Inverse Scattering in the Recovery of a Single Concave and a Finite Union of Disjoint Convex Obstacles

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Vacation Research Scholarships are funded jointly by the Department of Education, Skills and Employment and the Australian Mathematical Sciences Institute.

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1 Prelude

1.1 Abstract

This report aims to give a concise overview of billiard scattering and inverse scattering techniques in Euclidean spaces. It has previously been shown that finite disjoint unions of strictly convex obstacles can be uniquely recovered from their travelling time spectra, and we wish to extend this analysis to include a single concave obstacle. We have shown that certain restricted cases of concave obstacles and some finite disjoint union of circles have unique travelling time spectra, which would allow these obstacles to be uniquely recovered from this distribution alone.

1.2 Introduction

Problems involving scattering from obstacles draw inspiration from physical phenomena such as the reflection of light from a reflective surface or the scattering of a wave function from a potential. These problems aim to determine the trajectory of an incoming particle or wave after some interaction with an obstacle. In our problem, we consider point-like particles which behave like "billiard balls" in a Euclidean space, that is, particles travel in straight lines in free space and obey the law of reflection at obstacle boundaries. The law of reflection states that the angle of reflection is equal to the angle of reflection.

Our problem is of an inverse nature, that is, we wish to determine an obstacle from the observable scattering patterns. Specifically, we wish to determine the boundary of the obstacle based off the travelling times of the scattered rays. Several cases of obstacle sets have been considered in the past, including star-shaped obstacles in [5] and finite disjoint unions of strictly convex obstacles in [4]. There has also been progress in generating algorithms for constructively recovering the obstacle set in the case of the finite disjoint union of strictly convex obstacles in $\mathbb{R}^2$, by [1]. In this project, we wish to extend this analysis to include a single concave obstacle within a finite disjoint union of strictly convex obstacles.

The report is structured as follows: Section 2 provides necessary background information, definitions and theorems from scattering theory and invere scattering. Section 3 then applies the existing knowledge to obstacles involving concave sections. Finally, Section 4 provides a summary of the main results and recommends areas for future research.

1.3 Statement of Authorship

This project was conceived by my supervisor, Luchezar Stoyanov, who wanted to extend his analysis of inverse scattering to include concave obstacles. Definitions, theorems and concepts that have been sourced from existing literature have been referenced appropriately. Any remaining claims or proofs were completed by myself, with input and proofing from my supervisor.

2 Scattering Problems and Inverse Scattering

Here we will present some important results from Scattering Theory which will be applied to our problem.

2.1 Scattering from Obstacles

In this section, we will provide formal definitions for a number of concepts we will reference throughout the report. All definitions in this section are as reported in [1].

Definition 2.1. *Let an obstacle K in $\mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ such that the boundary of K, denoted ∂K, is a smooth manifold of dimension $n-1$ and such that $\mathbb{R}^n \setminus K$ is connected.*

Definition 2.2. *For an obstacle K in $\mathbb{R}^n$, let a bounding sphere, denoted S_0, be a sphere in $\mathbb{R}^n$ which is the boundary of an open ball O such that $K \subset O$.*

For any $x', y' \in \mathbb{R}^n$, we denote the line segment

$$\{p \in \mathbb{R}^n : ||p - x'|| + ||p - y'|| = ||x' - y'||\} \tag{1}$$

as $x' \to y'$. We can then construct connected line segments as $(x' \to y') \cup (y' \to z')$ and we denote these as $x' \to y' \to z'$.

Definition 2.3. *For an obstacle K and bounding sphere S_0, for $x, y \in S_0$, an (x,y)-reflecting ray, denoted γ, is a sequence of connected line segments in $\mathbb{R}^n$, i.e. $\gamma = x \to x_1 \to ... \to x_n \to y$ for some $n \in \mathbb{N}$. The ray γ satisfies the following conditions:*

1. $\gamma \cap \partial K = \{x_1, ..., x_n\}$
2. $\gamma \cap S_0 = \{x, y\}$
3. *Every consecutive pair of line segments satisfies the reflection law on ∂K, such that $x_{i-1} \to x_i$ and $x_i \to x_{i+1}$ are symmetric with respect to the outward unit normal vector $n_K(x_i)$ of ∂K at x_i, that is*

$$\left\langle \frac{x_{i-1} - x_i}{||x_{i-1} - x_i||}, n_K(x_i) \right\rangle = -\left\langle \frac{x_i - x_{i+1}}{||x_i - x_{i+1}||}, n_K(x_i) \right\rangle \tag{2}$$

Definition 2.4. *For γ an (x,y)-reflecting ray, the length of γ (or the travelling time of γ) is denoted $\ell(\gamma)$ and is such that $\ell(\gamma) \geq 0$. For multiple reflections, $\ell(\gamma)$ is the sum of the lengths of each of the line segments, i.e. if $\gamma = x_0 \to x_1 \to ... \to x_n \to x_{n+1}$, then*

$$\ell(\gamma) = \sum_{i=0}^{i=n} ||x_{i+1} - x_i|| \tag{3}$$

Definition 2.5. *For any obstacle K with bounding sphere S_0, the travelling time spectrum of K, denoted T_K, is a set-valued function on $S_0 \times S_0$, where for $(x,y) \in S_0 \times S_0$,*

$$T_K(x,y) = \{\tau : \tau = \ell(\gamma) \text{ for } \gamma \text{ some } (x,y)\text{-reflecting ray}\} \tag{4}$$

Definition 2.6. *A convex obstacle K is an obstacle such that for any two points $x, y \in K$, the set $\{\lambda x+(1-\lambda)y : 0 \leq \lambda \leq 1\}$ is a subset of K. A strictly convex obstacle is a convex obstacle whose boundary does not contain any straight line segments.*

Definition 2.7. *For an obstacle K in $\mathbb{R}^n$, let $\Omega_K = \overline{\mathbb{R}^n \setminus K}$, i.e. Ω_K is the closure of $\mathbb{R}^n \setminus K$. Then the cotangent bundle of Ω_K, denoted $T^*(\Omega_K)$ is the set $\{(x, v) : x \in \Omega_K, v \in \mathbb{R}^n\}$. The cosphere bundle of Ω_K is denoted $S^*(\Omega_K)$ and is the subset of $T^*(\Omega_K)$ given by $\{(x, v) : x \in \Omega_K, v \in \mathbb{S}^{n-1}\}$.*

Definition 2.8. *Let γ be an (x, y)-reflecting ray and let $u \in \mathbb{R}^n$, $u \neq 0$ be tangent to γ at x (so $u = \lambda(x_1 - x)$ where x_1 is the first reflection point of γ and lambda > 0). Also let $v \in \mathbb{R}^n$, $v \neq 0$ be tangent to γ at y (so $v = \mu(y - x_n)$ where x_n is the final reflection point of γ and $\mu > 0$) such that $||v|| = ||u||$. Suppose that $\ell(\gamma) = t$, we define the generalised geodesic flow, denoted $\mathcal{F}_t^K : T^*(\Omega_K) \to T^*(\Omega_K)$ by $\mathcal{F}_t^K(x, u) = (y, v)$. We also define $\mathcal{F}_{-t}^K(y, v) = (x, u)$ and $\mathcal{F}_t^K(y, -v) = (x, -u)$. Also we set $\mathcal{F}_0^K(x, \omega) = (x, \omega)$ for any $(x, \omega) \in T^*(\Omega_K) \setminus \{0\}$.*

Definition 2.9. *Let $\sigma = (x, \omega) \in T^*(\Omega_K)$ and let pr_1 and pr_2 be the projections onto the first and second coordinates respectively. We will say σ is non-trapped if $\{pr_1(\mathcal{F}_t^K(\sigma) : t \geq 0\}$ and $\{pr_1(\mathcal{F}_t^K(\sigma) : t \leq 0\}$ are unbounded curves in $\mathbb{R}^n$. A trapped point, $\sigma \in T^*(\Omega_K)$ is one where the forward or backward trajectories are bounded. Let $Trap(\Omega_K)$ be the set of trapped points of Ω_K.*

Trapped points severely hinder the likelihood that a given obstacle has a unique travelling time spectrum. One such example is the Livshits' obstacle, described in Section 5.4 of [3] and shown in Figure 1. A known property of ellipses is that if we place a billiard at one foci of the ellipse and reflect it off the interior boundary, the reflected ray is directed towards the other foci. Any ray that enters the obstacle between A and B will hence be reflected out through the same segment A to B. Hence there are no rays which have common points with ∂K in the regions $D \to A$ and $C \to B$. We observe that the trapped points of Ω_K contain an open subset of $S^*(\Omega_K)$, these are from the bounded points of Ω_K bounded by ∂K and the segments $D \to A$ and $C \to B$. This means that there are uncountably many obstacles that have the same travelling time spectrum T_K.

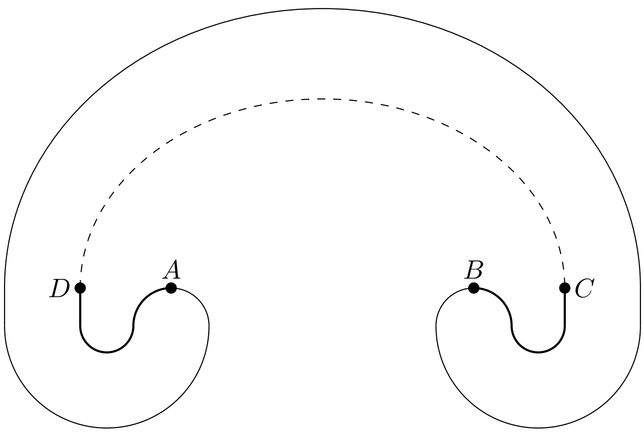


Figure 1: Livshits' obstacle. The dashed segment is a half-ellipse with foci at A and B.

2.2 Inverse Scattering

In this section we will provide a number of theorems and results that we will make use of throughout the report. One relatively trivial result we will apply is the uniqueness of a point determined by a single reflection ray. Let us show this claim.

Claim 2.1. *Consider an $(x, y)-$reflecting ray γ with known initial direction $\overrightarrow{u}$. If γ is known to have one reflection point, then the travelling time of γ uniquely determines the reflection point $z \in \partial K$.*

Proof. Given that the coordinates x, y are known, the Euclidean distance between them, L, is known. We can also construct the vector between the two points as $\overrightarrow{xy} = L\overrightarrow{v}$, with $||\overrightarrow{v}|| = 1$. Let us consider the line defined by the initial coordinate x and the direction $\overrightarrow{u}$; we will take $||\overrightarrow{u}|| = 1$. We parameterise the length along this line by the parameter λ.

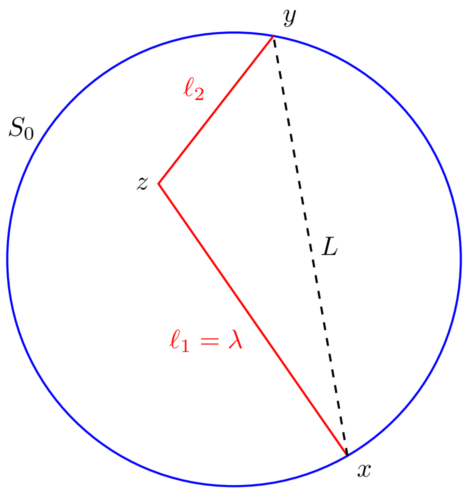


Figure 2: Uniqueness of a single reflection point

Hence we can consider the travelling time as a function of λ, that is: $\tau : \mathbb{R} \to \mathbb{R}$, $\tau = \tau(\lambda)$. We wish to show that this function is one-to-one, that is, for $\lambda_1 \neq \lambda_2$, we have $\tau(\lambda_1) \neq \tau(\lambda_2)$. Let us first construct the function $\tau(\lambda)$. τ represents the total length of the reflected ray. Given that it is a single reflection ray, it can be broken into two straight line segments of lengths ℓ_1, ℓ_2. ℓ_1 is simply the parameter λ and we can use the cosine rule to determine ℓ_2.

$$\begin{aligned} \tau &= \ell_1 + \ell_2 \\ \tau &= \lambda + \sqrt{\lambda^2 + L^2 - 2\lambda L \cos\theta} \end{aligned} \tag{5}$$

Here, $\cos\theta$ is determined by $\cos\theta = \overrightarrow{u} \cdot \overrightarrow{v}$. Some rearrangement shows that there is an explicit expression for the inverse function $\lambda(\tau)$, as given by

$$\lambda = \frac{\tau^2 - L^2}{2(\tau - L\cos\theta)} \tag{6}$$

The existence of a continuous inverse function implies that $\tau(\lambda)$ is a bijection, hence implying that it is in fact one-to-one. Therefore, λ defines a unique point $z = \lambda\overrightarrow{u}$ along the line defined by the initial point x and direction $\overrightarrow{u}$. $\square$

This argument can be generalised to any finite number of reflections where all reflection points except one are known. We will prove this claim below.

Claim 2.2. *Consider an (x, y)-reflecting ray γ with known initial direction $\overrightarrow{u}$. If γ is known to have N reflection points, where only $N-1$ points are known, then the travelling time of γ will uniquely determine the unknown reflection point.*

Proof. For our (x, y)-reflecting ray with N reflection points $\{x_1, ..., x_N\}$, there are $N+1$ line segments, $\ell_i : 1 \leq i \leq N+1$, of which $N-1$ are known. We will denote these two line segments as ℓ_j and ℓ_{j+1} and the unknown reflection point as x_j. The travelling time of γ can then be expressed as:

$$\tau = \sum_{i=1}^{N+1} \ell_i = \ell_j + \ell_{j+1} + \sum_{i \neq j, j+1} \ell_i \tag{7}$$

We will introduce the parameter L to represent the Euclidean distance between the reflection points x_j and x_{j+2}. The direction of x_{j+2} from x_j will be specified by the unit vector $\overrightarrow{v}$, such that $x_{j+2} = x_j + L\overrightarrow{v}$. As the reflection points $\{x_1, ..., x_{j-1}\}$ are known as well as the initial direction $\overrightarrow{u_1}$, the successive reflection directions can be evaluated by the law of reflection, such that $\{\overrightarrow{u_1}, ..., \overrightarrow{u}_{j-1}\}$ are also known, and we will take these to be unit vectors. We are now able to express ℓ_{j+1} in terms of ℓ_j through the use of the cosine rule:

$$\ell_{j+1}^2 = \ell_j^2 + L^2 - 2\ell_j L \cos\theta \tag{8}$$

where $\cos\theta = \overrightarrow{u}_{j-1} \cdot \overrightarrow{v}$. Hence we can express τ as a function, $\tau : \mathbb{R} \to \mathbb{R}$, $\tau = \tau(\ell_j)$:

$$\tau = \ell_j + \sqrt{\ell_j^2 + L^2 - 2\ell_j L \cos\theta} + \sum_{i \neq j, j+1} \ell_i \tag{9}$$

As before, we will show that there exists an analytical expression for the inverse function $\ell_j(\tau)$, implying that $\tau(\ell_j)$ is a bijection and hence an injection. Some rearrangement of Equation (9) yields the following:

$$\ell_j = \frac{\left[\tau - \sum_{i \neq j, j+1} \ell_i\right]^2 - L^2}{2\left(\tau - \sum_{i \neq j, j+1} \ell_i - L\cos\theta\right)} \tag{10}$$

The existence of a continuous inverse function shows that $\tau(\ell_j)$ is a bijection, and is therefore one-to-one. Since ℓ_j is uniquely determined, this implies that ℓ_{j+1} is also uniquely determined. We can then evaluate the uniquely determined reflection point x_j by evaluating $x_j = x_{j-1} + \ell_j \overrightarrow{u}_j$. Hence x_j is uniquely determined. $\square$

One application of single reflection rays is the notion of *back-scatter rays*, which prove useful in resolving the obstacle K from its travelling time spectrum T_K.

Definition 2.10. *A back-scatter ray γ is an (x, x)-reflecting ray that has a reflection point $x_m \in \partial K$ such that the ray reflects perpendicular to ∂K at x_m.*

Back-scatter rays occupy a subset of the total travelling time spectrum, $\cup_{x \in S_0} T_K(x,x)$, however not all rays in this subset are necessarily back-scatter rays. As a simple counter example, there may exist (x,x)-reflecting rays which do not pass through x at the same angle they left at. We are most interested in single reflection back-scatter rays; those which have a single reflection point $x_1 \in \partial K$.

Definition 2.11. *For γ an (x,y)-reflecting ray, γ is called simply reflecting if γ has no tangencies to ∂K.*

Lemma 2.1. *Let γ be a regular, simply-reflecting (x_0,y_0)-ray and let W be an open neighbourhood of $S_0 \times S_0$ about (x_0,y_0) such that for $(x,y) \in W$, there is a unique (x,y)-ray, $\gamma(x,y)$. Then $\ell(\gamma(x,y))$ is a smooth, real-valued function on W and for a,b tangent vectors to S_0 at x and y respectively and for q and w the unit vectors in the outgoing and incoming directions of $\gamma(x,y)$ at x and y respectively, then*

$$d\ell(\gamma(x,y))(a,b) = \langle b,q \rangle - \langle a,w \rangle \tag{11}$$

Consequently, the derivative of the travelling time function gives the outgoing and incoming directions of regular, simply-reflecting rays.

Here we will define the notion of *accessibility*, which refers to the ability for rays to reach a given section of the obstacle K. Accessibility has been used in [5] to show that strongly accessible obstacles have unique travelling time spectra.

Definition 2.12. *Given an obstacle K, fix a countable set $\{M_i\}$ of submanifolds of $S^*_{S_0}(\Omega_K)$. A smooth curve $\sigma(s)$, $0 \le s \le a$ (for some $a > 0$) will be called regular if it has the following properties:*

(i) $\sigma(0)$ generates a free ray in Ω_K, i.e. a ray without any common points with ∂K.

(ii) $\sigma(a) \notin \bigcup_i M_i$

(iii) $\sigma(s) \notin Trap(\Omega_K)$ for all $s \in [0,a]$.

(iv) if $\sigma(s) \in M_i$ for some i and $s \in [0,a]$, then σ is transversal to M_i at $\sigma(s)$ and $\sigma(s) \notin M_j$ for any submanifold $M_j \neq M_i$.

From this definition, we can make some comments regarding the regular curve. (ii) and (iii) imply that $\sigma(a)$ generates a simply reflecting ray, while (iii) and (iv) give that every $\sigma(s)$ generates a scattering ray with at most one tangent point to ∂K and the tangency (if any) is of first order only. We will define the recursive sequence $\partial K^{(1)} \subset \partial K^{(2)} \subset ... \subset \partial K^{(m)} \subset ...$ as follows. Denote by $\partial K^{(1)}$ as the set of those $x \in \partial K$ for which there exists a regular curve $\sigma(s)$ $(0 \le s \le a)$ in $S^*_{S_0}(\Omega_K)$ such that $x \in \gamma(\sigma(a))$ and for every $s \in [0,a]$ the ray $\gamma(\sigma(s))$ has at most one common point with ∂K. For convenience, define $\partial K^{(0)} = \emptyset$. We define the strongly accessible part of ∂K by

$$\partial K^{(\infty)} = \overline{\bigcup_{m=1}^{\infty} \partial K^{(m)}} \tag{12}$$

The obstacle will be called strongly accessible if $\partial K^{(\infty)} = \partial K$. Considering again the Livshits' example from Figure 1, we can now say that the sections $D \to A$ and $C \to B$ are not accessible as no incoming ray has common points with ∂K in these segments.

Theorem 2.1. *Assume that obstacles K, L have almost the same travelling time spectrum. Then $\partial K^{(m)} = \partial L^{(m)}$ for all $m \geq 0$, and therefore $\partial K^{(\infty)} = \partial L^{(\infty)}$. If K is strongly accessible, then $L = K \cup L'$ for some connected component L' of L with $L' \cap K = \emptyset$. Additionally, if L has the property that any connected component of it can be reached by a ray $\gamma_L(\rho)$ generated by an accessible point $\rho \in S^*_{S_0}(\Omega_K) \setminus Trap(\Omega_L)$, then $K = L$.*

It has previously been shown in [4] that any finite disjoint union of strictly convex obstacles is uniquely recoverable from the travelling time spectrum. This result is summarised in Theorem 2.2.

Theorem 2.2. *Let K and L be obstacles in $\mathbb{R}^n$, $(n \geq 2)$, and each of K and L can be represented as the finite, disjoint union of strictly convex obstacle components with C^3 boundaries. If K and L have almost the same travelling time spectrum, then $K = L$.*

3 Application to Concave Obstacles

We now wish to extend the analysis from convex obstacles to a set of obstacles which includes a single concave surface. Our work will focus mostly on $\mathbb{R}^2$, although the results are readily generalisable to $\mathbb{R}^n$.

3.1 A Simple Example

We begin by considering a relatively simple combination of a single concave obstacle and a single convex obstacle. Let our obstacle set $K = K_1 \cup K_2$ consist of K_1, a circle of radius r centred at $(0, 0)$ and K_2, an obstacle which contains an inner circular arc of radius R subtended by an angle α, centred at $(0, 0)$. Such an obstacle is illustrated in Figure 3.

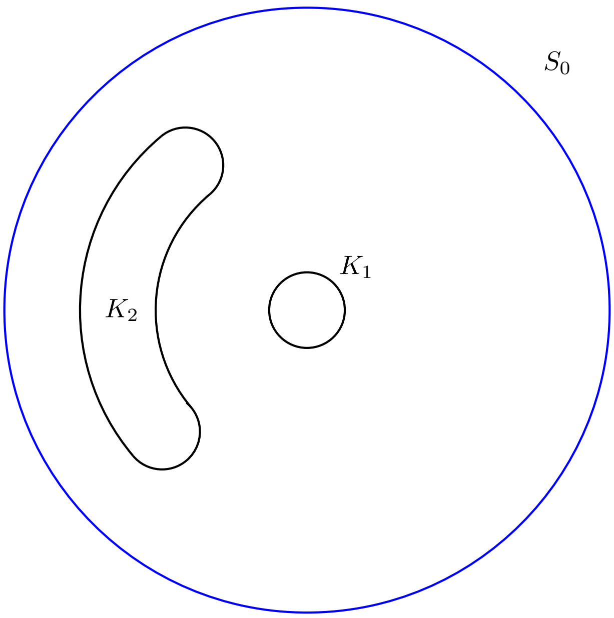


Figure 3: Obstacle set K and bounding sphere S_0 for our simple example.

We wish to determine whether the travelling time spectrum will uniquely determine this obstacle. To resolve a point on the obstacle K_2, we require that the inner circular arc is accessible by single-reflection rays. This will place an upper bound on the angle subtended by the inner circular arc from K_2. This maximum bound is reached when the incident ray is tangential to K_1 and reflects from K_2 at the end of the arc. Upon reflection, the ray will be incident on the opposite end of the arc while being tangent to K_1. This situation is shown in Figure 4. Some simple trigonometry shows that the maximum subtending angle is given by

$$\beta_{max} = 2 \arccos\left(\frac{r}{R}\right) \tag{13}$$

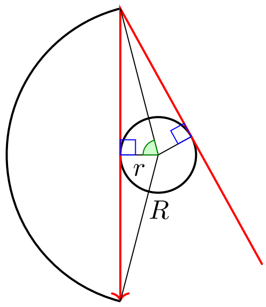


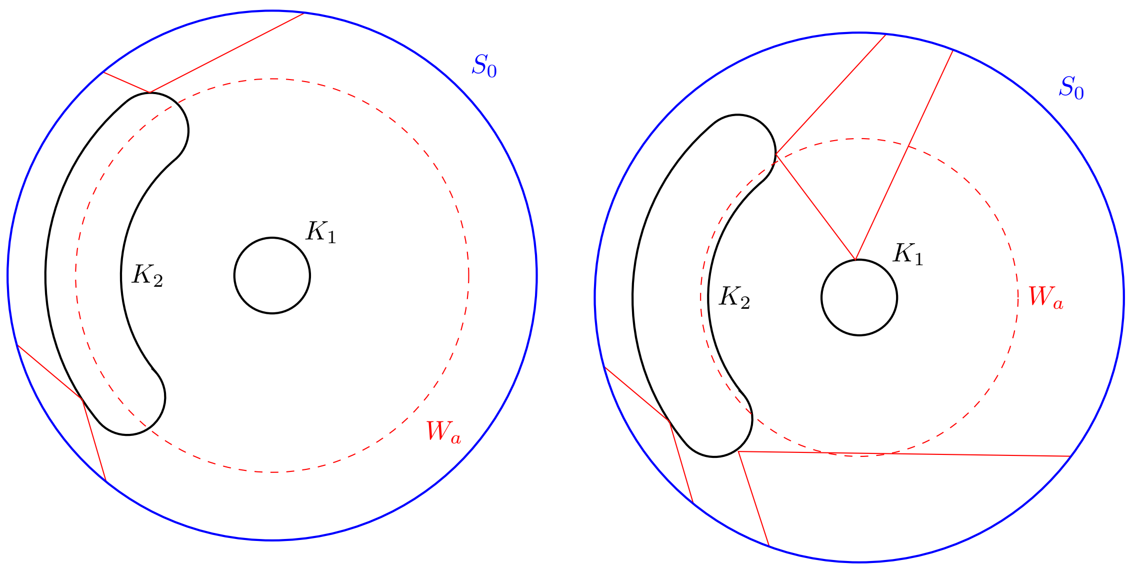
Figure 4: Limiting case for resolvable inner circular arcs.

Claim 3.1. *For an obstacle K such as that shown in Figure 3 with an inner circular arc subtended by an angle less than β_{max}, the travelling time spectrum T_K will uniquely determine the obstacle.*

Proof. Let us consider two obstacles, K and L. We will assume that the travelling time spectrum is identical for both obstacle sets, $T_K = T_L$, and we will show that this leads to $K = L$. Furthermore, we will assume that the obstacle K is known and is as shown in Figure 3, while obstacle L is only known to consist of obstacles similar in shape to K.

We begin by considering the set $W_r = \{y : ||y|| > r\}$ and note the infimum $a = \inf\{R > 0 : W_r \cap \partial K \subset \partial L, \forall r > R\}$. We claim that for radii $r > a$ that $W_a \cap \partial K = W_a \cap \partial L$. This is trivially true for large a as both obstacles would be completely encapsulated within W_a, hence $W_a \cap \partial K = W_a \cap \partial L = \emptyset$. We now wish to show that $a = 0$. We starting with such a large a that W_a contains both obstacles K and L. We then shrink a until we reach the point that $W_a \cap \partial K \neq \emptyset$. From Figure 3, we can observe that this point will occur along ∂K_2. From the local convexity of ∂K_2 and the absence of other obstacles, we can observe that all scattering rays will in fact be single reflection rays. This is illustrated in Figure 5 (a). By Claim 2.1, single reflection rays will uniquely determine a point on the obstacle, and so we have that $W_a \cap \partial K = W_a \cap \partial L$ in this region. As we continue to decrease a, we reach a point where multiple reflections from ∂K are possible. This is illustrated in Figure 5 (b).

Beyond this point, we make use of the fact that the subtending angle of the inner circular arc is less than β_{max}, implying the entire inner circular arc is accessible by single-reflection rays. Hence there exists a subset $T_S \subset T_K$ consisting of only single-reflection rays which have reflection points along the inner circular arc of



(a) W_a such that all scattered rays are single reflection rays only.

(b) W_a such that multiple reflection rays are possible.

Figure 5: Different configurations of W_a that permit only single reflection rays (a) and both single and multiple reflection rays (b).

∂K_2. From Claim 2.1, this implies that the entire inner circular arc of ∂K_2 is uniquely determined by these single-reflection rays. As these points are uniquely determined, this means that we may further shrink W_a as in this region $W_a \cap \partial K = W_a \cap \partial L$. We now reach the point $a = r$, where $K_1 \subseteq W_a$. Here we will make use of the fact that obstacle K is known. We will consider only the subset of single reflection back-scatter rays. Given that $T_K = T_L$, the subsets will be identical for both obstacles. Figure 6 depicts some examples of possible back-scatter rays from K_1. Quite a large proportion of ∂K_1 is recoverable from back-scatter rays and it is possible to reconstruct the entirety of ∂K_1 from these rays alone. Given that it is known K_1 is a circle, being able to determine the centre and radius is all that is required to uniquely define the obstacle; given that the travelling time spectra are identical, this will infer that $\partial K_1 = \partial L_1$. For the geometry established in Figure 6, for any given back-scatter ray with travelling time τ_i, we will have the relation,

$$\tau_i = 2(R - r) \tag{14}$$

which for a known bounding sphere S_0 of radius R, is easily inverted to obtain the radius of the obstacle;

$$r = R - \frac{\tau_i}{2} \tag{15}$$

The centre of the circle can then be obtained by picking any $x \in S_0$ which produces a back-scattering ray and travelling along the ray direction $\overrightarrow{u}$;

$$O = x + r\overrightarrow{u} \tag{16}$$

Hence the obstacle K_1 is uniquely determined by the travelling time spectra, which implies that $\partial K_1 = \partial L_1$. Therefore, we have shown that we can shrink $a \to 0$ and obtain $\partial K = \partial L$. $\square$

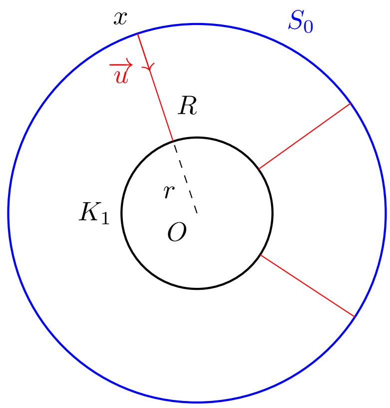


Figure 6: Examples of back-scatter rays from K_1.

We can now make use of notions of accessibility to relax the constraint on the subtending angle β_{max}. We will once again consider a concave surface as a circular arc subtended by some angle β. Such an obstacle can be seen in Figure 7.

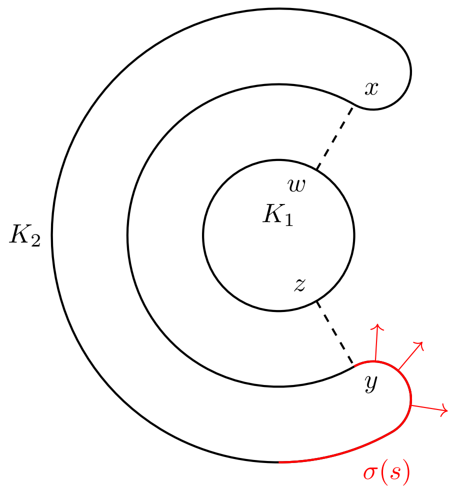


Figure 7: A circular concavity which is not completely accessible by single reflection rays. An example of a regular curve along ∂K is also illustrated which generates rays perpendicular to ∂K. The edges of the circular arc are indicated by the dashed lines.

Claim 3.2. *For concave obstacles with circular concavity C subtended by some angle $\beta > \beta_{max}$, we have that $\partial K \setminus C = \partial K^{(\infty)}$, and hence have a unique travelling time spectrum.*

Proof. We will argue that we can construct regular curves $\sigma(s) \subset \partial K \setminus C$ which satisfy the conditions in Definition 2.12. We will assume that every regular curve will generate rays $\nu_N(\sigma(s))$ which are perpendicular to ∂K for all $s \in [0, a]$. We have that any point along the exterior boundary of ∂K_2 will generate free rays, which satisfies condition (i). Due to the presence of only two obstacles, the maximum number of tangency points from the generated rays is two. Constructing a line between these two tangency points will determine at most one point in ∂K which generates a ray which has two tangency points. As such, this point would be excluded from $\bigcup_m \partial K^{(m)}$, however it would be included in the closure of this set. Any other tangencies with the generated ray will hence be at most first order, and so will be included in $\bigcup_m \partial K^{(m)}$. We wish to show that there exists $x, y \in \partial K_2$ and $w, z \in \partial K_1$ such that $\{x, y, w, z\} \in Trap(\Omega_K)$. These points will define the boundary of the circular arcs in both ∂K_1 and ∂K_2, which cannot be accessed by back-scatter rays. By geometrical reasoning, two concentric circles will have common perpendicular lines, which in our case will be represented by $\nu_N(x)$ and $\nu_N(y)$. As these rays are perpendicular to both ∂K_1 and ∂K_2, then the rays will be trapped in the reflection loops $x \to w \to x \to ...$ and $y \to z \to y \to ...$ and hence cannot escape to the bounding sphere. Therefore, these rays do not belong in the travelling time spectra and hence $\{x, y, w, z\} \in Trap(\Omega_K)$. Now for any $b \in C$, we have that $\nu_N(b) \in Trap(\Omega_K)$, and so all regular curves will satisfy $\sigma(s) \notin C$, $\forall s \in [0, a]$. Thus we have $\sigma(s) \subset \partial K \setminus C$ $\forall s \in [0, a]$, and so $\partial K^{(\infty)} = \partial K \setminus C$. Hence we can reconstruct ∂K as the combination $C \cup \partial K^{(\infty)}$, and as the obstacle now has a known boundary, its travelling time spectrum will be unique. □

3.2 A More General Case

We now wish to include a finite disjoint union of convex obstacles within the obstacle set. As a simplification, we will only consider circular obstacles and the concave obstacle will once again have its concavity defined by a circular arc subtended by some angle $\beta < \pi$. We will also make use of the Ikawa no-eclipse condition, which states for $i \neq j \neq k$, we have that the convex hull of $K_i \cup K_j$, denoted $\mathrm{conv}(K_i \cup K_j)$, has no common points with K_k, that is, $\mathrm{conv}(K_i \cup K_j) \cap K_k = \emptyset$ [2]. We will exclude the concave obstacle from the no-eclipse condition and require that all circular obstacles fall outside the defining circle of the concavity. Such an obstacle set is shown in Figure 8.

Claim 3.3. *For an obstacle set K which consists of a single concave obstacle and a finite disjoint union of equally-sized circular obstacles which satisfy the no-eclipse condition, the travelling time spectrum will uniquely determine the obstacle.*

Proof. Let us consider two obstacle sets, K and L, and let us assume that they have identical travelling time spectra. We construct the set W_r, as was defined in the proof of Claim 3.1 and again define the value $a = \inf\{R : W_r \cap \partial K = W_r \cap \partial L, r > R\}$. We will show that $a = 0$ and hence that $K = L$ everywhere.

Trivially, we have $W_a \cap \partial K = W_a \cap \partial L$. As we begin to shrink a, we reach a point where $W_a \cap \partial K \neq \emptyset$, let us define this radius as b. Let us say that $W_b \cap \partial K = \{z\}$. For $\epsilon > 0$, if we were to further reduce the radius to $b - \epsilon$, we would expose a small neighbourhood around z, say $\mathcal{O}(z)$. We can construct backscatter rays $\gamma_i : x_i \to z_i \to x_i$, where $x_i \in S_0$ and $z_i \in \mathcal{O}(z)$. By noting the travelling time of the ray γ_i and the initial direction from $x_i \in S_0$, we can fully reconstruct the radius and centre of the circular obstacle. We can thus further reduce a, exposing more circular obstacles as a decreases. As these circular obstacles are exposed, one can draw a unit normal vector $\nu_N(z)$ at the exposed point z. This vector defines the back-scatter ray which has two potential outcomes;

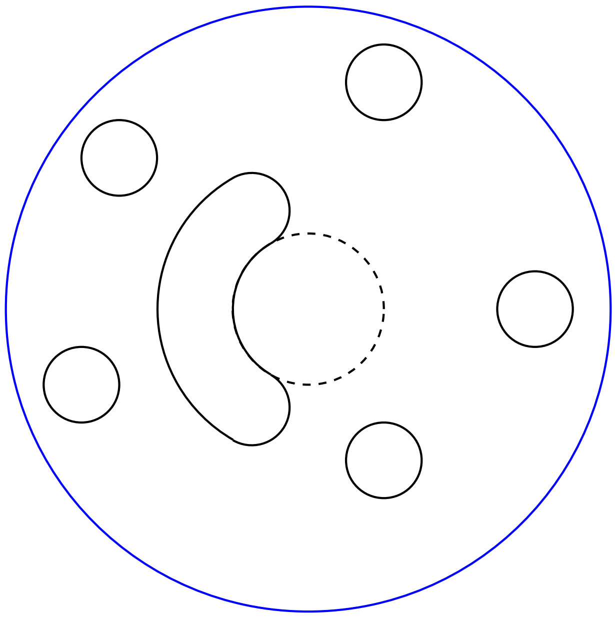


Figure 8: Obstacle set K and bounding sphere for the more general case. All circular obstacles obey the no-eclipse condition.

(i) The ray does not intersect ∂K again, meaning it is a single reflection back-scatter ray and is hence the point z is uniquely defined, as per Claim 2.1.

(ii) The ray is obstructed by a further obstacle. This may cause the ray to reflect back in towards the region where it is unknown whether the obstacle sets are identical.

Case (ii) can create uncertainty as to whether the circular obstacles can be recovered by using back-scatter rays. We will use the no-eclipse condition to make an argument that back-scatter rays exist for all circular obstacles.

Lemma 3.1. *For equal sized circular obstacles, the no-eclipse condition guarantees that back-scatter rays exist for all circular obstacles.*

Proof. We consider three circular obstacles K_1, K_2, K_3 with equal radii r. We apply the no-eclipse condition such that $\text{conv}(K_i \cup K_j) \cap K_k = \emptyset$ for $i \neq j \neq k$. We consider drawing two lines, T_1, T_2, originating at the centre of obstacle K_i and being tangent to K_j and K_k respectively. We have that $T_1 \cap \partial K_j = \{a\}$ and $T_2 \cap \partial K_k = \{b\}$. This set up is illustrated in Figure 9. Let us define some minor circular arc $c(s)$, $0 \leq s \leq 1$, along ∂K_i defined by $c(0) = \partial K_i \cap T_1$, $c(1) = \partial K_i \cap T_2$. Let us assume that $c(s) = \emptyset$, i.e. no back-scatter rays exist from ∂K_i. This would imply that $c(0) = c(1)$, or $\partial K_i \cap T_1 = \partial K_i \cap T_2$. As a result, we have that $T_1 = T_2$. Thus, $T_1 \cap \partial K_j = T_2 \cap \partial K_k$, i.e. $a = b$. However this would imply that $\text{conv}(K_i \cup K_j) \cap K_k = \{a\}$, which is a contradiction to the no-eclipse condition. Therefore we must have that $c(s) \neq \emptyset$, and hence back-scatter rays must exist for all circular obstacles. $\square$

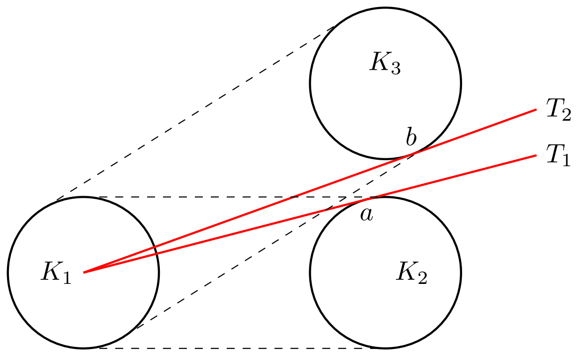


Figure 9: Proof of existence of back-scatter rays under the no-eclipse condition. The red lines indicate the allowable region for back-scatter rays, and dashed lines indicate the convex hull of pairs of obstacles.

So far we have shown that we can shrink W_a such that only the concave obstacle is unknown. As we continue to shrink a, the local convexity of the concave obstacle implies all scattered rays will be reflected back into the region where the obstacle is known. This leaves the reflection from z as a single unknown reflection point and from either Claim 2.1 or Claim 2.2, the point is uniquely recoverable. As we shrink further down, we may have that back scatter rays pass through the region $\mathbb{R}^n \setminus W_a$, where it is unknown whether the two obstacles are the same. By construction, the defining circle of the circular arc does not contain any obstacles inside, and so rays that pass through this region will not have any reflection points. As we shrink a to the radius of the defining circle, by choice of the subtending angle $\beta < \pi$, we will have no multiple reflection back scatter rays along the circular arc. This once again allows for unique definition of these points by their travelling times. Having resolved the defining circle, we are now free to shrink $a \to 0$ as there remain no other obstacles in the set. Hence we have showed that $a = 0$ and so $K = L$. $\square$

4 Discussion and Conclusion

Through this work, we have been able to show that obstacles which possess a restricted concavity are able to be uniquely determined by their travelling time distributions. By using arguments similar to that presented in [5], we have shown that a concave obstacle with its concavity defined by a circular arc and some collection of circular obstacles possess a unique travelling time spectra. However, several simplifying assumptions have been made which can form the basis for future work, such questions are as follows:

(i) Can we consider more general concavities other than circular arcs?

(ii) Can we consider strictly convex obstacles other than equally-sized circles?

(iii) Can we remove the no-eclipse condition on the convex obstacles?

Resolving these questions would open up a broader class of obstacles in $\mathbb{R}^n$ which have unique travelling time distributions and hence are uniquely recoverable.

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