of Γ. If Γ contains an embedded circuit

$$w_0, w_1, w_2, w_3, w_4, w_5, w_0$$

of length six, then the subgraph consisting of the circuit together will all geodesic paths in Γ connecting w_i to w_j is either (i) a complete graph (ii) $(K_4)_{0,0,0,1}$ (iii) third type.



(a) Complete graph K_6

(b) $(K_4)_{0,0,0,1}$

(c) Third type

Figure 1: Cases for 6-cycles in Lemma 1.

Corollary 2 (Stemple for 5-cycles)**.** *Let Γ be a simple, undirected, geodetic graph. If Γ contains an embedded circuit*

$$w_0, w_1, w_2, w_3, w_4, w_0$$

of length five, then these vertices either lie in a complete subgraph of Γ, or else is an isometrically embedded circuit.

Proof. Suppose it is not an isometrically embedded circuit. Then there is an edge from w_i to $w_{i+2 \mod 5}$. But this makes a 4-cycle so by Lemma 1 this must be filled by a K_4, and then this means lots more 4-cycles which get filled, and we get a K_5. □

Notation 3. Let Γ be a geodetic graph with vertices $u_i, i \in I$ some index set and a chosen basepoint $1 = u_i$. We say a vertex u_j is in *level* r if $d(1, u_j) = r$. The notation $(u_j)_r$ denotes the unique level r vertex on the geodesic from 1 to u_j.

Here is the lemma we attempted to prove. While there remain unfinished cases, we believe with more time we can eliminate these and obtain the result.

Lemma 4. *Let Γ be a locally-finite simple geodetic graph. If ρ is an embedded circuit of diameter exceeding three and that has minimal length among all such paths in Γ, then ρ contains a geodesic subpath of length four.*

Proof. Let ρ be an embedded circuit of diameter exceeding three and that has minimal length among all such paths in Γ. Suppose that ρ does not contain a geodesic subpath of length 4. Since ρ has diameter at least four, there exist vertices 1 and x visited by ρ such that $d(1, x) = 4$[1]. Now I will make an extra assumption. Assume that $1, x$ is chosen so that the number of edges along ρ between them is minimal. This assumption is valid since you can consider all pairs of vertices a, b on ρ with $d_\Gamma(a, b) = 4$ and so that the length of the (shorter) subpath of ρ joining them is minimised over all such (a, b). We choose a basepoint (the vertex 1), an orientation of ρ, and label the vertices visited by ρ in order

$$1, u_1, u_2, \ldots, u_m = x = v_n, v_{n-1}, \ldots, v_1, 1.$$

For each vertex $w \in \Gamma$, we say that w is in *level* $d(w, 1)$.

Note that $m, n \geq 4$ since $d(1, x) = 4$. If $m = 4$ or $n = 4$ then ρ has a geodesic subpath of length 4 which contradicts our assumption. So $m, n > 4$. Also by our new assumption in red, setting $k = \min\{m, n\}$ we know that $u_4, \ldots, u_{k-1}, v_4, \ldots, v_{k-1}$ are in level at most 3. This is because if not we could find another pair $1', x'$ that have a shorter arc between them along ρ

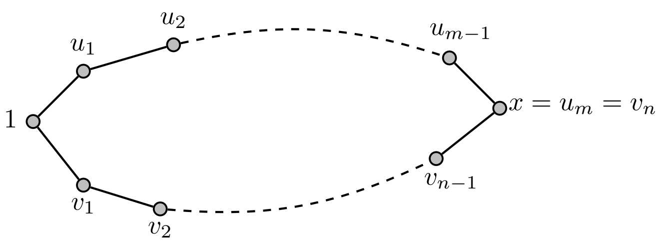


Figure 2: The embedded circuit ρ in Lemma 4.

We note that, since ρ is an embedded circuit, the vertices

$$1, \ldots u_{m-1}, v_1, \ldots, v_{n-1}, x$$

[1]Proof: if there are vertices $1, y$ with $d(1, y) > 4$, then move edge-by-edge along ρ from y towards 1, each step the distance changes by $0, 1$ or -1. So the existence of such an x is guaranteed.

are distinct. Since 1 and u_1 are distinct, u_1 is in level 1. Similarly v_1 is in level 1.

Claim 0: u_i, v_j are in level at least 2 for $2 \leq i \leq m, 2 \leq j \leq n$.

Suppose u_i is in level 1 for $1 \leq i < m$, then replace the path $2, u_1, \ldots, u_{i-1}, u_i$ by $1, u_i$ to obtain a shorter embedded circuit that still visits $x = u_m = v_n$. Similarly if v_i is in level 1, replace $1, \ldots, v_i$ by $1, v_i$ to obtain a shorter path that visits x.

It now follows that with $k = \min\{m, n\}$ we know that $u_4, \ldots, u_{k-1}, v_4, \ldots, v_{k-1}$ are either in level 2 or 3.

Claim 1: The geodesic from 1 to x of length 4 is either

- $1, u_1, p, v_{n-1}, x$
- $1, u_1, u_2, v_{n-1}, x$
- $1, u_1, v_{n-2}, v_{n-1}, x$
- $1, u_1, v_j, u_{m-1}, x$
- $1, v_1, p, u_{m-1}, x$
- $1, v_1, v_2, u_{m-1}, x$
- $1, v_1, u_{m-2}, u_{m-1}, x$
- $1, v_1, u_i, v_{n-1}, x$

where p is a vertex that does not lie on ρ and $2 \leq i \leq m-1, 2 \leq j \leq n-1$. [2]

Proof of Claim 1: Suppose the geodesic γ from 1 to x does not visit any v_j vertices except $x = v_n$. Then we can replace $1, u_1, \ldots, u_{m-1}, x$ by γ and obtain an embedded circuit $\gamma, v_{n-1}, \ldots, v_1, 1$ which visits 1 and x and is shorter than ρ (since $m > 4$ and $|\gamma| = 4$). This contradicts the choice of ρ. Similarly if γ does not visit any u_i vertices except $x = u_m$, then we can replace $1, v_1, \ldots, v_{m-1}, x$ by γ and find an embedded circuit shorter than ρ which visits 1 and x.

Thus γ must visit both u_i and v_j for some $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$.

Suppose the first vertex on γ after 1 is not u_1 or v_1. Then by Claim 0 it does not lie on ρ at all. Then $\gamma = 1, p, u_i, v_j, x$ or $\gamma = 1, p, v_j, u_i, x$. If $\gamma = 1, p, u_i, v_j, x$ and $i > 2$ then we can replace $1, u_1, u_2, \ldots, u_i$ by $1, p, u_i$ and obtain an embedded circuit shorter than ρ. If $i = 2$ then we have two distinct geodesics from 1 to u_2 (via u_1 or p) contradicting geodecity. Thus $i = 1$ but this is a contradiction since u_1 is in level 1 and $1, p, u_i$ is a geodesic. Similarly $\gamma = 1, p, v_j, u_i, x$ is not possible. So the first vertex on γ after 1 is u_1 or v_1.

Next we show the last vertex on γ before x is either u_{m-1} or v_{n-1}. If γ contains u_i, p, x, a geodesic of length 2, then $i \neq m-1$ since u_{m-1}, x is an edge. If $i < m-2$ we can replace $u_i, u_{i+1}, \ldots u_{m-1}, x$ by u_i, p, x and obtain a shorter embedded circuit. Thus $i = m-2$ but then this gives two distinct geodesics of length 2 from u_{m-2} to x. Similarly γ cannot contain v_j, p, x.

Lastly, if γ contains two adjacent vertices u_i, u_k (resp. v_j, v_k) then $k - i = 1$ (resp. $k - j = 1$) otherwise we could replace $u_i, \ldots, u_k$ by a single edge and obtain an embedded circuit shorter than ρ (resp. replace $v_j, \ldots v_k$). This gives us the cases remaining as shown in the claim (noting that $x = u_m = v_n$). □

[2](Note this list does not include $1, v_1, v_2, v_3, x = v_4$ or $1, u_1, u_2, u_3, x = u_4$ since we already assumed if $m, n = 4$ then we are done.)

Claim 2: u_2 is in level 2. Suppose that u_2 is not in level 2. Then it is either in level 0 or 1, but $u_2 \neq 1$ so it must be in level 1. This implies that u_2 is adjacent to 1, and omitting u_1 from ρ yields a shorter embedded circuit of diameter exceeding two. This contradicts the choice of ρ as a minimal length example, and hence proves that u_2 is in level 2.

A symmetric argument shows that v_2 is in level 2.

Since Γ is geodetic, u_1 is the *unique* level-1 vertex adjacent to u_2. It follows that u_3 is in level 2 or level 3. Similarly, v_3 is in level 2 or level 3.

Claim 3: At least one of u_3, v_3 is in level 3.

Suppose that u_3 and v_3 are both in level 2. Let $a = (u_3)_1$ be the unique vertex in level 1 that is adjacent to u_3; let $b = (v_3)_1$ be the unique vertex in level 1 that is adjacent to v_3. If a is not on ρ then let ρ' be obtained from ρ by replacing $1, u_1, u_2, u_3$ by $1, a, u_3$. Since ρ does not visit a, we know that ρ' is an embedded circuit. Since ρ' still visits a vertex (x) in level 4, it still has diameter at least 4, and we contradict the minimality of ρ. Similarly if b is not on ρ then let ρ' be obtained from ρ by replacing $1, v_1, v_2, v_3$ by $1, b, v_3$ to obtain a shorted embedded circuit that visits x. Thus both a, b lie on ρ and are in level 1 so they must be either u_1 or v_1 (because of Claim 0).

1. If $a = u_1$ then we can omit u_2 and obtain a shorter embedded circuit that visits x, contradicting the minimality of ρ.

2. If $b = v_1$ then we can omit v_2 and obtain a shorter embedded circuit that visits x, contradicting the minimality of ρ.

3. Else $a = v_1$ and $b = u_1$ (see Figure 3). Let ρ' be obtained from ρ by replacing $1, u_1, u_2, u_3$ by $1, v_1, u_3$, and replacing $v_3, v_2, v_1, 1$ by $v_3, u_1, 1$. Since ρ' visits only vertices visited by ρ, and 1 is the only vertex visited twice, we know that ρ' is an embedded circuit. Since the only vertices from ρ omitted were in levels 1 and 2, we know that ρ' still visits a vertex in level 4 (the vertex x), and hence it still has diameter 4.

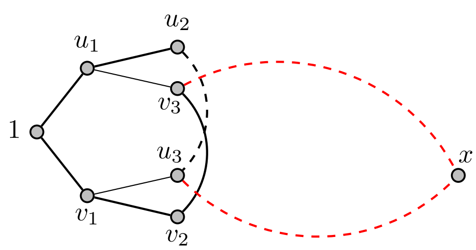


Figure 3: Case $a = v_1$ and $b = u_1$ in Lemma 4.

Thus we know that one of u_3, v_3 must be in level 3. Assume without loss of generality that v_3 is in level 3 (and u_3 is either in level 2 or 3).

Now if one of u_4, v_4 is in level 4 we have found a subpath of ρ of length 4 that is geodesic, contradicting our first assumption of the lemma. So assume neither u_4, v_4 are in level 4. Since v_3 is in level 3 and v_2 is in level 2, v_2 is the unique level 2 vertex adjacent to v_3, so we have v_4 must be in level 3. And u_3 could be in level 2 or 3. Recall u_2 is in level 2.

Case 1: First suppose u_3 in level 2.

Let $a = (u_3)_1$ be the unique vertex in level 1 that is adjacent to u_3.

If a is not on ρ then let ρ' be obtained from ρ by replacing $1, u_1, u_2, u_3$ by $1, a, u_3$. Since ρ does not visit a, we know that ρ' is an embedded circuit. Since ρ' still visits a vertex (x) in level 4, it still has diameter at least 4, and we contradict the minimality of ρ.

If ρ visits a, then either $a = u_1$ or $a = v_1$ (by Claim 0).

1. If $a = u_1$ then we can omit u_2 and obtain a shorter embedded circuit that visits x, contradicting the minimality of ρ.

2. If $a = v_1$, then let $b = (v_4)_2$ be the unique vertex in level 2 that lies on the geodesic from 1 to v_4. Either $b = u_2, b = u_3, b = v_2$, $b = u_i, 4 \leq i \leq m-2$, $v = v_i, 5 \leq i \leq n-2$ or b does not lie on ρ. (Note $v_n = x = u_m$ is in level at least 4, so $b = v_i$ and $b = u_i$ cannot be less than 2 steps aways from x. Hence the ranges shown here.)

Case 1.1 $b = u_2$. See Figure 4. An embedded circuit that visits 1 and a vertex in level 4 and is shorter than ρ can be shown. Then

$$1, v_1, u_3, \ldots, u_{m-1}, x, \ldots, v_4, u_2, u_1, 1$$

is an embedded circuit that is shorter than ρ (since it omits v_2, v_3) and visits $1, x$.

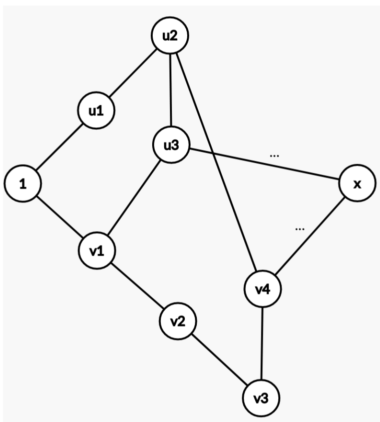


Figure 4: Case 1.1: $a = v_1$ and $b = u_2$ in Lemma 4.

Case 1.2 $b = u_3$. See Figure 5. Consider the distance in Γ from u_1 to v_3. Since there are two paths joining them of length 4, the distance is at most 3. If $d(u_1, v_3) = 1$ then we have a path of length 2 from 1

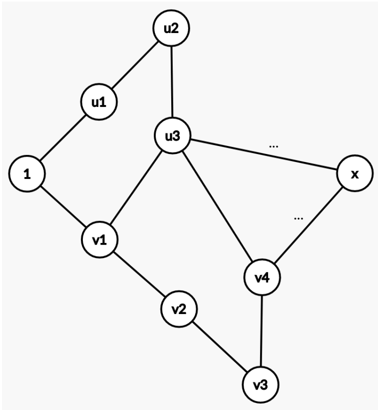


Figure 5: Case 1.2: $a = v_1$ and $b = u_3$ in Lemma 4.

to v_3, contradicting v_3 is in level 2. If $d(u_1, v_3) = 2$ then there are two paths of length 3 from 1 to v_3, contradicting geodecity of Δ. So $d(u_1, v_3) = 3$.

Now consider the geodesic from 1 to x. It is one of the 8 cases in Claim 1.

Case 1: $1, u_1, p, v_{n-1}, x$, we have $1, u_1, p, v_{n-1}, x, u_{m-1} \ldots, u_3, v_1, 1$ is shorter than ρ and an embedded circuit that visits 1 and x. (note $n > 4$ so $v_{n-1} \neq v_1$).

Case 2: $1, u_1, v_{n-2}, v_{n-1}, x$, we have $1, u_1, v_{n-2}, v_{n-1}, x, u_{m-1} \ldots, u_3, v_1, 1$ is shorter than ρ and an embedded circuit that visits 1 and x. (note $n > 4$ so $v_{n-1}, v_{n-2} \neq v_1$).

Case 3: $1, u_1, u_2, v_{n-1}, x$, we have $1, u_1, u_2, v_{n-1}, x, u_{m-1} \ldots, u_3, v_1, 1$ is shorter than ρ and an embedded circuit that visits 1 and x. (note $n > 4$ so $v_{n-1} \neq v_1$).

Case 4: $1, u_1, v_j, u_{m-1}, x$. If $j = 1$ we have a complete K_5 in the 5-cycle containing 1, so shorten u_1, u_2, u_3 to u_1, u_3. If $j = 2, 3, 4$ then $d(u_1, v_3) < 3$ contradicts the above observation. So $j \geq 5$. We have $1, u_1, v_j, v_{j+1}, \ldots, x, u_{m-1}, \ldots, u_3, v_1, 1$ is a shorter embedded circuit than ρ which visits $1, x$. (It is shorter because it misses u_2 and $v_2, \ldots, v_{j-1}$.)

Case 5: $1, v_1, p, u_{m-1}, x$. u_{m-1} cannot be u_3, u_2, u_1 since $m > 4$. Replace $1, u_1, u_2, ..., u_{m-1}$ by $1, v_1, p, u_{m-1}$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, u_3, u_2, u_1, 1$ to obtain a shorter embedded circuit that visits 1 and x.

Case 6: $1, v_1, u_{m-2}, u_{m_1}, x$. If $m - 2 > 3$, replace $1, u_1, u_2, ..., u_{m-1}$ by $1, v_1, u_{m-2}, u_{m-1}$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, u_3, u_2, u_1, 1$ to obtain a shorter embedded circuit that visits 1 and x. Else $m - 2 \leq 3$ so $4 < m \leq 5$ so $m = 5$ and $u_{m-2} = u_3$. Now note that $u_1, u_2, u_3, u_4, x = u_5$ is not a geodesic otherwise we have a geodesic subpath of ρ of length 4, contradiction – so there is a path of length ≤ 3. Let δ be this path. If δ does not visit any v_j vertex, then we can replace $1, u_1, \ldots, u_5 = x$ by δ and obtain a shorter embedded circuit. Thus δ visits at least one v_j vertex. If δ visits v_1 then it cannot be the first vertex of δ or we get a K_5 in the 5-cycle containing 1, and it can't be later since then v_1, u_3, u_4, u_5 is not geodesic. So δ does not visit v_1. Also δ does not visit 1 because it would have to be the first vertex it visits, and then $1, v_1, u_3, u_4, u_5$ would not be geodesic. So summary: δ

does not visit $1, v_1$ but must visit some $v_j, j \geq 2$. If δ visits u_3 then since u_3, u_4, x is a geodesic, and δ visits v_j, there are too many vertices for δ to be length ≤ 3. If δ visits u_4, then it must go u_1, v_j, u_4, x – so we have an embedded circuit $1, \delta, v_{n-1}, \ldots, v_4 u_3, v_1, 1$. If δ visits u_2 then it must go u_1, u_2, v_j, x so we have an embedded circuit $1, u_1, u_2, v_j, \ldots, v_{n-1}, x, u_4, u_3, v_1, 1$ which is shorter than ρ.

Thus δ does not visit any u_i vertex, and we have $1, \delta, u_4, u_3, v_1, 1$ is a shorter embedded circuit for ρ.

Case 7: $1, v_1, v_2, u_{m-1}, x$. u_{m-1} cannot be u_3, u_2, u_1 as $m > 4$. If $u_{m-1} = u_4$ then we get a 4-cycle $v_1 v_2, u_4, u_3, v_1$ so it is a K_4, contradicting that γ is geodesic. Else $m - 1 \geq 5$. Replace $1, u_1, u_2, u_3, ..., u_{m-1}, x$ by $1, u_1, u_2, u_3, v_4, \ldots, x$ and $x, \ldots, v_4, v_3, v_2, v_1, 1$ by $x, u_{m-1}, v_2, v_1, 1$ to obtain a shorter embedded circuit that visits 1 and x.

Case 8: $1, v_1, u_i, v_{n-1}, x$. If $i = 1, 2$ makes the 5-cycle at 1 a K_5 and we can shorten u_1, u_2, u_3 to u_1, u_3. If $i > 3$ then

$$1, v_1, u_i, u_{i+1}, \ldots, u_{m-1}, x, v_{n-1}, \ldots, v_4, u_3, u_2, u_1, 1$$

is a shorter embedded circuit. Thus $i = 3$.

If $m = 5$ then we have a 6-cycle $v_1, u_3, v_5, v_4, v_3, v_2, v_1$ with an edge between u_3 and v_4, so by Stemple this is filled either with K_6 (which would make v_4 in level 2, contradiction) or there is a path of length 2 from v_2 to v_5 If the midpoint of that path of length 2 is u_1 or u_2 then

$$1, (u_1, u_2), v_5, v_6, \ldots, x, u_{m-1}, \ldots, u_3, v_1, 1$$

is an embedded circuit shorter than ρ, if the midpoint is 1 then v_2 is in level 1 which is a contradiction, else if it is $u_i, i > 3$ then we have an embedded circuit

$$1, v_1, v_2, u_i, u_{i+1}, \ldots, x, v_{n-1}, \ldots v_5, u_3, u_2, u_1, 1$$

which is shorter than ρ, else $v_j, j > 5$ then replace $v_2, \ldots v_j$ by v_2, v_j, and otherwise it is not on ρ so we can shorten v_2, v_3, v_4, v_5 to v_2, p, v_5.

Finally, if $m > 5$ consider the geodesic from u_1 to v_3. Call it $\delta = u_1, p, q, v_3$.

i. If $p = v_1, u_3$ then the top 5-cycle is filled by K_5, contradicting u_2 is in level 2.

ii. If $p = 1$ then v_3 is in level 2, contradiction.

iii. If $p = v_2, v_4$ then $d(u_1, v_3) = 2$ contradiction.

iv. If $p = u_i, i \geq 4$ then replace $u_1, u_2, \ldots, u_i$ by u_1, u_i and shorten ρ.

v. If $p = v_j, j \geq 5$ then $1, u_1, v_j, v_{j+1}, \ldots, x, u_{m-1}, \ldots, u_3, v_1, 1$ is shorter than ρ.

vi. If $p = u_2$ then $d(u_2, v_3) = 2$, and q is the vertex in the middle of this geodesic.

   A. If $q = v_4$ then we get a shorter embedded circuit

$$1, u_1, u_2, v_4, \ldots x, u_{m-1}, \ldots, u_3, v_1, 1$$

   B. If $q = v_2$ then u_2, v_2, v_1, u_3, u_2 is a 4-cycle so is filled by K_4 and we get two paths of length 2 from 1 to u_2, contradiction.

C. If $q = u_3$ then there is a 4-cycle v_1, u_3, v_3, v_2, v_1 which is filled by K_4 and contradicts v_3 in level 3.

D. q cannot equal $1, u_1, v_1$ since q is adjacent to v_3 in level 3.

E. If $q = u_i, i \geq 4$ then we can replace $u_2, u_3, \ldots, u_i$ by u_2, u_i

F. if $q = v_j, j \geq 4$ we get an embedded circuit

$$1, u_1, u_2, v_j, v_{j+1} \ldots, x, \ldots u_3, v_1, 1$$

G. Else q does not lie on ρ and we get an embedded circuit

$$1, u_1, u_2, q, v_3, v_4, \ldots, x, \ldots u_3, v_1, 1$$

(missing v_2 so shorter by one)

vii. Else p does not lie on ρ.

A. If $q = u_3$ then we get two geodesics of length 2 between u_1 and u_3, contradiction.

B. q cannot equal $1, u_1, v_1$ since q is adjacent to v_3 in level 3.

C. If $q = u_i, i \geq 4$ then we can replace $u_1, u_2, u_3, \ldots, u_i$ by u_1, p, u_i which is shorter

D. if $q = v_j, j \geq 4$ we get an embedded circuit

$$1, u_1, p, v_j, v_{j+1} \ldots, x, \ldots u_3, v_1, 1$$

E. If q does not lie on ρ and we get an embedded circuit

$$1, u_1, p, q, v_3, v_4, \ldots, x, \ldots u_3, v_1, 1$$

(missing v_2 so shorter by one)

F. If $q = v_2$ then

$$1, u_1, u_2, u_3, v_1, v_2, p, 1$$

is a 6-cycle, so Stemple applies. Stemple (I) produces two paths of length 2 from 1 to v_2 (additional path through u_1), contradicting it being in level 2.

For Stemple type II, there is a path of length 2 already from u_1 to v_1 via 1, impliying either an edge from p to u_3 or from u_2 to v_2. If there is an edge from p to u_3, u_1, u_2, u_3, p is a 4-cycle, so by Stemple this must be K_4, giving an additional path of length 2 from 1 to u_3 through u_1, contradicting it being in level 2.If there is an edge from u_2 to v_2, u_1, u_2, v_2, p is a 4-cycle, so by Stemple this must be K_4, giving an additional path of length 2 from 1 to v_2 through u_1, contradicting it being in level 2.

Stemple (III) produces paths of length 2 from p to u_3 and from u_2 to v_2, with two distinct intermediate vertices. Consider the path from u_2 to v_2. Let x be the intermediate vertex of this path. Then $1, u_1, u_2, x, v_2, v_1$ is a 6-cycle. There are already paths of length 2 from u_1 to v_2 (through p) and from u_2 to v_1 (through u_3), so this must be a Stemple type III configuration, and there is a path of length 2 from 1 to x. Then x must not be on ρ since otherwise it would be a level 2 vertex, contradiction. The same applies for the intermediate vertex of the path of length 2 from 1 to x, otherwise it would be a level 1 vertex, and it cannot be u_1 or v_1. Case to be done.

Case 1.3 $b = v_2$. See Figure 6. Replace v_2, v_3, v_4 by v_2, v_4 to get a shorter embedded circuit than ρ which still visits the vertex x.

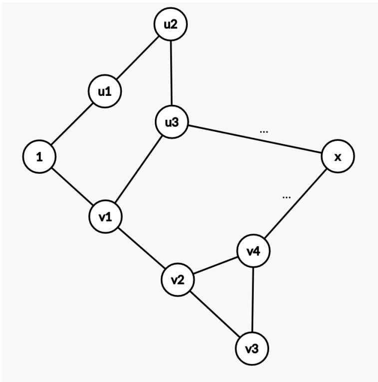


Figure 6: Case 1.3: $a = v_1$ and $b = v_2$ in Lemma 4.

Case 1.4 $b = u_i, 4 \leq i < m$. Let $c = (b)_1$ be the unique level 1 vertex adjacent to b. Either c does not lie on ρ or is equal to u_1 or v_1 by Claim 0. If $c = u_1$ or is not on ρ, then replace $1, u_1, \ldots, u_{i-1}, u_i$ by $1, c, u_i$ to obtain a shorter embedded circuit that visits 1 and x. Else $c = v_1$ and we have Figure ??.

Note that $i < m - 1$ else we have a path of length 3 from 1 to x, contradiction.

If $i = 4$ then we have a circuit of length 6 with vertices $1, u_1, u_2, u_3, u_4, v_1$, with an edge from v_1 to u_3. By Stemple, either this is filed by K_6 (contradicting u_2, u_3 are in level 2) or there must exist a path of length 2 between u_1 and u_4. Let u_* be the immediate vertex of this path. Then u_* can only be u_j with $5 \leq j < m$, v_k with $2 \leq k < n$, or u_* is not on ρ.

If u_* is not on ρ, replace $1, u_1, u_2, u_3, u_4$ by $1, u_1, u_*, u_4$ to obtain a shorter embedded circuit that visits 1 and x.

If u_* is u_j with $5 \leq j < m$, replace $1, u_1, u_2, ..., u_j$ by $1, u_1, u_j$ to obtain a shorter embedded circuit that visits 1 and x.

If u_* is v_k with $2 \leq k < n$, replace $1, v_1, ..., v_k$ by $1, u_1, v_k$ and $u_i, u_3, u_2, u_1, 1$ by $u_i, v_1, 1$ to obtain a shorter embedded circuit that visits 1 and x.

If $i = 5$ then there is a 4-cycle with v_1, u_3, u_4, u_5, which by Stemple must be K_4. Replace $1, u_1, u_2, u_3, u_4, u_5$ by $1, u_1, u_2, u_3, u_5$ to obtain a shorter embedded circuit that visits 1 and x.

If $i \geq 6$, consider the 8 cases for the geodesic from 1 to x.

- $1, u_1, p, v_{n-1}, x$: Then we have a shorter embedded circuit

$$1, u_1, p, v_{n-1}x, u_{m-1}, \ldots, u_i, v_1, 1$$

- $1, u_1, v_{n-2}, v_{n-1}, x$: Then we have a shorter embedded circuit

$$1, u_1, v_{n-2}, v_{n-1}x, u_{m-1}, \ldots, u_i, v_1, 1$$

- $1, u_1, u_2, v_{n-1}, x$: Then we have a shorter embedded circuit

$$1, u_1, u_2, v_{n-1}x, u_{m-1}, \ldots, u_i, v_1, 1$$

- $1, u_1, v_j, u_{m-1}, x$: If $j > 1$ then we have

$$1, u_1, v_j, v_{j+1}, \ldots, x, u_{m-1}, \ldots, u_i, v_1, 1$$

  If $j - 1$ then the 5-cycle $1, u_1, u_2, u_3, v_1, 1$ is now K_5 contradicting u_2, u_3 are in level 2.

- $1, v_1, p, u_{m-1}, x$ – recall $i < m - 1$, and we have a shorter embedded circuit

$$1, u_1, u_2, \ldots, u_i, v_4, \ldots, v_{n-1}, x, u_{m-1}, v_1, 1$$

- $1, v_1, u_{m-2}, u_{m-1}, x$: if $i < m - 2$ then we have a shorter embedded circuit

$$1, v_1, u_{m-2}, u_{m-1}, x, v_{n-1}, \ldots, v_4, u_i, \ldots, u_2, u_1, 1$$

  If $i = m - 2$. Case to be done.

- $1, v_1, v_2, u_{m-1}, x$ then we have a shorter embedded circuit

$$1, v_1, v_2, u_{m-1}, x, v_{n-1}, \ldots, v_4, u_i, \ldots, u_2, u_1, 1$$

- $1, v_1, u_k, v_{n-1}, x$ then $k = 1, 2, 3, \ldots, i, \ldots, m - 2$ to consider. Case to be done.

Case 1.5 $b = v_i, 5 \leq i \leq n - 2$. Let $c = (v_4)_1 = (b)_1$. If c does not lie on ρ then replace $1, v_1, \ldots, v_i$ by $1, c, v_i$ and get a shorter embedded circuit which visits 1 and x. If $c = v_1$ then replace $v_1, v_2, \ldots, v_i$ by v_1, v_i and get a shorter embedded circuit which visits 1 and x. Else $c = u_1$ (by Claim 0) and in this case we have a shorter embedded circuit $1, u_1, v_i, \ldots, v_{n-1}, x, u_{m-1}, \ldots, u_3, v_1, 1$ which visits 1 and x.

Case 1.6 b does not lie on ρ. Let $c = (b)_1 = (v_4)_1$ be the unique level 1 vertex adjacent to b. If c doesn't lie on ρ then $1, c, b, v_4$ replaces $1, v_1, v_2, v_3, v_4$ and gives an embedded shorter circuit than ρ. Else $c = u_1$ or $c = v_1$ by Claim 0. If $c = v_1$ then there are two geodesic paths from v_1 to v_4, contradiction. So $c = u_1$. Then we have Figure 7 - a shorter embedded circuit is $1, u_1, b, v_4, \ldots, v_{n-1}, x, u_{m-1}, \ldots, u_3, v_1, 1$.

Case 2: Now suppose u_3 in level 3. Then u_4, v_4 must be at level 3 since u_2, v_2 are the unique level 2 vertices adjacent to u_3, v_3 respectively, and u_4, v_4 cannot be in level 4 by assumption (for contradiction).

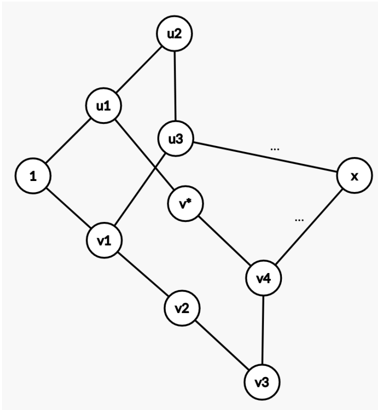


Figure 7: Case 1.6: $a = v_1$ and b new and $c = u_1$ in Lemma 4.

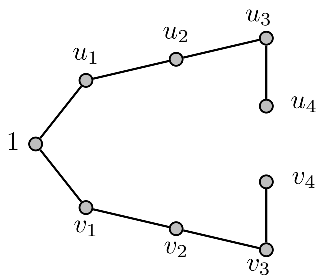


Figure 8: Case 2: u_4, v_4 both in level 3 in Lemma 4.

Let $a = (u_4)_2$ and $b = (v_4)_2$ and $c = (a)_1$ and $d = (b)_1$. If $a = u_2$ then we have a shortcut: replace u_2, u_3, u_4 by u_2, u_4. Similarly if $b = v_2$. This leaves $v_2, u_5, v_i; 5 \leq i \leq n-2$, or not on ρ as possible positions of a, similarly $u_2, u_i; 5 \leq i \leq m-2, v_5$, or not on ρ are possible positions of b. By Claim 0, c and d can only be either u_1, v_1 or not on ρ.

If $a = u_5$ and c is u_1 or not on ρ, then we have a shortcut: replace $1, u_1, u_2, u_3, u_4, a$ by $1, c, a$. So $c = v_1$. Likewise, $c = u_1$ if $a = v_i; 5 \leq i \leq n-2$. This also applies to b and d. If a is not on ρ and c is u_1 or not on ρ, replace $1, u_1, u_2, u_3, u_4$ by $1, c, a, u_4$ to get a shortcut. So $c = v_1$. Similarly $d = u_1$ if b is not on ρ.

We have the following cases for a and b.

Case 2.1 $a = v_2, b = u_2$. Replace $u_2, u_3, u_4, ..., x, ..., v_4, v_3, v_2$ by $u_2, v_4, ..., x, ..., u_4, v_2$ to get a shorter circuit than ρ.

Case 2.2 $a = v_2, b = u_i, 5 \leq i \leq m-2$.

$d = v_1$. If $i = 5$, u_4, b, d, a is a 4-cycle, so it must be K_4. Then u_4 is in level 2, contradiction. If $i > 5$, replace $1, u_1, u_2, u_3, ..., u_i$ by $1, v_1, u_i$ and $v_2, v_1, 1$ by $v_2, u_4, u_3, u_2, u_1, 1$ to get a shorter circuit than ρ.

Case 2.3 $a = v_2, b = v_5$.

$d = u_1$. Replace $1, u_1, u_2, u_3, u_4$ by $1, v_1, v_2, u_4$ and $v_5, v_4, v_3, v_2, v_1, 1$ by $v_i, u_1, 1$ to get a shorter circuit than ρ.

Case 2.4 $a = v_2, b$ not on ρ.

$d = u_1$. Replace $1, u_1, u_2, u_3, u_4$ by $1, v_1, v_2, u_4$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, b, u_1, 1$ to get a shorter circuit than ρ.

Case 2.5 $a = u_5$ and $b = u_2$. Symmetrical to case 2.3.

$c = v_1$. Replace $1, u_1, u_2, u_3, u_4, u_5$ by $1, v_1, u_5$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, u_2, u_1, 1$ to get a shorter circuit than ρ.

Case 2.6 $a = u_5$ and $b = u_j; 5 \leq j \leq m-2$. Case to be done.

Case 2.7 $a = u_5$ and $b = v_5$.

$c = v_1$, $d = u_1$. Replace $1, u_1, u_2, u_3, u_4, u_5$ by $1, v_1, u_5$ and $v_5, v_4, v_3, v_2, v_1, 1$ by $v_5, u_1, 1$ to get a shorter circuit than ρ.

Case 2.8 $a = u_5$ and b not on ρ.

$c = v_1$, $d = u_1$. Replace $1, u_1, u_2, u_3, u_4, u_5$ by $1, v_1, u_5$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, b, u_1, 1$ to get a shorter circuit than ρ.

Case 2.9 $a = v_i; 5 \leq i \leq n-2$ and $b = u_2$. Symmetrical to case 2.2.

$c = u_1$. If $i = 5$, v_4, b, c, a is a 4-cycle, so it must be K_4. Then v_4 is in level 2, contradiction. If $i > 5$, replace $1, v_1, v_2, v_3, ..., v_i$ by $1, u_1, v_i$ and $u_2, u_1, 1$ by $u_2, v_4, v_3, v_2, v_1, 1$ to get a shorter circuit than ρ.

Case 2.10 $a = v_i; 5 \leq i \leq n-2$ and $b = u_j; 5 \leq j \leq m-2$

$c = u_1$, $d = v_1$. Replace $1, u_1, u_2, u_3, ..., u_j$ by $1, v_1, u_j$ and $v_j, ..., v_3, v_2, v_1, 1$ by $v_j, u_1, 1$ to get a shorter circuit than ρ.

Case 2.11 $a = v_i; 5 \leq i \leq n-2$ and $b = v_5$. Symmetrical to case 2.6.

Case 2.12 $a = v_i; 5 \leq i \leq n-2$ and b does not lie on ρ. Case to be done.

Case 2.13 a does not lie on ρ, $b = u_2$. This is symmetrical to case 2.4.

Case 2.14 a does not lie on ρ, $b = u_i, 5 \leq i \leq m-2$. Symmetrical to case 2.12.

Case 2.15 a does not lie on ρ, $b = v_5$. Symmetrical to case 2.8.

Case 2.16 a does not lie on ρ, b does not lie on ρ.

$c = v_1$, $d = u_1$. Replace $1, u_1, u_2, u_3, u_4$ by $1, v_1, a, u_4$ and $v_4, v_3, v_2, v_1, 1$ by $v_4, b, u_1, 1$ to get a shorter circuit than ρ.

$\square$

3 Enumeration of Geodetic Graphs

Below is the main function to check for the geodetic property, written in Python and using the SageMath environment. Note that G is an instance of SageMath's Graph object, and vertices() and neighbors() are its methods. The algorithm is based on work of Parthasarathy and Srinivasan [5] from 1982.

```
# for each pair of vertices in G
for pair in itertools.combinations(G.vertices(), 2):
  (u, v) = pair
  # find any neighbour i of v that has dist(u, i) = dist(u, v) - 1 excluding u
  predecessor = [i for i in G.neighbors(v) if dist[u][i] == dist[u][v] - 1 and i != u]
  # if there is more than one such neighbour
  if len(predecessor) > 1:
    return false
  return true
```

Using this program, all geodetic graphs and all biconnected geodetic graphs of up to 11 vertices have been successfully enumerated and stored. Aside from the biconnected geodetic graphs belonging to known families that we already knew, no new biconnected graph was found for up to 11 vertices.

From the enumerated graphs, the two sequences for the number of biconnected geodetic graphs and all geodetic graphs are constructed. We have published these on the On-line Encyclopedia of Integer Sequences as A337178 and A337179 [6, 7].

4 Future Directions and Conclusion

If remaining cases are successfully eliminated, we could try to prove a version of [2, Lemma 8] and then proceed to extend [2, Theorem 2].

Alternatively, we could also try extending the result to this conjecture, and try to either prove it or construct (probably with the help of a computer) a counterexample:

Conjecture 5. *Let Γ be a locally-finite simple geodetic graph, and $n \in \mathbb{N}$. If ρ is an embedded circuit of diameter exceeding n and that has minimal length among all such paths in Γ, then ρ contains a geodesic subpath of length $n+1$.*

Comparing the number cases requiring proof of [2, Lemma 6] and that of our project, and also the difficulty in proving them, we can conclude that further extending the graph theory result using the same contradiction method is not feasible due to the combinatorial explosion that occurs.

In the process of developing the program, the unexpected and main challenge was again combinatorial explosion, in the number and complexity of graphs as the number of vertices increases (see Appendix B). At the moment this report was written, enumeration for graphs of 12 vertices is still in progress.

5 Acknowledgements

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A Graph theory definitions

A simple undirected graph Δ is a pair comprising a nonempty set $V(\Delta)$, the set of *vertices*, and a set of two-element subsets $E(\Delta)$, the set of *edges*. The vertices that form an edge are said to be *adjacent*. All graphs considered in this paper will be simple and undirected. For the remainder of this section, fix a simple undirected graph Δ.

A *path* of length n in Δ from a vertex u to a vertex v is a sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ with the property that u_{i-1} and u_i are adjacent for $i = 1, \ldots, n$. A path from u and v is called a *geodesic* if there is no shorter path in Δ from u to v. If for each pair (u, v) of distinct vertices in Δ there is at least one path in Δ from u to v, we say that Δ is *connected*; if for each pair (u, v) of distinct vertices in Δ there exists a unique geodesic from u to v, we say that Δ is *geodetic*. If Δ is connected, there is a natural metric d on the vertex set of Δ such that $d(u, v)$ is the length of a shortest path in Δ from u to v.

A path in Δ is an *embedded circuit* if the vertices $u_0, \ldots, u_{n-1}$ are distinct and $u_0 = u_n$, and in this case we say its length is n. An embedded circuit in Δ is *isometrically embedded* if the subgraph comprising the vertices in the circuit and the edges between consecutive vertices is convex in Δ; that is, $d(u_i, u_j) = \min\{j - i, n + i - j\}$ for all $0 \leq i < j < n$. We note that if u, v are adjacent vertices in Δ, then the path u, v, u is an isometrically embedded circuit of length two. We also note that in a geodetic graph, the unique geodesic joining two vertices of an isometrically embedded circuit is a subpath of the isometrically embedded circuit.

A vertex v in Δ is a *cut vertex* if Δ is connected, but the graph obtained from Δ by removing v and the edges incident to v is disconnected. A graph is *biconnected* if it is connected and has no cut vertices. The maximal two-connected subgraphs of a graph Γ are called *blocks*. It follows immediately from the maximality of blocks that any block B in Δ is the subgraph of Δ induced by the vertex set of B. In a connected graph having at least two vertices, each block has at least two vertices. The following well-known characterisation of blocks (see, for example, [3, Theorem 5.4.3, p. 87]) is useful.

Lemma 6. *Let Δ be a simple undirected graph. Two vertices u, v of Δ lie in the same block if and only if there exists an embedded circuit in Δ that visits both.*

That is, to understand geodetic graphs it suffices to understand geodetic blocks.

B Number of graphs with n vertices

Here is a table of the number of simple unlabeled graphs with n vertices, from `https://oeis.org/A001349/`.

n	$a(n)$
0	1
1	1
2	1
3	2
4	6
5	21
6	112
7	853
8	11,117
9	261,080
10	11,716,571
11	1,006,700,565
12	164,059,830,476
13	50,335,907,869,219
14	29,003,487,462,848,061
15	31,397,381,142,761,241,960
16	63,969,560,113,225,176,176,277
17	245,871,831,682,084,026,519,528,568
18	1,787,331,725,248,899,088,890,200,576,580
19	24,636,021,429,399,867,655,322,650,759,681,644

Table 1: Table of the number of simple unlabeled graphs with n vertices. This is sequence A001349 on the On-line Encyclopedia of Integer Sequences.