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Trisecting Hyperbolic 4-manifolds

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Abstract

The Heegaard genus of a 3-manifold has extensively been studied in terms of algebraic, geometric, and topological properties of 3-manifolds. The generalisation of this notion to a trisection of a 4-manifold by Gay and Kirby prompts the question of determining natural families of trisections of 4-manifolds, and their trisections of minimal genus. There are certain methods which help one determine trisections, and for the known hyperbolic 4–manifolds of smallest volume, these methods leave a large gap in complexity bounds on the trisection genus. The aim of this project is to close this gap, and to determine at least the order of magnitude of the trisection genus for a number of hyperbolic 4–manifolds.

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1 Introduction

In the area of topology, there are techniques that can reduce many problems in high-dimensional manifold theory into purely algebraic problems. Geometric topology then is a unification of the areas of topology where these techniques in general are not used. A *manifold* is a topological space that locally looks flat – each point has a neighbourhood which can be 'deformed' (is homeomorphic to) Euclidean space $\mathbb{R}^n$. In this case, we call it an n-manifold. Geometric topology refers largely to manifolds of low dimensions, that is, dimensions up to four.

Fundamental questions in the area involve finding topological characterisations of these low-dimensional manifolds. In particular, relevant to this project, the philosophy of decomposing manifolds into simple pieces can reveal a lot of their geometric and topological structure. An example of this is Thurston's geometrisation conjecture, whereby through a decomposition of any closed 3-manifold, it can be shown that it admits one of only eight possible types of geometries.

The scope of the project relies on decomposition methods of 4-manifolds. Going one dimension down, a Heegaard splitting decomposes a 3-manifold into two simpler parts. This technique was recently generalised by Gay-Kirby to a trisection of 4-manifolds. To this trisection one can associate a 2-manifold, which has its *genus*, or number of holes, as one of its topological invariants. The following question is then natural to ask:

Given a 4-manifold, what is its minimal trisection genus?

In this report we first build up the necessary conceptual background to understand trisections, done so from sections 2 through to 4. We start with the theory of 2-manifolds, also known as surfaces, and discuss the visualisation and computation of topological invariants such as genus. We then move on to 3-manifolds, considering first the ideas of handlebody decompositions before using this to motivate the theory surrounding Heegaard splittings. Throughout we also assume basic knowledge of topological theory and terminology.

In Section 4 we examine the generalisation of Heegaard splittings to 4-manifolds, and explore a computational method to determine the trisection genus using the notion of a tricolouring. Following this we give examples of computing the trisection genus for specific manifolds in order to demonstrate its relative inefficiency. For example, a naive construction of the Davis manifold M_D gives an upper bound of 864,000, which in [CT] was reduced to

$$96 \leq g(M_D) \leq 5,621.$$

The goal of this project was to attempt to reduce these bounds to within an order of magnitude, and possibly find more efficient constructions of other specific manifolds.

The results presented in this report are combined findings by both the author and Tony Wang from the University of Sydney. The theory presented is entirely based on previous work done by authors as cited throughout.

2 2-manifolds (surfaces)

A 2-manifold, also known as a *surface*, is a manifold that locally looks like 2-dimensional space. An entity living 'in' the surface will only see two dimensions of space – an example of this is the hollow sphere, S^2, seen in Figure 1. In general, S^n is the n-sphere. So for example, S^1 is a circle, and S^3 is the 3-sphere.

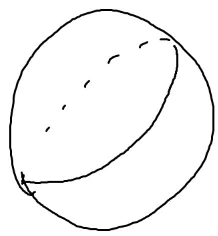

Figure 1: The 2-sphere S^2.

2-manifolds, and in general n-manifolds, can be built through taking the product of two other manifolds. An example in surfaces is the construction of the hollow torus T^2 through the product of two circles, as seen in Figure 2.

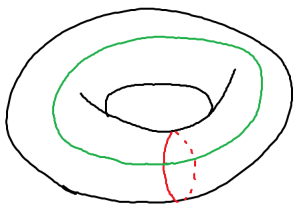

Figure 2: The 2-torus $T^2 = S^1 \times S^1$.

Another way to visualise the torus is by identifying opposite sides of a square to each other, as seen in Figure 3. One imagines cutting up and unrolling the torus to reach the square, or otherwise gluing together two (green) sides of the square to form a cylinder, and then gluing the two bounding (red) circles together to form the torus. The original square has 4 edges and 4 vertices, whereas the new identified flat torus has 2 edges and 1 vertex.

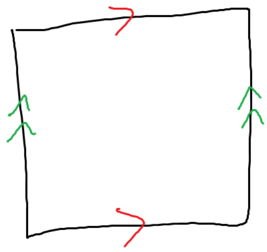

Figure 3: The 2-torus represented as a flat torus.

The last way we consider building manifolds is through a *connected sum*, or a gluing of pre-existing manifolds. We can build a 2-holed torus by gluing together two one holed tori, by cutting out small disks in each of them and identifying their boundaries together. This is demonstrated in Figure 4, but in general we can sum up any number of tori or other manifold.

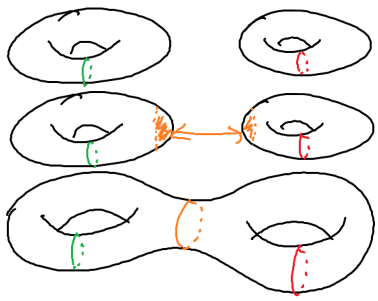

Figure 4: Connected sum of two tori.

Theorem 2.1. (Möbius, Jordan, 1860s) Every connected closed orientable surface is homeomorphic to a g holed torus (with a 0 genus surface being a sphere). The number g is the genus of the surface – it is a topological invariant.

In this way, we can associate to each surface a topological invariant that stays constant regardless of how we deform it. In line with the philosophy of decomposing manifolds, we can imagine decomposing a surface by pulling out simpler manifolds, 1-holed tori, in order to analyse certain topological characterisations.

3 3-manifolds

A 3-manifold locally looks like 3-dimensional space. A common example is the three torus, which is given by the product $T^3 = S^1 \times S^1 \times ^1$. To visualise this we recall how the hollow torus T^2 was represented by a square through identifying opposite sides. For T^3, we identify opposite sides of a cube, as in Figure 5. The A's, B's, and C's on opposite sides look the same. Of note, from the cube which has 6 faces, 12 edges, and 8 vertices, the 3-torus now has 3 faces, 4 edges, and 1 vertex.

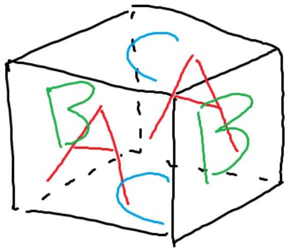

Figure 5: A 3-torus.

3.1 Handle decompositions

In the spirit of decomposing manifolds we present a way of doing so with 3-manifolds, using the concept of handles. Much of the theory here can be found in [Schultens].

Definition 3.1.1. A k-handle is an n-ball, $[0,1]^n$, attached to some manifold along $[0,1]^{3-k} \times \partial[0,1]^k$.

The visualisations of these can be seen in Figure 6, with the attaching regions highlighted in orange. In particular, a 0-handle in 3-dimensions is a 3-ball attached to the empty set. A 1-handle is like a cylinder that attaches via its bounding disks. A 2-handle attaches along its annular surface, and a 3-handle attaches along its entire spherical boundary. Using these handles one can build the 'basic' manifolds for decomposing a 3-manifold.

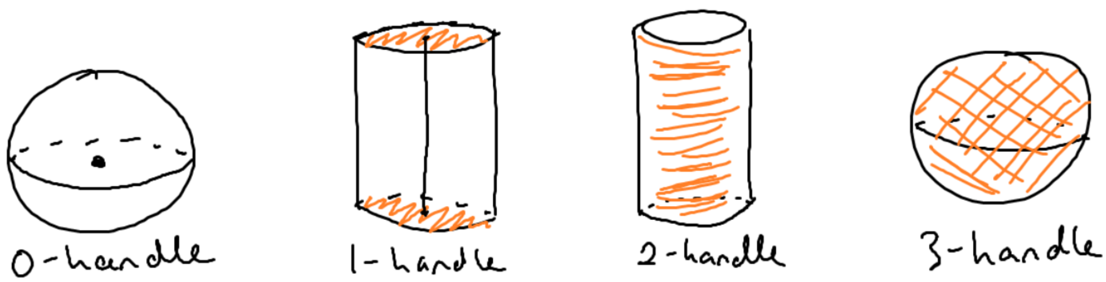

Figure 6: k-handles in 3 dimensions.

Definition 3.1.2. A *handlebody* is a compact connected orientable 3-manifold that can be decomposed entirely into 0-handles and 1-handles.

An example of such a handlebody is given in Figure 7, where the handles themselves are attached again by the parts highlighted in orange.

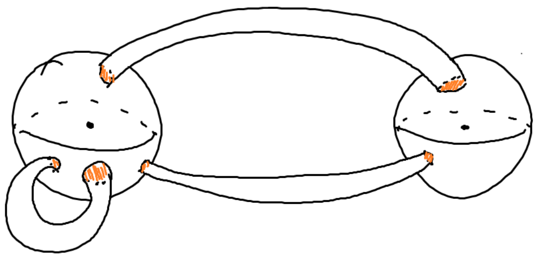

Figure 7: A handlebody composed of two 0-handles and three 1-handles.

3.2 Heegaard Splittings

The particular method of decomposition of a 3-manifold into handlebodies is a Heegaard splitting, which bisects the manifold into two handlebodies. This is visualised through Figure 8.

Definition 3.2.1. A *Heegaard splitting* is a decomposition

$$M = H_0 \cup_\Sigma H_1,$$

where each of the H_i are 1-handlebodies, and $\Sigma = H_0 \cap H_1$ is a 2-manifold, or surface, termed the *Heegaard surface*.

Figure 8: Heegaard splitting.

Theorem 3.2.2. **(Heegaard, 1898)** Every closed orientable 3-manifold admits a Heegaard splitting.

From this result we can associate to each 3-manifold a Heegaard surface, which (recalling from Section 2) means that in turn the genus of Σ can be associated to the 3-manifold. Moreover, two splittings are *equivalent* if their Heegaard surfaces are isotopic.

Definition 3.2.3. Let $M = H_0 \cup_\Sigma H_1$ be a Heegaard splitting, and let $S^3 = V \cup_T W$ be a genus 1 Heegaard splitting of the 3-sphere. The connected sum $(M, S)\#(S^3, T)$ defines a Heegaard splitting of M, given by

$$M = V' \cup_{S'} W'.$$

This is an *elementary stabilisation* of the initial Heegaard splitting.

A *stabilisation* of a Heegaard splitting is a splitting obtained from a finite number of elementary stabilisations. Stabilising a Heegaard splitting increases the genus of the Heegaard surface, which at first appears to complicate matters. This is not entirely true.

Theorem 3.2.4. (Reidemeister-Singer) Any two Heegaard splittings of a 3-manifold become equivalent after a finite number of stabilisations.

In particular, when we now consider the genus of the Heegaard surface, we consider the minimal genus of the splitting surface, also called the *Heegaard genus* of a 3-manifold. In this way the natural question, and precursor to the question given in the introduction, is as follows:

Given a 3-manifold, what is its Heegaard genus?

4 4-manifolds

4.1 Trisections

The generalisation of Heegaard splittings to 4-manifolds as a trisection was introduced in Gay-Kirby, and is visualised through Figure 9.

Definition 4.1.1. A *trisection* is a decomposition

$$M = H_0 \cup H_1 \cup H_2$$

such that

- each H_i is a 4-dimensional 1-handlebody
- $H_i \cap H_j$ is a 3-dimensional 1-handlebody $(i \neq j)$
- $\Sigma = H_0 \cap H_1 \cap H_2$ the *central surface.*

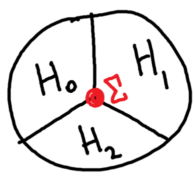

Figure 9: Trisection.

Theorem 4.1.2. (Gay-Kirby, 2016) Every 4-manifold has a trisection.

In a similar fashion, we associate to a trisection a genus, and can ask the same question of what possible minimal trisection genus may be for a given 4-manifold. Low complexity manifolds have been classified already by Meier and Zupan, however for other 4-manifolds the answer is not so clear.

An algorithmic method of determining bounds on trisection genus first begins with constructions the trisection. One way of doing so is presented in [BHRT] which involves the notion of tricolouring the vertices of a 4-simplex, the generalisation of a 3-simplex (tetrahedron) to 4-dimensions. Specifically, by colouring each 4-simplex (pentachoron) with three colours in such a way that two colours have two vertices and the third is a single vertex, and considering how these relate to trisections, they obtained the following result.

Theorem 4.1.3. (Bell, Hass, Rubinstein, Tillmann, 2018) Given an arbitrary triangulation of a closed orientable 4-manifold into n 4-simplices, there exists a triangulation with $120n$ 4-simplices that gives a trisection of the manifold.

Corollary 4.1.4. (BHRT) Given an arbitrary triangulation of a closed orientable 4-manifold into m 4-simplices, there is an associated trisection with genus *at most* $60m$.

5 Examples

5.1 Davis Manifold

The 120-cell is the 4-dimensional analogue of the regular dodecahedron in 3-space. It is a polytope consisting of 120 cells (dodecahedra) with 3 around each edge, and 4 meeting at each vertex. The 120-cell itself also has 720 pentagonal faces, 1200 edges, and 600 vertices. The Davis manifold M_D is constructed from the 120-cell by identifying opposite pairs of cells in the 120-cell via their reflecting hyperplanes in hyperbolic space, resulting in 60 dodecahedra, 144 pentagonal faces, 60 edges, and a single vertex. This identification is similar to what was done with the 2 and 3-tori.

A triangulation of M_D can be obtained by placing a vertex v_0 at the centre of the 120-cell, v_1 at the centre of a dodecahedral face, v_2 at the centre of a pentagonal face, v_1 at the barycentre of an edge of a pentagon, and v_0 at a vertex of this edge. The resulting 4-simplex is a *Coxeter 4-simplex* denoted Δ_3, which tiles both the 120-cell as well as the entire hyperbolic space as well.

From previous discussion, there are $1 \cdot 120 \cdot 12 \cdot 5 \cdot 2 = 14400$ of these 4-simplices, each of which are copies of the *fundamental domain* Δ_3. One can then imagine a group acting on Δ_3 via reflections to tile the 120-cell and $\mathbb{H}^4$. In any case, from Corollary 4.1.4 we can see that the upper bound on the trisection genus of this triangulation is $60 \cdot 14400$. The lower bound computed from previous work in

[CT] is 96, hence giving the bounds

$$864,000 \geq g(M_D) \geq 96.$$

The tricolouring introduced in [BHRT] allows an algorithm that computes the upper bound and improves it significantly – the following discussion described this. Colour the v_0 and v_1 with one colour (green), v_2 with another (blue), and v_3 and v_4 with the fourth (red). Taking pairs of 4-simplices and performing a move that converts it to four 4-simplices together allows one to 'join up' the isolated blue vertices, after which it can be proved that the resulting coloured triangulation also results in a trisection. The number of pentachora increases to 28800. The genus computed from this is 7201, from which we obtain

$$7,201 \geq g(M_D) \geq 96.$$

By reducing the number of choices of vertices to pick more efficient triangulations (in particular, removing the necessity for choosing v_1 at barycentres of edges of each pentagonal face), this upper bound can be reduced further to 5621.

6 Discussion and Conclusion

The approach of Section 5.1 using Coxeter simplices can potentially be used in other hyperbolic manifolds in order to determine tighter bounds on their minimal trisection genus. Other directions to take this project are to find better lower bounds on manifolds using their geometry. We did not manage to find a better upper bound for the Davis Manifold due to time constraints, with the main limitation being checking if more efficient tricolourings can still obtain the appropriate conditions to get a trisection. It would be nice to also obtain concrete combinatorial constructions of other hyperbolic 4-manifolds in order to explore efficient constructions to find upper bounds for those ones.

7 Acknowledgments

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All diagrams were hand drawn.

8 References

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