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Gravitational Waves: A Mathematical Analysis

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Abstract

With the recent discovery of Gravitational Waves, this project aims to understand the propagation of such waves in a vacuum. To do this, we first explore the mathematical fundamentals underpinning a description of space-time. We introduce tensors, specifically a metric tensor, to describe the relative structure between tangent planes at points on a manifold. We then define a natural derivative operator for a given metric tensor and hence arrive at a general description of curvature. Using these, we linearise Einstein's equation in a vacuum with weak gravity. Finally, we study the gravitational waves predicted by this model and their plane wave solutions.

1 Introduction

Einstein's field equations describe gravity as fluctuations of curvature in space-time. These fluctuations can propagate through space-time at the speed of light as Gravitational Waves (Abramovici et al. 1992, p. 4). Recently, gravitational waves have been directly observed by the LIGO detectors in the United States, which are extremely large and sensitive interferometers (LIGO 2016). This observation of gravitational waves has opened up a new way of looking at the universe and has motivated this report. Here, we construct a mathematical model for the curvature of space time, with an introduction to tensor calculus. We then build a notion of derivatives and curvature, culminating in simplifying Einstein's equation under the assumption of weak gravity. This model uses a rudimentary linearisation of Einstein's equation in a vacuum, but still predicts the existence of gravitational waves. The information presented is based of Wald's General Relativity, published in 1984, with details of derivations and further explanations provided by myself. A working understanding of multi-variable calculus, linear algebra and special relativity is assumed, and can be obtained from any undergraduate course in these topics.

2 Tensor Fields

2.1 Tangent Vectors on Manifolds

For our purposes, it suffices to say that manifolds are a locally flat structure with unknown large-scale behaviour and therefore no global coordinate system. Earth is a 2-manifold as it is locally flat and two-dimensional, but globally a sphere. As such, any local coordinate system has no natural generalisation to cover the entire globe. Space-time is a 4-manifold.

A vector space V_p can be constructed at each point p on the manifold from tangent vectors, but an explicit understanding of the construction is not necessary. These spaces are required for the later use of the metric tensor; details can be found in Chapters 2.1 and 2.2 of Hawking and Ellis (1973).

2.2 Tensors

A tensor is a multi-linear map into the real numbers. We will use tensors to describe metrics, curvature, and subsequently gravitation.

Definition 2.2.1. Let V be a finite dimensional vector space with basis $(v^1, \ldots v^n)$ and define V^* to be its corresponding dual vector space with basis $(v_1^*, \ldots, v_n^*)$, where $v_i^*(v^j) = \delta_i^j$. Then, a tensor T of type (k, l) is the map:

$$T : \underbrace{V^* \times \ldots \times V^*}_{k \text{ entries}} \times \underbrace{V \times \ldots \times V}_{l \text{ entries}} \to \mathbb{R}\,.$$

T is linear with respect to each slot, where the ith slot refers to ith entry in the domain.

It can be considered as an assignment of a value to a collection of vectors and dual vectors, similar to a height function if T was of type $(0, 2)$. In physics, where vectors commonly represent locations and dual vectors represent forces at given locations, a tensor provides a value that arises from the combination of certain forces at certain locations. An assignment of a tensor for each V_p on the manifold is called a tensor field[1].

2.2.1 Tensor Operations

Addition and scalar multiplication: Quite natural, operates on the results of the tensors. If T returns a and T' returns b, then $T + T' = a + b$. For any real number k, $kT = ka$.

Thus, a tensor has a vector space structure, with a basis defined by its value when operating on the bases of V and V^*. A tensor of type (k, l) is of dimension n^{k+l} where n is the dimension of the vector space V. Hence, one way of defining a tensor is by taking a linear combination of the basis vectors.

Outer product: The outer product, denoted $\otimes$, of tensors T of type (k, l) and T' of type (m, n) is a new tensor of type $(k + m, l + n)$ and is the multiplication of the results. If T returns a and T' returns b, then $T \otimes T' = ab$.

[1] A tensor field T is smooth if for all smooth inputs, T is smooth. The smoothness of dual vectors is defined similarly by operation on vectors. We will henceforth assume all vectors and therefore tensors are smooth.

Contraction: A contraction is the summation over basis elements in the ith dual vector slot and the jth vector slot. It is defined as a map $C : \mathcal{T}(k,l) \to \mathcal{T}(k-1,l-1)$, where $\mathcal{T}(k,l)$ is the space of all tensors of type (k,l). If $T \in \mathcal{T}(k,l)$, then

$$C(T) = \sum_{\sigma=1}^{n} T(\ldots, \underbrace{v_\sigma^*}_{i\text{th slot}}, \ldots; \ldots, \underbrace{v_\sigma}_{j\text{th slot}}, \ldots),$$

where $\{v_\sigma\}$ is a basis for the vector space and $\{v_\sigma^*\}$ is a basis for the dual vector space. This generalises taking the trace when applied to one tensor, and is akin to the operation of one tensor on another when contracting across separate tensors.

2.3 Abstract Index Notation

This description of a tensor as operations on elements in a basis is generally disadvantageous, and instead we will use the Abstract Index Notation. This captures the essence of the tensor whilst distancing it from coordinate systems.

A tensor T of type (k,l) is denoted using superscripts for dual vector slots and subscripts for vector slots, such as $T^{a_1 a_2 \ldots a_k}{}_{b_1 b_2 \ldots b_l}$ where T operates on k dual vectors and l vectors. The tensor $T^{abc}{}_{de}$ is therefore of type $(3,2)$. They can be replaced arbitrarily, and hence, tenors are specified by the character and not the indices.

In this notation, ω_a is a dual vector as it acts on a vector and v^a is an tensor that acts on a dual vector. However, we can take any dual vector ω_a and define $v^b(\omega_a) \equiv \omega_a(v^b)$, and so there is a natural isomorphism between tensors of type $(1,0)$ and vectors. Hence, vectors will be denoted as v^a. Note that v^a and v^z are the same vector and the index has no meaning in isolation.

When tensors are paired with a coordinate system, each index can range from 1 to n where n is the dimension of the system. These represent the components of the tensors with respect to a basis of the coordinate system. Then ω_a has components $(\omega_1, \ldots, \omega_n)$ and v^a has components $(v^1, \ldots, v^n)$. Note that the components $\omega_1, \ldots, \omega_n; v^1, \ldots, v^n$ have no meaning unless paired with a coordinate system.

2.3.1 Operations in the Abstract Index Notation

Outer Product: We omit the $\otimes$ and simply concatenate tensors. The outer product of $S^{ab}{}_c{}^d$ of type $(3,1)$ and $S_e{}^f{}_{gh}$ of type $(1,3)$ is denoted as $S^{ab}{}_c{}^d S_e{}^f{}_{gh}$ of type $(4,4)$.

Contraction: This is done by repeating the index in which the contraction is conducted over. $T^{abc}{}_b$ is a tensor of type $(2,0)$ obtained by contracting $T^{abc}{}_d$ on the 2nd dual vector and 1st vector.

Finally, note that we have been using Latin characters for these indices. The convention is Latin script for abstract use and Greek letters if the tensor is in a coordinate system.

2.3.2 Symmetry and Antisymmetry

Finally we introduce notation to simplify expressions that can arise when manipulating tensors. We define:

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \,,$$

and

$$T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}) \,.$$

To generalise, the round brackets sum over all permutations of the indices inside, while the square brackets add the even permutations and subtract the odd permutations. These are then both divided by the number of permutations as follows.

$$T_{(a_1 \ldots a_n)} = \frac{1}{n!}(\sum_{\sigma} T_{a_{\sigma}(1)} \ldots T_{a_{\sigma}(n)} \,,$$

$$T_{[a_1 \ldots a_n]} = \frac{1}{n!}(\sum_{\sigma} \delta_{\sigma} T_{a_{\sigma}(1)} \ldots T_{a_{\sigma}(n)} \,,$$

where σ represents all permutations and δ_σ denotes parity of permutation $\operatorname{sgn}(\sigma)$.

2.4 The Metric Tensor

A metric is an expression of the relationship between two inputs, and is how we will relate points on our manifold. Taking vectors as inputs, a metric $g_{ab} : V_p \times V_p \to \mathbb{R}$ is a symmetric tensor of type $(0, 2)$. It is similar to an inner product, except without the positive definite requirement. Let v^a and v^b be two vectors on some vector space. Then to take the metric g_{ab} of these two vectors we write $g_{ab}v^a v^b$. Note that the initial indices can be anything, but when we take the metric they must be correctly repeated as shown.

A metric can be represented by a matrix for computations in a similar manner to that of the inner product. An example of a metric g_{ab} is the distance metric in 3-dimensional Euclidean space with standard coordinates,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

This is equivalent to

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \,.$$

These are both valid representations of the same metric. In this fashion, g_{ab} can be thought of a matrix with components indexed by values of a and b as follows:

$$g_{ab} \equiv \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \\ g_{n1} & & g_{nn} \end{bmatrix} ,$$

although this is an abuse of notation as g is not necessarily a matrix.

2.4.1 Additional Convention for the Metric Tensor

As all metrics are of type $(0, 2)$, we can define related metrics as follows without ambiguity. Let g_{ab} be a metric tensor. We denote the inverse of g, $(g_{ab})^{-1}$ simply as g^{ab}. This can be done as metrics do not operate on dual vectors. Furthermore, the contraction of a metric tensor and a vector, $g_{ab}v^b$ is denoted v_a. This explicitly defines the isomorphism between V and V^*. Subsequently, the raising or lowering of an index refers to the application of the metric tensor on the index as follows:

$$g^{ab} g_{bc} = \delta^a{}_c \,,$$

$$T^a{}_b{}^c{}_{de} = g_{bf} g^{eh} T^{afc}{}_{dh} \,.$$

3 Curvature

We seek a description of the curvature of a manifold generated purely from the metric, as it is the only quantifier available to us. Curvature is closely related to *parallel transport*, the idea of moving a vector while keeping it parallel to itself.

Definition 3.0.1. A geodesic is a curve whose tangent is parallel transported along itself.

This can be understood as the curve always being in the same 'direction' as the projection of its tangent onto the surface. Then, a space is curved if two initially parallel geodesics do not stay parallel, implying that parallel transport is path-dependant. To define our liberal use of 'parallel', we need a notion of taking a derivative.

3.1 Derivative Operators

A global coordinate system does not exist on a manifold and so we must forego the usual notion of a derivative. Instead, let ∇ be a derivative operator. We want this to be a natural extension of the classical directional derivative, ∂. For some linear function $y = Ax$, we know $\frac{\partial y}{\partial x} = A$. Taking the tensor analogue, if $y = A_a x^a$, then we want $\nabla y = A_a$. And so, let ∇ be an operator that takes a tensor field of type (k, l) to a tensor field of type $(k, l + 1)$. It is useful to denote this operator with a subscript index, ∇_a, as it takes an extra vector as an input.

Furthermore, we want it to obey the following generalisations of the ordinary derivative.

1. Linearity: For tensors S, T and real number α,

$$\nabla_c (T^{a_1 \dots a_k}{}_{b_1 \dots b_k}) + \alpha S^{a_1 \dots a_k}{}_{b_1 \dots b_k} = \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_k} + \alpha \nabla_c S^{a_1 \dots a_k}{}_{b_1 \dots b_k} \, .$$

2. Leibniz (chain rule):

$$\nabla_c (T^{a_1 \dots a_k}{}_{b_1 \dots b_k} S^{a_1 \dots a_k}{}_{b_1 \dots b_k}) = \nabla_c (T^{a_1 \dots a_k}{}_{b_1 \dots b_k}) S^{a_1 \dots a_k}{}_{b_1 \dots b_k} + T^{a_1 \dots a_k}{}_{b_1 \dots b_k} \nabla_c (S^{a_1 \dots a_k}{}_{b_1 \dots b_k}) \, .$$

3. Commutativity with contraction:

$$\nabla_c (T^{abc}{}_{dec}) = \nabla_c T^{abc}{}_{dec} \, .$$

4. Consistency with the notion of tangent vectors as directional derivatives on scalar fields. For the scalar field f,

$$t(f) = t^a \nabla_a f \, ,$$

where $t(f)$ is taking the directional derivative with respect to unit vector t, often denoted $\partial_t f$. Note that this subscript is not a tensor index, but rather a vector.

5. Torsion Free. For a scalar function f,

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f \, .$$

This condition is a requirement we will impose for our model of gravitation, but is not strictly required.

3.1.1 Existence and Uniqueness?

We must now show what these five requirements create. With some manifold magic,[2] we can show that for any point on a manifold, there locally exists a derivative operator that satisfy these conditions.

However, these conditions are not a unique specification. How do two derivative operators that satisfy the above differ? From condition (4), they must be identical on a scalar field f. To consider their difference on the next highest order tensor, let ω_a be a smooth dual vector field. Then $f\omega_a$ is a general quantity defined on the manifold. We take the difference $\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b)$ and use the Leibniz rule to show

$$\begin{aligned}\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b) &= (\tilde{\nabla}_a f)\omega_b + f\tilde{\nabla}_a\omega_b - (\nabla_a f)\omega_b - f\nabla_a\omega_b \\ &= f(\tilde{\nabla}_a\omega_b - \nabla_a\omega_b) \\ &= f(\tilde{\nabla}_a - \nabla_a)\omega_b\,.\end{aligned}$$

Therefore this $\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b)$ is independent of f and solely depends on the value of ω_a at a point. The left-hand side is an operation takes a dual vector ω_b and returns the right-hand side, a tensor-like object of type $(0,2)$ as ∇_a takes ω_b of type $(0,1)$ to type $(0,2)$. Hence $\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b)$ is an operation of type $(1,2)$. As this map is linear, we can consider it a tensor of type $(1,2)$ despite each ∇_a ordinarily not being a tensor.

Call this tensor $C^c{}_{ab}$. This tensor is symmetric in lower indices by property 5 earlier. The relationship between the effect of two different derivative operators on dual vectors is[3]

$$\nabla_a\omega_b = \tilde{\nabla}_a\omega_b - C^c{}_{ab}\omega_c\,. \tag{3.1}$$

Flat geometry has the ordinary partial derivative $\partial_a = (\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_n})$ associated with it. This always exists locally but necessarily globally, as per the definition of a manifold. When $\tilde{\nabla}_a$ is this ordinary partial derivative, the associated $C^c{}_{ab}$ is called the *Christoffel symbol* and denoted $\Gamma^c{}_{ab}$. This is a local specification as the ∂_a is local, but the relationship described in equation (3.1) is global.

Now we still have freedom in our choice of derivative operator, and have not yet related it to the metric. To rectify this, we impose that this ∇_a should satisfy the following *Levi-Civita* condition for any metric tensor g_{ab}:

$$\nabla_a g_{bc} = 0\,.$$

[2]For a region on the manifold, take the coordinate basis associated with the chart that covers the region. Then define $\partial_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ to have components which are the standard partial derivatives with respect to the basis. This trivially satisfies the above.

[3]We choose to subtract C to make calculations in appendix A easier, but this could be an addition instead.

This is a unique specification[4] and thus we will use this for the rest of the report.

3.2 The Riemann Curvature Tensor

Now we can begin describing curvature. We first consider the difference between taking successive derivative operators ∇_a. Using the Leibniz rule twice, we get

$$\begin{aligned}\nabla_a \nabla_b (f\omega_c) &= \nabla_a(\omega_c \nabla_b f + f\nabla_b \omega_c) \\ &= (\nabla_a \nabla_b f)\omega_c + \nabla_b f \nabla_a \omega_c + \nabla_a f \nabla_b \omega_c + f\nabla_a \nabla_b \omega_c \,.\end{aligned}$$

Similarly, by swapping indexes a and b, we get

$$\nabla_b \nabla_a (f\omega_c) = (\nabla_b \nabla_a f)\omega_c + \nabla_a f \nabla_b \omega_c + \nabla_b f \nabla_a \omega_c + f\nabla_b \nabla_a \omega_c \,.$$

Taking the difference between these equations, and noting that the derivatives commute on the scalar field f, we see that the first three terms on the right hand side cancel and we arrive at

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f\omega_c) = f(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c \,.$$

As with the derivation of equation (3.1), this now defines a tensor of type $(1, 3)$. It is known as the *Riemann curvature tensor* and denoted ${R_{abc}}^d$:

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = {R_{abc}}^d \omega_d \,. \tag{3.2}$$

A cumbersome derivation[5] shows that the transportation of a vector on a closed curve differs from the original vector in the second order by a term proportional to the Riemann tensor. Specifically, let δv^a represent the change in some vector v^a when parallel transported around a closed square with edges parametrised by t and s. Then

$$\delta v^a = \Delta t \Delta s v^d T^c S^b {R_{cbd}}^a$$

where T^c and S^b are tangent to curves of constant s and t respectively. Thus the Riemann tensor directly measures the path dependence of parallel transport and thus the curvature of a manifold. We further define the Ricci Tensor, R_{ac}, as

$$R_{ac} = {R_{abc}}^b \,,$$

and the scalar curvature, R, as the trace of the Ricci tensor:

$$R = g^{ab} R_{ab} = {R_a}^a \,.$$

[4] See appendix A.

[5] See appendix B.

4 General Relativity

4.1 Suitability of Tensors and General Covariance

Now that we've established the mathematical tools required, we can apply them to the physical world. All quantities in physics are measurable as numbers, sometimes with a location or direction, and so it makes sense that physical effects are represented as maps from vectors and dual vectors to the real numbers. Such a map can be decomposed into a linear combination of multi-linear maps via the Taylor expansion, and so tensors are a natural object for describing the laws of physics.

Next we will introduce the idea of *General Covariance*. General Covariance is the principle that in General Relativity, the only quantity pertaining to space that can appear in the laws of physics is the space-time metric. This is equivalent to the idea that there is no preferred coordinate system, as there cannot be a specific Christoffel symbol.

4.2 Einstein's Equation

Einstein's founded his theory of general theory of relativity on two main ideas:

1. That all bodies are influenced by gravity in the same way, and

2. That the structure of space-time is influenced by the presence of matter.

Together, these ideas propose that the force of gravity is actually the manifestation of the effect of a curved space-time due to the presence of mass. And so, an object 'at rest' on the surface of the Earth is not experiencing balanced forces as predicted by Newtonian mechanics. Rather, an unbalanced normal force prevents it from moving relative to Earth (Figure 1).

These ideas form the foundations upon which the Einstein equation is derived. From his 1915 paper, we are presented with the following equation:

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}\,, \tag{4.1}$$

where T_{ab} is the stress-energy-momentum tensor. Consider this to be the mass (and energy) in the area that G_{ab} is defined.

4.3 Linearised Gravity

The full non-linear Einstein's equation (4.1) is too complex to solve. Instead we will linearise Einstein's equation in an environment with weak-gravity. This is when the space-time metric g_{ab} is

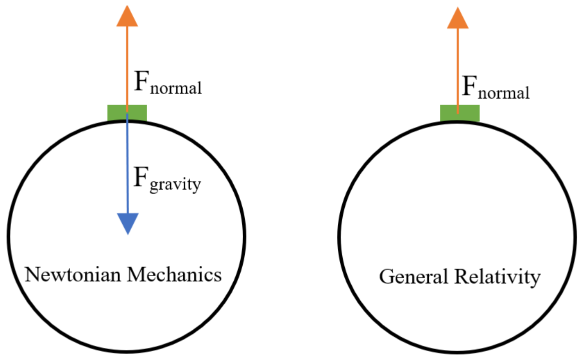

Figure 1: Newtonian Mechanics (Left) states that an object (green) 'at rest' on the surface of Earth (black) has two equal and opposite forces acting upon it, whereas General Relativity states that there is only one unbalanced force.

almost flat and can be written as the sum of the flat space-time metric in special relativity η_{ab} and some small perturbation η_{ab}:

$$g_{ab} = \eta_{ab} + \gamma_{ab}\,. \tag{4.2}$$

The metric η_{ab} corresponds to the matrix diag$(-1, 1, 1, 1)$. Along with this, we will denote ∂_a as the derivative operator associated with the flat η_{ab}, and any raising and lowering of indices as operated by η_{ab} instead of g_{ab}; e.g. $T_a{}^b = \eta^{bc}T_{ac}$.

We substitute the assumption (4.2) into equation (4.1) and only retain terms linear in γ_{ab}, hence 'linearisation'. After making the necessary computations[6], we arrive at the linearised Einstein tensor $G^{(l)}_{ab}$:

$$\begin{aligned} G^{(l)}_{ab} &= R^{(l)}_{ab} - \frac{l}{2}\eta_{ab}R^{(l)} \\ &= \partial^c\partial_{(b}\gamma_{a)c} - \frac{1}{2}\partial^c\partial_c\gamma_{ab} - \frac{1}{2}\partial_a\partial_b\gamma - \frac{1}{2}\eta_{ab}\left(\partial^c\partial^d\gamma_{cd} - \partial^c\partial_c\gamma\right)\,, \end{aligned} \tag{4.3}$$

noting that $\partial^c\partial_{(b}\gamma_{a)c}$ is symmetrised on indices a and b.
To simplify this slightly, let

$$\overline{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma\,. \tag{4.4}$$

Then the linearised Einstein equation (4.3) becomes

$$G^{(l)}_{ab} = -\frac{1}{2}\partial^c\partial_c\overline{\gamma}_{ab} + \partial^c\partial_{(b}\overline{\gamma}_{a)c} - \frac{1}{2}\eta_{ab}\partial^c\partial^d\overline{\gamma}_{cd} = 8\pi T_{ab}\,. \tag{4.5}$$

Our initial assumption (4.2) made no specification to the coordinates, or *gauge*, that our system is in. Because of the freedom given by tensors to choose a coordinate system, we may choose the most

[6] See appendix C.

appropriate gauge to simplify our equations. We note that two perturbations γ_{ab} and γ'_{ab} represent the same physical phenomenon under different coordinates if they differ by the infinitesimal diffeomorphism $\partial_a \xi_b + \partial_b \xi_a$ (Witten 1963, p. 61). This means that we can make the gauge transformation:

$$\overline{\gamma}_{ab} \to \overline{\gamma}_{ab} + \partial_a \xi_b + \partial_b \xi_a \,. \tag{4.6}$$

We will enforce the *de Donder* gauge choice of $\partial^b \overline{\gamma}_{ab} = 0$. This gauge holds for any ξ_a that satisfies $\partial^b \partial_b \xi_a = -\partial^b \overline{\gamma}_{ab}$ and $\partial^b \partial_a \xi_b = 0$. This can be seen by reversing the transformation (4.6) in the de Donder gauge. If $-\partial^b \overline{\gamma}_{ab} = \partial^b \partial_b \xi_a$ and $\partial^b \partial_a \xi_b = 0$, then:

$$\begin{aligned} 0 &= \partial^b \overline{\gamma}_{ab} + \partial^b \partial_a \xi_b + \partial^b \partial_b \xi_a \\ &= \partial^b (\overline{\gamma}_{ab} + \partial_a \xi_b + \partial_b \xi_a) \\ &= \partial^b \overline{\gamma}_{ab} \,. \end{aligned} \tag{4.7}$$

A proof of existence of such a ξ_a can be found in Schutz (1985, pg. 205). With the gauge condition (4.7), equation (4.3) becomes $\partial^c \partial_c \overline{\gamma}_{ab} = -16\pi T_{ab}$ as the second and third terms vanish. In a vacuum, $T_{ab} = 0$ and this becomes

$$\partial^c \partial_c \overline{\gamma}_{ab} = 0 \,. \tag{4.8}$$

4.3.1 Plane Wave Solutions to Gravitational Radiation

The linearised Einstein equation in a vacuum (4.8) is analogous to the 3D wave equation as shown. Note that $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the inverse of $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

$$\begin{aligned} 0 &= \partial^c \partial_c \overline{\gamma}_{ab} \\ &= \eta^{cd} \partial_d \partial_c \overline{\gamma}_{ab} \\ &= -\frac{\partial^2 \overline{\gamma}_{ab}}{\partial x_0^2} + \frac{\partial^2 \overline{\gamma}_{ab}}{\partial x_1^2} + \frac{\partial^2 \overline{\gamma}_{ab}}{\partial x_2^2} + \frac{\partial^2 \overline{\gamma}_{ab}}{\partial x_3^2} \\ \frac{\partial^2 \overline{\gamma}_{ab}}{\partial x_0^2} &= \nabla^2 \overline{\gamma}_{ab} \,, \end{aligned}$$

where ∇^2 is the spacial Laplacian.

Hence, we can analyse tensor plane wave solutions of the form

$$\overline{\gamma}_{ab} = H_{ab} \exp(i k_\mu x^\mu) \,,$$

where H_{ab} is a tensor with constant components and k_μ is a vector. This is the generalisation of the standard $y = A \exp(ikt)$. Substituting this into (4.8), we get the following[7]:

$$0 = -k^c k_c \overline{\gamma}_{ab} \,.$$

[7] See appendix D

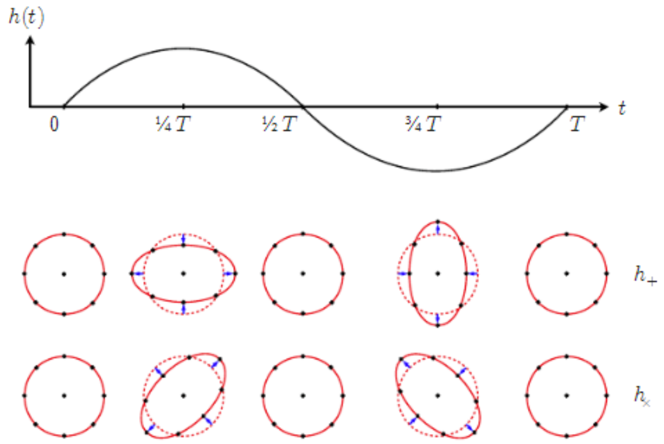

Figure 2: The effect of each basis wave on particles in the plane perpendicular to the direction of the wave. (Le Tiec and Novak 2017)

Therefore, for a non-trivial solution, k^c must be a null vector in Minkowski space, i.e.

$$k^c k_c = 0\,.$$

As k_c is the coefficient for x^μ, it translates into these waves propagating through space-time at the speed of light, as photons and other massless particles follow paths inscribed by null vectors (Schutz 1985).

There remains more degrees of freedom to fix. The radiation gauge $\gamma = 0$ can be made without violating (4.7) and enforces $\gamma_{0b} = 0$ (Wald 1984, p. 80). Let us now inspect the tensor H_{ab} represented as a 4×4 matrix. As $\exp(ik_\mu x^\mu)$ is never 0, the radiation gauge enforces that H_{ab} is of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_{11} & H_{12} & H_{13} \\ 0 & \times & H_{22} & H_{23} \\ 0 & \times & \times & H_{33} \end{bmatrix},$$

where the $\times$ are dependant other entries due to the symmetry of g_{ab}. The solution space for H_{ab} is a six-dimensional vector space, but the conditions $\gamma = 0$ and the de Donder gauge (4.7) enforce four[8] more conditions, reducing this space to two-dimensional. We can find a basis for this space using elemental linear algebra. By convention, the elements of this basis are labelled h_+ and $\mathrm{h}_\times$ (h-plus and h-cross), and have the effect of moving particles in a cross or plus shape (Figure 2) when viewed perpendicular to the motion of the wave (Le Tiec and Novak 2017, p. 41).

[8]The de Donder gauge is four equations, but one is redundant for index $a = 0$.

Using Fourier analysis, any gravitational wave can decomposed into a linear combination of these waves.

5 Discussion and Conclusion

In this report we have studied tensor calculus, developing a notion of curvature and laying the groundwork for understanding general relativity. By simplifying Einstein's equation in a vacuum to a wave equation, we derived properties of the plane wave solutions for gravitational waves. Further research would look into the energy radiated by such waves, and to understand origins of the waves in when described by a linearised equation. Techniques could also be explored to approach solving the full non-linear Einstein equation in a gravitational collapse scenario. Another avenue of study could be with regard to space-time containing one large body, described by the Schwarzschild metric. As gravitational waves have recently been discovered by LIGO, a comparison could be conducted between the energy in the waves and the loss of energy experienced by object causing the wave.

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7 References

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8 Appendix

A Unique Specification of the Derivative Operator

Theorem 8.1. There exists a ∇_a such that for any metric g_{ab}, $\nabla_a g_{bc} = 0$ (Wald 1984, pp. 35-36).

Proof. Let $\tilde{\nabla}_a$ be any derivative operator. To compute the difference between derivative operators, it is easier to consider the scalar $h^{ab}g_{ab}$. From applying the Leibniz rule on $h^{ab}g_{ab}$ substituted into equation (3.1) and then factoring out the h^{ab}, we arrive at

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} \,.$$

Then, by lowering the indices with g, we get

$$C_{cab} + C_{bac} = \tilde{\nabla}_a g_{bc} \,. \tag{8.1}$$

Making index substitutions, we get two more equations.

$$C_{cba} + C_{abc} = \tilde{\nabla}_b g_{ac} \,, \tag{8.2}$$

$$C_{bca} + C_{acb} = \tilde{\nabla}_c g_{ab} \,. \tag{8.3}$$

Then we take (8.1) + (8.2) - (8.3). We also note that $C^c{}_{ab} = C^c{}_{ba}$ by our choice of property 5 of derivative operators. Using this, we find that

$$2C_{cab} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} \,.$$

Rearranging, we get the following equation for $C^c{}_{ab}$:

$$C^c{}_{ab} = \frac{1}{2} g^{cd} [\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab}] \,. \tag{8.4}$$

This solves $\nabla_a g_{bc} = 0$ and is unique by construction and so concludes the proof. □

B Relation of Parallel Transport to the Curvature Tensor

Consider the closed square formed by travelling a small Δt upwards, Δs to the right, Δt downwards, and finally Δs to the left. Let v^a be any vector at the bottom left corner of this square. To compute the change in v^a, we will instead compute the change in the scalar $v^a \omega_a$ for any dual vector ω_a. The change in the first Δt is

$$\delta_1 = \Delta t \frac{\partial}{\partial t} (v^a \omega_a) \mid_{(\Delta t/2, 0)} \,,$$

where, by evaluating at the middle of the line segment, we ensure this term is accurate to the second order. Using property 4 of derivative operators from section **3.1**, we can write this as

$$\delta_1 = \Delta t T^b \nabla_b (v^a \omega_a) \mid_{(\Delta t/2,0)} , \tag{8.5}$$

where T^b is the vector in the direction of Δt. Using the Leibniz rule, this becomes

$$\delta_1 = \Delta t [\omega_a T^b \nabla_b v^a + v^a T^b \nabla_b \omega_a] \mid_{(\Delta t/2,0)} .$$

We note that for the parallel transport of v^a along curve T^b, $T^b \nabla_b v^a = 0$ and so this expression simplifies to

$$\delta_1 = \Delta t v^a T^b \nabla_b \omega_a \mid_{(\Delta t/2,0)} .$$

Similar expressions hold for the other three paths around the square, with negatives for the third and fourth paths. Let us consider the total change in v^a. Consider $\delta_1 + \delta_3$, noting that $\delta_2 + \delta_4$ is identical, save for a change of coordinates.

$$\delta_1 + \delta_3 = \Delta t \left(v^a T^b \nabla_b \omega_a \mid_{(\Delta t/2,0)} - v^a T^b \nabla_b \omega_a \mid_{(\Delta t/2,\Delta s)} \right) . \tag{8.6}$$

As $\Delta s \to 0$, the two terms in the brackets become equavilent and hence the total change is 0, and so there is no change in the first order evaluation. To consider the change in the second order, the change of the term in the brackets to first order Δt. We will approximate it by considering the change in v^a and $T^b \nabla_b \omega_a$ as we transport it from the midpoint of one edge to another, e.g. from $(\Delta t/2, 0)$ to $(\Delta t/2, \Delta s)$. We know from equation (8.6) that v^a does not change, but $T^b \nabla_b \omega_a$ changes by $\Delta s S^c \nabla_c (T^b \nabla_b \omega_a)$, obtained from substituting it into an adjusted (8.5) for δ_2. Hence to second order in Δt, Δs, we find

$$\delta_1 + \delta_3 = -\Delta t \Delta s v^a S^c \nabla_c (T^b \nabla_b \omega_a) .$$

Adding this to a similar term for $\delta_2 + \delta_4$, we get

$$\begin{aligned} \delta(v^a \omega_a) &= \delta_1 + \delta_2 + \delta_3 + \delta_4 \\ &= \Delta t \Delta s v^a \left[T^c \nabla_c \left(S^b \nabla_b \omega_a \right) - S^c \nabla_c \left(T^b \nabla_b \omega_a \right) \right] . \end{aligned}$$

As the vector fields T^a and S^a commute,

$$\begin{aligned} \delta(v^a \omega_a) &= \Delta t \Delta s v^a T^c S^b \left(\nabla_c \nabla_b - \nabla_b \nabla_c \right) \omega_a \\ &= \Delta t \Delta s v^a T^c S^b R_{cba}{}^d \omega_a , \end{aligned}$$

which holds for all ω_a if and only if

$$\delta v^a = \Delta t \Delta s v^d T^c S^b R_{cbd}{}^a .$$

C Computing the Curvature Tensor and Linearised Einstein Equation

To calculate the components of the curvature tensor, we must first select a coordinate system as a reference. We will use ∂_a from flat geometry as this reference. To compute the Christoffel Symbol, we substitute the assumption (4.2) into equation (8.4) as follows:

$$\Gamma^c{}_{ab} = \frac{1}{2}(\eta_{bd} + \gamma_{bd})[\partial_a(\eta_{bd} + \gamma_{bd}) + \partial_b(\eta_{ad} + \gamma_{ad}) - \partial_d(\eta_{ab} + \gamma_{ab})]\,.$$

But by Leibniz rule and linearity, we expand the first bracket and delete terms non-linear in γ_{ab}. Then, using the fact that $\partial_a \eta_{bc} = 0$, we arrive at:

$$\Gamma^{(l)}{}^c{}_{ab} = \frac{1}{2}\eta^{cd}(\partial_a\gamma_{bd} + \partial_b\gamma_{ad} - \partial_d\gamma_{ab})\,. \tag{8.7}$$

Next, to find an equation for Curvature. From equation (3.1), we get

$$\nabla_b\omega_c = \partial_b\omega_c - \Gamma^d{}_{bc}\omega_d\,.$$

Thus, in a similar manner to in Appendix A, we can take the ∇_a of this quantity and arrive at

$$\nabla_a\nabla_b\omega_c = \partial_a(\partial_b\omega_c - \Gamma^d{}_{bc}\omega_d) - \Gamma^e{}_{ab}(\partial_e\omega_c - \Gamma^d{}_{ec}\omega_d) - \Gamma^e{}_{ac}(\partial_b\omega_e - \Gamma^d{}_{be}\omega_d)\,.$$

To compute the curvature tensor, we take the difference of successive differentiations as per equation (3.2) and collect terms using anti-symmetry notation. The result is

$$R_{abc}{}^d\omega_d = [-2\partial_{[a}\Gamma^d{}_{b]c} + 2\Gamma^e{}_{c[a}\Gamma^d{}_{b]e}]\omega_d$$
$$R_{abc}{}^d = [-2\partial_{[a}\Gamma^d{}_{b]c} + 2\Gamma^e{}_{c[a}\Gamma^d{}_{b]e}]\,.$$

We note than the antisymmetry is on the lower indices in this equation and that this holds for all $\Gamma^c{}_{ab}$. Furthermore, the Christoffel symbol is due to the difference in derivative operator from η_{ab} due to γ_{ab} and so $\Gamma^c{}_{ab}$ is atleast linear in γ_{ab}. Hence, any product of more than one $\Gamma^c{}_{ab}$ is non-linear and discarded when deriving the linearised Einstein equation:

$$R^{(l)}{}_{abc}{}^d = [-2\partial_{[a}\Gamma^d{}_{b]c}]\,.$$

To find the Ricci Tensor, we contract on the second and fourth index to get the following expression:

$$R_{ac} = -\partial_a(\sum_b \Gamma^b{}_{bc}) + \sum_b \frac{\partial}{\partial x^b}\Gamma^b{}_{ac} + \sum_{b,e}(\Gamma^e{}_{ca}\Gamma^b{}_{be} - \Gamma^e{}_{cb}\Gamma^b{}_{ae})$$
$$R^{(l)}_{ac} = -\partial_a(\sum_b \Gamma^{(l)}{}^b{}_{bc}) + \sum_b \frac{\partial}{\partial x^b}\Gamma^{(l)}{}^b{}_{ac} \tag{8.8}$$

Finally, another contraction is done to find $R^{(l)}$ and both results are substituted back into the Einstein equation to obtain equation (4.3).

D Properties of k_μ in the Plane Wave Solution

After substituting $\overline{\gamma}_{ab} = H_{ab}\exp(ik_\mu x^\mu)$ into $\partial^c\partial_c\overline{\gamma}_{ab} = 0$, expanding using chain rule and pulling out constants, we get the following:

$$\begin{aligned} 0 &= \partial^c\partial_c(H_{ab}\exp(ik_\mu x^\mu)) \\ &= H_{ab}\eta^{cd}\partial_d[\partial_c(\exp(ik_\mu x^\mu)] \\ &= H_{ab}\eta^{cd}\partial_d\exp(ik_\mu x^\mu)\partial_c(ik_\mu x^\mu) \\ &= H_{ab}\eta^{cd}\partial_d\exp(ik_\mu x^\mu)ik_\mu\partial_c(x^\mu) \\ &= H_{ab}\eta^{cd}\partial_d\exp(ik_\mu x^\mu)ik_\mu\frac{\partial x^\mu}{\partial x^c}\,. \end{aligned}$$

Noting that since $\{x^0, \ldots, x^4\}$ is linearly independent,

$$\frac{\partial x^\mu}{\partial x^c} = \begin{cases} 1 & \mu = c \\ 0 & \text{otherwise} \end{cases}\,.$$

Hence,

$$\begin{aligned} 0 &= H_{ab}\eta^{cd}\partial_d\exp(ik_\mu x^\mu)ik_\mu\frac{\partial x^\mu}{\partial x^c} \\ &= H_{ab}\eta^{cd}\partial_d\exp(ik_\mu x^\mu)ik_c\,. \end{aligned}$$

We do the same for ∂_d, and arrive at:

$$\begin{aligned} 0 &= -H_{ab}\eta^{cd}\exp(ik_\mu x^\mu)k_d k_c \\ &= -k^c k_c\overline{\gamma}_{ab}\,. \end{aligned}$$