AMSI VACATION RESEARCH SCHOLARSHIPS 2018-2019

Analysis of Chances of Winning in a Poker Game: Combinatorial Probability and the Law of Large Numbers

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Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

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Contents

Abstract

The paper examines the chances of winning in Texas Hold'em Poker given different variables. Variables considered are starting hand ranges, starting money, likelihood of bluffing, number of players and rake. Proofs and applications of combinations and permutations, the Weak Law of Large Numbers are discussed. Computer simulations are used to investigate complex topics in the game. Several betting strategies are discussed and their distributions are compared using simulation results. No strategies are expected to be profitable when playing at an Australian casino.

1 Introduction

Early developments of probability theory arised from interests regarding games of chance. This report focuses on the chances inside the well-known gambling game, Texas Hold'em poker. The purpose of this report is to educate players on how to make wiser decisions in gambling using Texas Hold'em poker as an example. Combinatorial rules, combinations and permutations are given and applications are discussed. Proof of a prominent probability theorem, the Weak Law of Large Numbers is shown and applications such as computer simulations are discussed. The chance of winning by Poker Hand, Starting Hand given various numbers of players and the performance of several betting strategies are examined using computer simulations. The expected return and variance of return are the measures used to compare the performance of betting strategies. The betting strategies are simulated under different environments including the level of rake, amount of starting money and time. The betting strategies consider the likelihood of bluffing and starting hand range. No strategies are expected to be profitable in the long-run when playing at an Australian casino. The paper will begin by defining notations and terminologies used throughout the paper.

1.1 Notations

Texas Hold'em Poker uses one deck of playing cards with 13 ranks and 4 suits. The 4 suits are Diamonds, Clubs, Hearts and Spades, they are all equal in strength in this game. The 13 ranks, ordered from strongest to weakest are: Ace, King, Queen, Jack, Ten, Nine, Eight, Seven, Six, Five, Four, Three, Two. For the remainder of this report, the suits will be represented by their first letter: "d", "c", "h", "s". Similarly, the ranks will be represented by a single character: "A", "K", "Q", "J", "T", "9", "8", "7", "6", "5", "4", "3", "2". A combination of 2 *suited*/unsuited (off-suited) unpaired cards is denoted by "XYs"/ "XYo" respectively where "XY" are the ranks of the unpaired cards. Suited cards mean the cards share the same suit, vice versa. A pair is denoted by "ZZ" where "Z" is the rank of the pair.

Examples:

- "Ac" represents the Ace of Clubs
- "A3567" represents a combination of A,3,5,6 and 7 with the suits unspecified
- "As Ks" represents the unique combination of Ace of Spades and King of Spades.
- "AQs"/ "AQo" represents a suited/off-suited combination of A, Q respectively.
- "AA" represents a combination of paired Aces.

1.2 Terminologies and Rules

Starting Cards The 2 faced-down cards dealt to each player at the start of a hand

Community Cards The faced-up cards in the middle that can be used by any players, also known as the board.

Stages There are up to 4 stages in a hand, they are explained below.

Pre-flop Occurs after starting cards are dealt and *blinds* are posted

Flop 3 community cards are dealt

Turn 4^{th} community card is dealt

River 5^{th} community card is dealt

Each stage is followed by a *betting round* and the hand continues to the next phase if there is still more than one player in play. There are up to 4 phases in a hand, they are explained below.

Blinds Mandatory bets made at the start of each hand, before starting cards are dealt. Many blinds structure exist in practice. This report will focus on the traditional one small blind and one big blind structure. The player to the left of the *button* is the small blind, while the player to the left of the small blind is the big blind. The button is passed to the player to the left after each hand, so the small and big blinds rotate around the table as the game progresses.

Betting Round During the pre-flop betting round, the player to the left of the big blind acts first. During other betting rounds, the small blind acts first. Action rotates clockwise. When action is required, if there is a bigger bet to respond to, the player can choose to either call (match the bet), raise (increase the bet) or fold (give up the hand). When there isn't a bigger bet to respond to, the player can choose to either check (pass onto the next player) or make a bet. The betting round continues until either one player remains, everyone has put in all their chips (all-in) or matched the bet. If only one player remains, that player is declared the winner of the hand and take the *pot*. If there is still more than one player in play after the final betting round (after the River), *showdown* occurs.

Pot The money in the middle, total sum of the bets made in a hand

Showdown All player still in the hand reveals their starting cards, each player plays the best *poker hand* they can make with their starting cards and community cards. The player with the best hand wins the hand and take the pot. If the best hand is shared by multiple players, the pot is split equally between them.

Poker Hands A five-card combination. The 9 distinguishable types of poker hands are listed in Appendix A ordered from the strongest to the weakest.

Notes: "A" can be used as a "1" to form a 5-high straight. i.e. "A2345" is a valid straight. If the hand involves high cards, *kickers* are used to settle ties.

Kickers Used to break ties between hands of the same rank. E.g. "8d 8h Ah 3d 2c" is ranked as pair of 8s with kickers "A-3-2". It beats "8d 8h Kh 3d 2c" as "A-3-2" out-ranks (out-kicks) "K-3-2" but it would lose to "8d 8h As Tc 3d" as "A-T-3" out-kicks "A-3-2".

Bluffing Making a bet or raise in a situation where the player would normally check or fold to induce folds from other players

Let X be a discrete random variable.

Its **PF** (Probability Function) is defined as $f_X = P(X = x)$,

Its **CDF** (Cumulative Distribution Function) is defined as $F_X(x) = P(X \leq x)$.

The **Expected Value** of X is denoted as $E[X]$, $E[X] = \sum_x x\, f_X(x)$.

The **Variance** of X is denoted as $\mathrm{V}[X]$, $\mathrm{V}[X] = \sum_x x^2\, f_X(x) - \{E[X]\}^2$.

The **Standard Deviation** (StDev) of X is defined as $\sqrt{Variance(X)}$.

2 Permutations and Combinations

2.1 Proof

Let ${}^nP_r$, ${}^nC_r$ be the number of different permutations, combinations of n distinguishable objects taken r at at time. A permutation is an arrangement in a particular order while a combination is an arrangement without any reference to order.

$${}^nP_r = (\text{\# options for 1st object}) \cdot (\text{\# options for 2nd object}) \cdot$$
$$(\text{\# options for 3rd object}) \cdot \cdots \cdot (\text{\#options for rth object})$$
$$= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot \big(n-(r-1)\big)$$
$$= n(n-1)(n-2)\ldots(n-r+1) \cdot \frac{(n-r)!}{(n-r)!}$$
$$= \frac{n!}{(n-r)!}$$

For the derivation of ${}^nC_r$, consider the possible combinations and permutations of the digits 1-4, taken 3 at a time.

The possible combinations are: 123 124 134 234

Each of these combinations will give rise to $3! = 6$ different permutations as this is the number of ways you can rearrange each of these combinations.

It can be generalized that each combination of r objects will give rise to $r!$ unique permutations.

$${}^nP_r = {}^nC_r \cdot r!$$

$$\therefore {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

2.2 Application

Given that all outcomes are equally likely, the probability of an event can be found by:

$$\frac{Number\ of\ favourable\ outcomes}{Number\ of\ possible\ outcomes}$$

Let's examine a typical poker problem. The Flop has been dealt, the cards are "2h 3h Kh". Only Ann and Bob are still in the hand, Bob acts first and bets, now Ann has a decision to make. Ann's starting hand is "Kc Kd", although she currently has Triple Kings which is a very strong hand, she is worried that Bob might have at least one heart in his hand and will have a Flush at Showdown. What should Ann do?

5 cards are revealed already, so there are only 47 cards Bob can possibly have. 3 hearts are revealed already, so there are only 10 hearts Bob can possibly have. So, the probability that Bob has 2 hearts in his hand (already have a flush) is: $\frac{^{10}C_2}{^{47}C_2} = 0.04163$. That is unlikely to occur, let's examine the probability that Bob has any hearts in his hand (any possibility of a Flush at Showdown), that is: $\frac{^{47}C_2 - ^{37}C_2}{^{47}C_2} = 0.38390$. It is not as intimidating as players may think, as it is not guaranteed that Bob will have a Flush at Showdown. In fact, the probability that Bob will have a Flush at Showdown considering all cases is $\frac{^{10}C_2}{^{47}C_2} + 10 \cdot \frac{37}{^{47}C_2} \cdot \left(1 - \frac{^{36}C_2}{^{45}C_2}\right) = 0.16609$. This implies around 84% of the time, Bob will not have a Flush at Showdown, even if he does, Ann may make a Full House or Quad which ranks higher than Flush. Therefore, Ann has a very good chance of winning this hand and shouldn't be thinking of folding the hand away at this stage.

3 The Weak Law of Large Numbers

Let $X, X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with finite mean, μ. Let $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. Then $\frac{S_n}{n}$ converges to μ in probability as $n \to \infty$.

3.1 Proof

Let $Y_k|n = \begin{cases} X_k & ; \quad |X_n| \leq n\epsilon^3 \\ 0 & ; \quad o/w \end{cases} = X_k I\{|X_k| \leq n\epsilon^3\}$ and $S_n' = \sum_{k=1}^{n} Y_k|n$ where $\epsilon > 0$ and $k = 1,2,\ldots,n, n \geq 1$. We assume that $E[X_k] = 0$, which implies $E[Y_k|n] = 0$ and $V[Y_k|n] = E[Y_1^2|n]$.

The truncated Chebyshev inequality - If $X_1, \ldots, X_n$ are indepdent random variables, with $Y_k = X_k \cdot I\{|X_k| \leq b_k\}$, $S_n' = \sum_{k=1}^{n} Y_k$. Then, for $x > 0$:

$$P(|S_n - E[S_n']| > x) \leq \frac{\sum_{k=1}^{n} \mathrm{V}[Y_k]}{x^2} + \sum_{k=1}^{n} P(|X_k| > b_k).$$

If $b_k = b\ \forall k$, then the RHS of the inequality reduces to $\frac{nV[Y_1]}{x^2} + nP(|X_1| > b)$.

Using this inequality, we have the following:

$$\begin{aligned} P(|S_n - E[S_n']| > n\epsilon) &\leq \frac{1}{n\epsilon^2}\mathrm{V}[Y_1|n] + nP(|X| > n\epsilon^3) \\ &= \frac{1}{n\epsilon^2} E[Y_1^2|n] + nP(|X| > n\epsilon^3) \end{aligned}$$

$$= \frac{1}{n\epsilon^2} E(X^2 I\{|X| \le n\epsilon^3\}) + nP(|X| > n\epsilon^3)$$
$$\le \epsilon E|X| I\{|X| \le n\epsilon^3\} + nP(|X| > n\epsilon^3)$$
$$\le \epsilon E|X| + nP(|X| > n\epsilon^3).$$

Intuitively, $E[X^r] < \infty$ implies $\lim_{x\to\infty} x^r P(X > x) = 0$. Therefore, $E[X] = 0 < \infty$ implies $\lim_{n\to\infty} nP(X > n) = 0$, which implies $\lim_{n\to\infty} nP(|X| > n\epsilon^3) = 0$ as ϵ is some arbitrary number.

Therefore, $\limsup_{n\to\infty} P(|S_n - E[S'_n]| > n\epsilon) \le \epsilon E[X]$, which implies $\frac{(S_n - E[S'_n])}{n}$ converges to 0 in probability as $n \to \infty$.

Since $E[X] = 0$, $|E[S'_n]| = |nE[X] \cdot I\{|X| \le n\epsilon^3\}| \le nE[|X|] \cdot I\{|X| > n\epsilon^3\}$, therefore $\frac{E[S'_n]}{n} \to 0$ as $n \to \infty$ which is the required result to prove the theorem.

3.2 Application

A straight-forward application of the theorem is related to simulations, the next section of this report. The weak law of large numbers states that the arithemetic mean of a sample with size n will converge in probability to the population mean as n is getting larger. Therefore, if we simulate a random variable n times, provided n is large enough, the arithmetic mean of the simulated sample is a good estimate of the expectation of the random variable. How do we decide whether n is large enough though? This would depend entirely on the size of the sample space, which is finite but still very large, in the case of random varaibles in poker games. If a certain level of precision is required, we can use the Chebyshev's inequality to find the appropriate level of n to reach the desired level of precision.

Chebyshev's inequality - Suppose that $\mathrm{V}[X] < \infty$. Then:

$$P(|X - E[X]| > x) \le \frac{\mathrm{V}[X]}{x^2}; x > 0.$$

Let $X_1, X_2, \ldots, X_n$ be a sample from the distribution F with CDF $F(x)$ and the empirical distribution function (EDF) $F_n(x)$ be defined as: $F_n(x) = \frac{1}{n}\sum_{k=1}^{n} I\{X_k \le x\}$, which is equal to the proportion of observations out of the first n that are at most equal to x. By the weak law of large numbers, $F_n(x)$ converges in probability to $F(x)$ as $n \to \infty$, implying $E[F_n(x)] = F(x)$. $\mathrm{V}[F_n(x)] = \frac{F(x)(1-F(x))}{n} \le 1/4n$ as $nF_n(x) \in Binomial\,(n, F(x))$ and $F(x)(1 - F(x)) \le 1/4$ since $F(x) \in (0,1)$ as it is a CDF. Therefore, by applying Chebyshev's inequality:

$$P(|F_n(x) - F(x)| > \epsilon) \leq \frac{1}{4n\epsilon^2}$$

Note that the RHS of this inequality approaches 0 as n approaches infinity. This inequality provides a lower bound for the required n to reach the desired level of precision, and is indepdent of x.

4 Simulations

Exact probability can be obtained by going through every possibility but is very resource demanding for most scenarios in poker. Therefore, simulations are used to investigate winning chances in poker given different variables, all simulations in this project are performed using the R software.

The chances associated with each type of poker hands are estimated for different numbers of players. The chance of winning for each type of starting hand is estimated for 2 players, this information allows us to rank the starting hands, the ranking is then used in betting strategies. Various betting strategies are tested against each other under specific conditions.

4.1 Poker Hands

The chance of making, winning and tying with each poker hand is investigated. The total number of possible outcomes for a hand of poker between n players is given by: $\left(\prod_{i=0}^{n-1} {}^{52-2i}C_2\right) \times {}^{52-2n}C_5$, which is 2.781×10^{12} for just 2 players. Since it isn't realistic to go through all possible outcomes, a simulation size of 100 000 is chosen.

High Card	**Pair**	**Two Pairs**	**Triple**	**Straight**	**Flush**	**Full House**	**Quad**	**Straight Flush**
17.51%	43.78%	23.36%	5.01%	4.59%	2.91%	2.61%	0.19%	0.03%

Figure 1: Chance of making each poker hand at Showdown

The shape of the distribution is mostly expected, except for the fact that High Card is rarer than Pair and Two Pairs when High Card is ranked lower than those 2 poker hands. It implies that a player should not expect to win many hands with just High Card. It is worth noting that approximately 85% of the time, a player will hit nothing higher than Two Pairs after all cards are dealt. It implies that a player should not expect to hit anything higher than Two Pairs most of the time. The odds of making Quad and Straight Flush is extremely low, approximately at a rate of 1 in

500 hands and 1 in 3000 hands respectively. Therefore, a player should not expect to make a Quad or Straight Flush every session of poker.

The chances of winning and tying with each poker hand are also worth investigating. Intuitively, these chances are dependent on the number of players unlike the chance of making each poker hand. Therefore, 100 000 simulations are run to investigate the chances of winning and tying given different numbers of players. The numbers of players considered are 2 to 6.

See Appendix B for the chance of winning with each poker hand. As expected, the simulations show the chance of winning declines for each poker hand as the number of players increases, except for Quad and Straight Flush. The unexpected patterns for Quad and Straight Flush is likely due to sampling errors arising from low occurance rates of these 2 hands seen in Figure 1. Theoretically, these 2 hands should also show a declining chance of winning as the number of players increase. This problem can be resolved by running more simulations or by using a better simulation technique.

	High Card	Pair	2pairs	Triple	Straight	Flush	Full House	Quad	Straight Flush
Chance of Win	16.16%	40.60%	64.04%	72.86%	80.02%	85.59%	87.95%	88.48%	93.10%
Chance of Win + Tie	18.76%	43.63%	68.98%	76.16%	92.66%	88.07%	94.68%	94.76%	100.00%
CDF of Making	17.51%	61.29%	84.65%	89.66%	94.25%	97.16%	99.77%	99.96%	99.99%

Figure 2: Various chances associated with each poker hand in a 2-player game

Although both the chance of win + tie and the cumulative probability of making share a similar trend, they do not perfectly resemble one another. This implies the cumulative probability of making each poker hand is not a perfect indication of their respective win rates. It is interesting that the chance of win+tie of Flush is lower than those of Straight despite Flush is ranked higher than Straight. This is mostly due to the fact it is much easier to get out-kicked with Flush than with a Straight.

Most of time when a player hit a Flush or Straight, the hand only requires one card from the player's starting hand. In the case of Flush, this implies there are 4 cards of that suit on the board, which means there are 9 cards of that suit other players might have. Each of these 9 cards have a unique rank, which means there are up to 8 cards that can out-kick a player with a Flush in this case. In the case of Straight, it is only possible to be out-kicked when there is an open-ended 4-card sequence on the board. For example, if the board is "3-4-5-6-K", a player that have a 2 in their starting hand to form a 6-High Straight can be out-kicked by a player that have a 7 in their starting hand which forms a 7-High Straight. There are 4 of each rank in a deck, therefore there are only 4 cards that can out-kick a player with a lower straight. Therefore, Flush has a lower win+tie rate than Straight as Flush has more out-kicked opportunities than Straight.

See Appendix C for the chance of tying with each poker hand. The tie rates have a strictly increasing or decreasing trend for most poker hands as the number of players increases. Again, due to low occurance rates, the results for Quad and Straight Flush are not meaningful. Intuitively, the chance of tying is expected to decline as the number of player increases, but this does not seem to hold true for Straight and Full House, their tie rates have an increasing trend instead. Straight and Full House are the only 2 poker hands that do not have kickers, they are fully specified. It explains why their tie rates increase with the number of players as the probability of another player also having a card that make the same Straight or Full House increases with the number of players too. For Quad and Straight Flush, as they both have kickers in their 5-card hand, so they both should have a strictly decreasing trend as the number of players increase too. Again, the sampling errors can be eliminated by either increasing the simulation size or improving the simulation technique.

4.2 Starting Hands

Out of the 1326 (${}^{52}C_2$) possible starting hands, there are 169 distinct types of starting hands. Since there are 13 ranks, there are 13 distinct pocket pairs (e.g. AA), 78 (${}^{13}C_2$) distinct suited 2-card combinations (e.g. AKs) and 78 (${}^{13}C_2$) distinct unsuited 2-card combinations (e.g. AKo). It is intended to calculate the pre-flop (before any community cards are dealt) win rate for each distinct 2-card combination. For each distinct combination, there are $\left(\prod_{i=1}^{n-1} {}^{52-2i}C_2\right) \times {}^{52-2n}C_5$ possible outcomes for n players. The R software would need to run through more than 350 billion outcomes to determine the exact win rate of each starting hand. Like the previous section, a smaller simulation size is chosen to circumvent the lack of computational power. A simulation size of 10 000 is chosen for this section. See Appendix D for the simulated win rates for each distinct starting hand in a 2-player game.

	Chance of Being Dealt This Hand (%)	Average (%)	Median (%)	Min (%)	Max (%)
Pocket Pair	5.88	68.63	69.065	50.055	84.84
Suited	23.53	50.99	50.665	36.405	67.44
Unsuited	70.59	48.21	48.2025	31.795	65.46
Contains Ace or King	28.66	60.13	59.4925	49.415	84.84
Does not Contain Ace or King	71.34	47.47	46.32	31.795	80.195

Figure 3: Descriptive Statistics for the simulated win rates of starting hands in a 2-player game

Intuitively, in a fair game between 2 players, it is expected that each player wins exactly 50% of the games played. Out of the 169 starting hands, 88 of them have a win rate greater than 50%. Pocket pairs have a distinct advantage over unpaired starting hands, with all pocket pairs having a win rate greater than 50%. Suited Hands have a slight advantage (around 2%) over Unsuited Hands due to the greater likelihood of completing a Flush. However, it is important to note that out of every 100 games, Suited Hands are expected to win only 2 extra games compared to their Unsuited counterpart, Suited Hands are just more aesthetically appealing in the other 98 games. Starting hand containing an Ace or a King has a significant edge over hands that does not. It shows that high ranks matter a lot more than suits in starting hands.

4.3 Betting Strategies

The games simulated in this project have several conditions placed to make it simple enough to simulate various betting strategies. It is assumed that there is one betting round, which occurs at pre-flop (before any community cards are dealt and after starting cards are dealt). Only up to two raises can be made in each betting round and each raise is 10 times the previous bet.

6 strategies are tested in this project, the first 3 are simple strategies while the last 3 are more advanced, they are defined below.

1. Call Station – Only Call
2. Maniac – Only Raise
3. Random – Call, Fold and Raise with equal probability

	Call Range	Raise Range	Re-Raise Range	Bluff Likelihood
4 (Tight)	Top 55 Starting Hands	Top 20 Starting Hands	Top 3 Starting Hands	10%
5 (Balanced)	Top 90 Starting Hands	Top 45 Starting Hands	Top 15 Starting Hands	40%
6 (Loose)	Top 125 Starting Hands	Top 75 Starting Hands	Top 30 Starting Hands	70%

Figure 4: Definitions for Strategy 4 to 6.

The 6 strategies are tested against each other in a 6-player game with the aforementioned conditions. They are tested under two different settings. Setting 1 assumes each player has infinite wealth, each player plays for 50 000 hands. The Small Blind and Big Blind is 50 and 100 respectively. Setting 2 assumes each player starts with 100 Big Blinds, game terminates once all players but one have been knocked out or if 24 hands have been played. Setting 2 has more realistic assumptions than Setting 1. Setting 2 is tested under 2 *rake* assumptions, 5% of the pot and no rake. Rake is a scaled commission fee taken by the house as the house does not receive any income from running the game otherwise. The simulation size for Setting 2 is 10 000.

See Appendix E, Appendix F for the descriptive statistics, raw results of Setting 1 respectively. In the long-run, the 3 simple strategies are losing to the 3 advanced strategies in terms of expected winnings. All 3 advanced strategies win in the long run while all 3 simple strategies lose in the long run, demonstrating the power of playing selective hands. There is a correlation between average gain per hand and standard deviation of gain per hand for the last 3 strategies. Loose has the highest expected return but it also has the highest variance. Tight has the lowest expected return out of the 3 advanced strategies, but it also has the lowest variance. This is the tradeoff between risk and return, high risk is rewarded by high return.

See Appendix G for the descriptive statistics of Setting 2 assuming no rake. Unlike Setting 1, Setting 2 assumes players have finite wealth and finite time to play poker. A similar pattern to Setting 1 is witnessed for the variance of each strategy, but a different pattern is observed for the expected gain of each strategy. Loose has the highest expected gain under Setting 1 but have the lowest expected gain out of all winning strategies under Setting 2. The risk-return tradeoff no longer holds true as Balanced has the highest expected gain but has the second lowest variance. Maniac, a losing strategy under Setting 1, is a winning strategy under Setting 2. It is unclear to why the results changed this way, it warrants further investigations.

See Appendix H for the descriptive statistics of Setting 2 assuming a rake of 5%. After a rake of 5% is applied to the game, all strategies became unprofitable. Most Australian casinos charge a rake of 10% in poker games, therefore, the strategies discussed in this report will certainly lose in the long-run when playing at these casinos. Interestingly, Maniac is now much worse than Loose, which indicates the unexpected result for Maniac previously in Figure 8 is likely due to sampling errors. It might be implying that the simulation size of 10 000 is not large enough for the long-run equilibrium to be reached under Setting 2.

5 Conclusion

In this report, we have discussed probabilities in the gambling game Teaxs Hold'em Poker. The Combinations and Permutations rules were discussed and were employed to solve simpler probabilistic problems. The mathematical proof of the Weak Law of Large Numbers was shown and some applications are discussed including simulations. Simulations were performed to answer some complex probabilistic problems. Practical implications of these results were discussed, informing players on how to make wiser decisions in poker. All betting strategies discussed in this report are unprofitable in the long-run when playing at Australian casinos. Simulation results can be improved upon or can be extended further. Ways of improving the accuracy of the simulation results include improving code effiency to allow an increase in simulation size, using a better sampling technique such as importance sampling which is generating samples from a different distribution other than the distribution of interest. In the future, it would be interesting to implement betting algorithms for all betting rounds, re-evaluate strategies discussed in this report and evaluate other possible strategies, investigate chances of making each partial hand at various stages of a hand. Of course, it is not expected to find a profitable strategy in the long-run when playing at a Australian casino.

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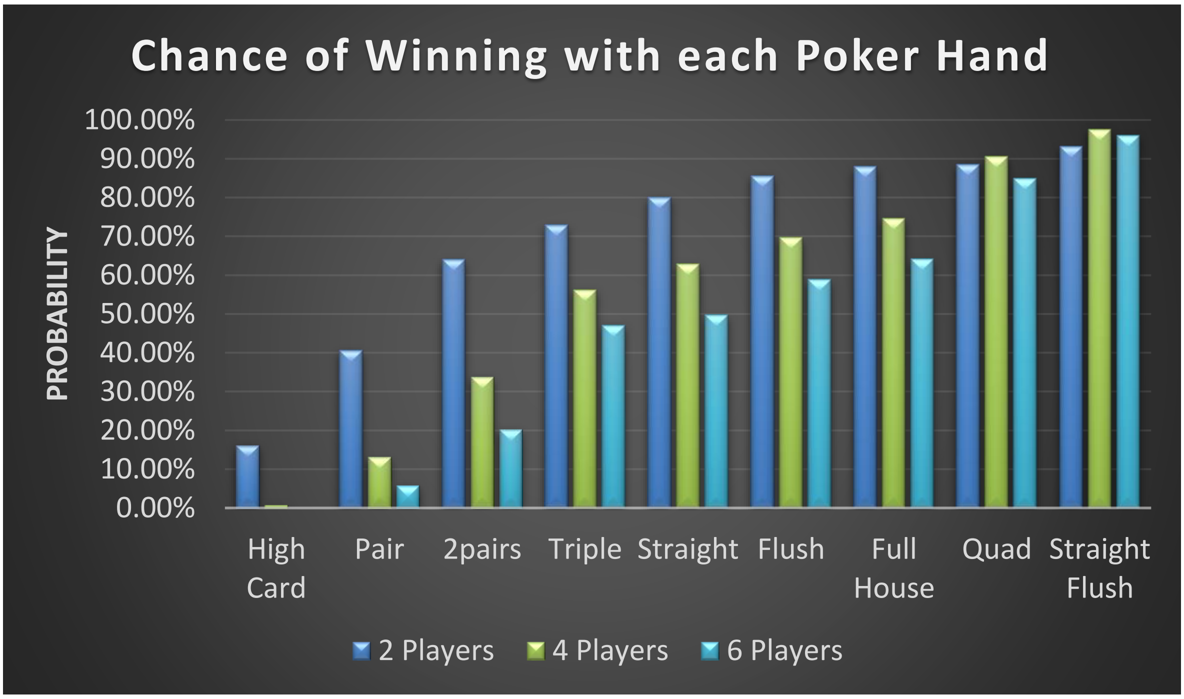
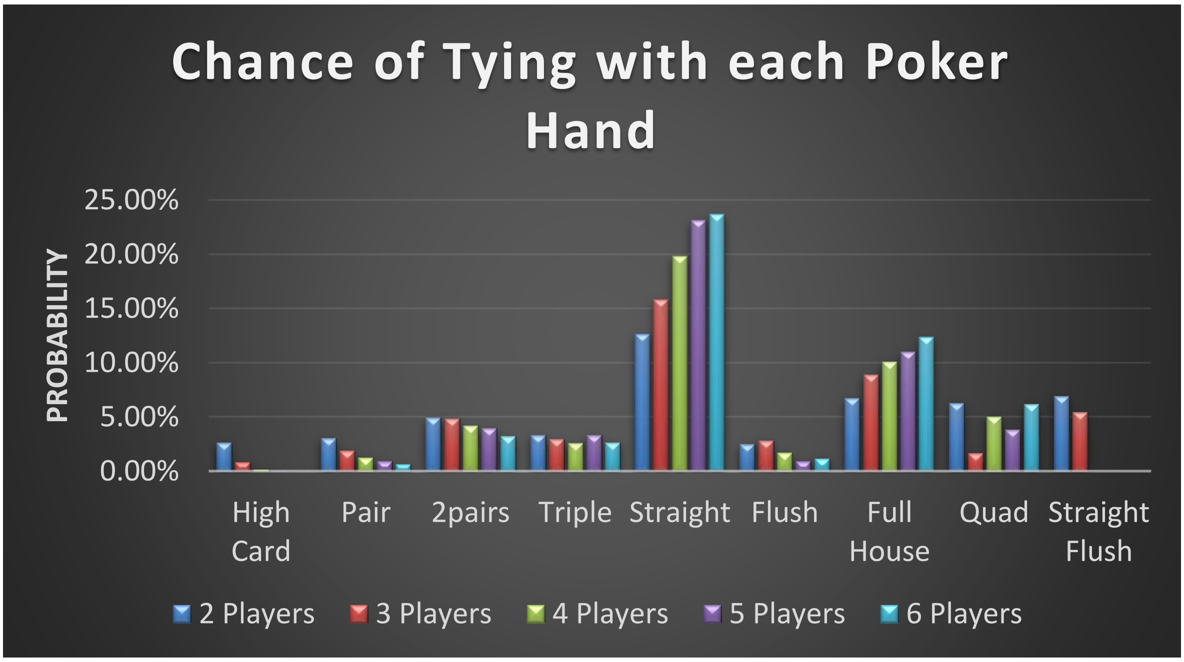
Appendix A: Poker Hand Definitions

Rank	Hand Name	Example	Description
1	Straight Flush	As 2s 3s 4s 5s	5 in a sequence and all in one suit
2	Quad	7d 7c 7s 7h 8d	4 of the same rank + 1 High Card
3	Full House	3d 3h 3s 2s 2d	A Three of a Kind + A Pair
4	Flush	2h 8h 9h Jh Kh	5 of the same suit, not in a sequence
5	Straight	7s 8h 9s Tc Jd	5 in a sequence, not all in one suit
6	Three of a Kind	7s 7h 7d Tc Ad	3 of the same rank + 2 High Cards
7	Two Pairs	8s 8h 9s 9d Ah	2 Pairs + 1 High Card
8	One Pair	8d 8h Ah 3d 2c	2 of the same rank + 3 High Cards
9	High Card	2d 8d Th Jc Ks	5 different ranks, doesn’t satisfy any hands above

Appendix B: Winning Chances by Poker Hand



Appendix C: Tying Chances by Poker Hand

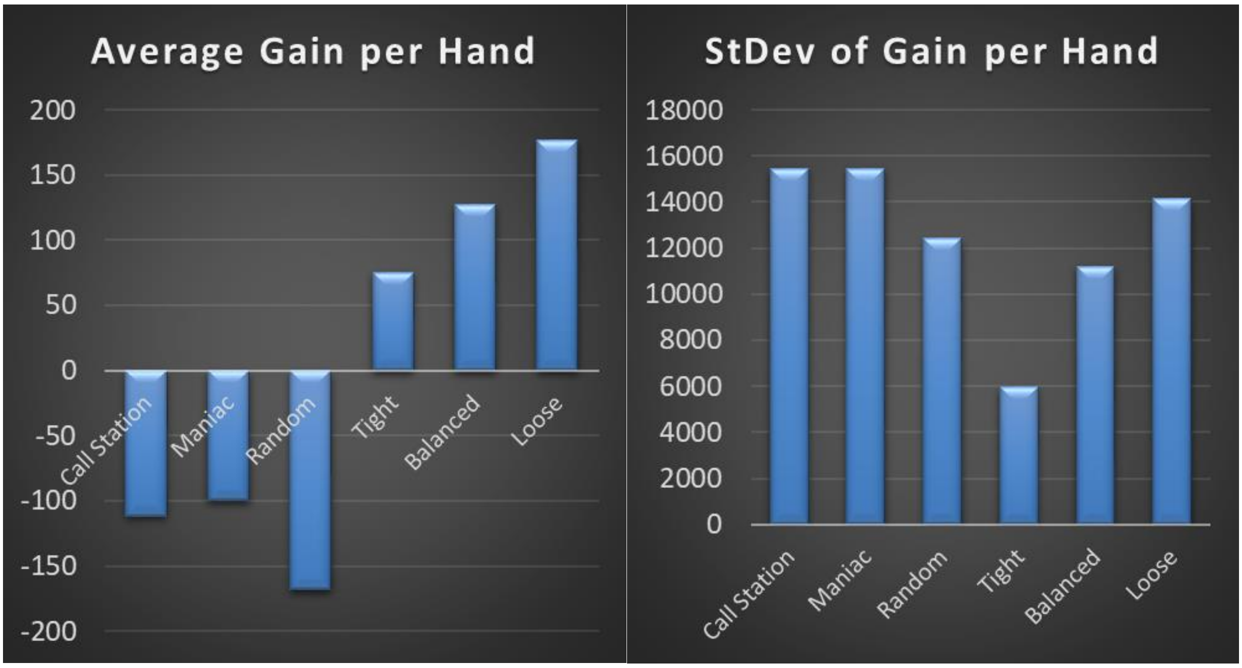


Appendix D: Starting Hand Win Rates

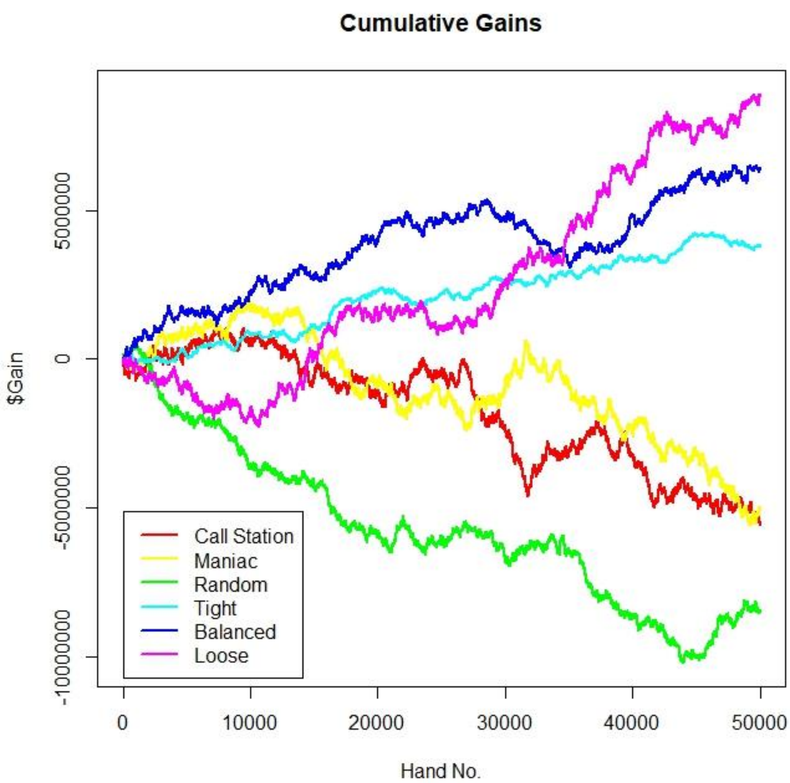
Rank	Hand	Simulated Win% for 2 players	Rank	Hand	Simulated Win% for 2 players	Rank	Hand	Simulated Win% for 2 players
1	AA	84.84	36	QTs	58.935	71	T8s	53.13
2	KK	83.045	37	K9o	58.695	72	J9o	52.86
3	QQ	80.195	38	A7o	58.535	73	Q5s	52.685
4	JJ	77.465	39	A6o	58.51	74	Q4s	52.635
5	TT	74.67	40	K7s	58.4	75	K3o	51.935
6	99	71.855	41	QJo	58.345	76	J7s	51.905
7	88	69.065	42	A2s	58.01	77	K4o	51.885
8	AKs	67.44	43	A3s	57.965	78	Q7o	51.875
9	77	66.67	44	JTs	57.665	79	T9o	51.78
10	AQs	66.47	45	Q9s	57.65	80	98s	51.515
11	AJs	65.66	46	A4o	57.54	81	Q6o	51.295
12	AKo	65.46	47	QTo	57.22	82	Q2s	50.87
13	AQo	64.71	48	A5o	57.17	83	J8o	50.505
14	KQs	64.03	49	44	56.775	84	J6s	50.46
15	ATs	63.98	50	K6s	56.345	85	Q3s	50.425
16	AJo	63.315	51	Q8s	56.32	86	T7s	50.15
17	66	63.215	52	J9s	56.13	87	22	50.055
18	KJs	63.18	53	K8o	55.825	88	97s	50.045
19	KQo	62.64	54	K7o	55.82	89	T8o	49.97
20	A9s	62.38	55	A3o	55.785	90	J5s	49.8
21	ATo	61.95	56	K5s	55.365	91	Q5o	49.475
22	KTs	61.855	57	Q9o	55.18	92	J7o	49.435
23	A7s	61.44	58	K6o	55.135	93	K2o	49.415
24	A8s	61.16	59	K4s	55.075	94	T6s	49.38
25	KJo	61.095	60	A2o	55	95	Q4o	49.38
26	55	60.705	61	JTo	54.955	96	87s	48.96
27	KTo	60.525	62	J8s	54.73	97	J4s	48.535
28	A9o	60.485	63	Q7s	54.12	98	J3s	48.51
29	QJs	60.32	64	T9s	54.055	99	Q3o	48.47
30	A6s	60.035	65	K2s	53.685	100	T7o	48.3
31	K9s	59.89	66	33	53.645	101	98o	48.105
32	A4s	59.725	67	K3s	53.59	102	Q2o	47.745
33	A8o	59.58	68	Q8o	53.48	103	J2s	47.735
34	A5s	59.405	69	Q6s	53.46	104	96s	47.73
35	K8s	59.035	70	K5o	53.45	105	J6o	47.61

Rank	Hand	Simulated Win% for 2 players	Rank	Hand	Simulated Win% for 2 players
106	J5o	47.57	143	75o	40.03
107	T5s	47.28	144	93o	39.88
108	T4s	46.49	145	43s	39.48
109	86s	46.32	146	73s	39.475
110	T3s	46.185	147	63s	39.215
111	J4o	46	148	53s	39.005
112	T6o	45.825	149	84o	38.855
113	97o	45.575	150	92o	38.525
114	95s	45.55	151	74o	38.32
115	J3o	45.17	152	64o	38.26
116	76s	45.06	153	62s	38.12
117	J2o	44.715	154	83o	38.045
118	85s	44.6	155	54o	37.89
119	87o	44.565	156	72s	37.8
120	T2s	44.535	157	52s	37.635
121	94s	44.13	158	82o	37.32
122	T4o	44.13	159	42s	37.255
123	T5o	44.095	160	73o	36.66
124	96o	44.045	161	32s	36.405
125	86o	43.78	162	63o	36.335
126	75s	43.66	163	53o	36.02
127	76o	43.43	164	43o	35.62
128	65s	43.425	165	52o	34.92
129	93s	43.15	166	72o	34.615
130	92s	42.99	167	62o	33.63
131	T3o	42.955	168	42o	33.455
132	95o	42.635	169	32o	31.795
133	84s	42.46			
134	85o	41.77			
135	T2o	41.705			
136	64s	41.33			
137	74s	41.29			
138	94o	41.265			
139	54s	41.1			
140	83s	40.88			
141	82s	40.56			
142	65o	40.345			

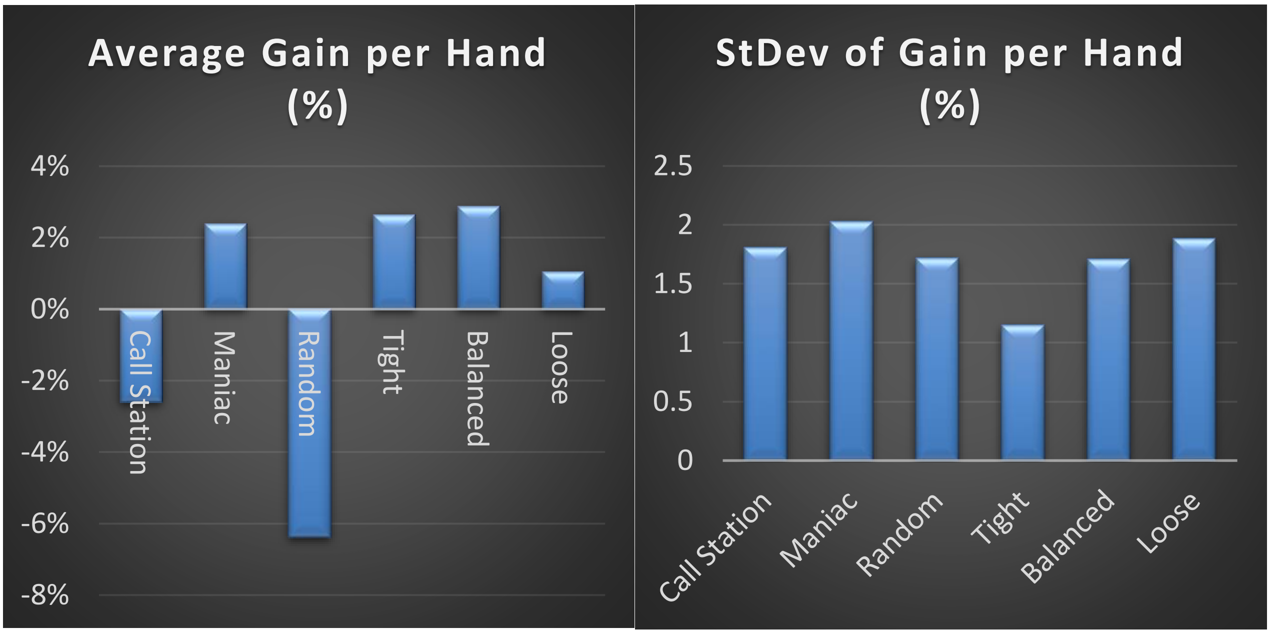
Appendix E: Setting 1 Descriptive Statistics



Appendix F: Setting 1 Simulated Path



Appendix G: Setting 2 (No Rake) Descriptive Statistics



Appendix H: Setting 2 (Rake = 5%) Average Gain per Hand