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The Shape of a Drum

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Abstract

We prove a lower bound on the number of corners of a simple polygon that one can spectrally determine and review the literature surrounding the famous problem "Can one hear the shape of a drum?", with discussions made into some of the current open problems.

1. Introduction

An interesting problem of spectral geometry concerns itself with the relationship between the eigenvalues of the Laplace-Beltrami operator on a compact Riemannian manifold (M, g) and the metric tensor g. More specifically, if the Laplace spectra of two compact Riemannian manifolds coincide, what can one say about their geometries? In 1964, John Milnor [10] determined that the Laplace spectrum does not uniquely determine the geometry of M by using a construction of two self-dual lattices L_1 and L_2 in $\mathbb{R}^{16}$ by Witt [15] such that the corresponding Riemannian manifolds $\mathbb{R}^{16}/L_1$ and $\mathbb{R}^{16}/L_2$ are isospectral but not isometric. However, there do exist geometric properties that are fixed by the spectrum called *spectral invariants*.

Although the problem for general Riemannian manifolds is interesting, we restrict our attention to planar domains where the question can be intuitively thought of as "Can one hear the shape of a drum?", which Mark Kac popularised in 1966 [6]. Since we are dealing with a drum, we require Dirichlet boundary conditions. There are a number of positive results: one can hear the area of the domain, the length of its boundary, and the number of holes. In general the answer to this question is negative, as shown by Gordon, Webb, and Wolpert in 1992 [4]. They constructed two isospectral polygonal domains which are not congruent using a group theoretic method known as the Sunada method [13].

Recently in 2015, Zhiqin Lu and Julie Rowlett discovered a new spectral invariant; one can hear corners [8]. That is, any simply connected planar domain with piecewise smooth Lipschitz boundary and at least one corner cannot be isospectral to any connected planar domain, of any genus, that has smooth boundary.

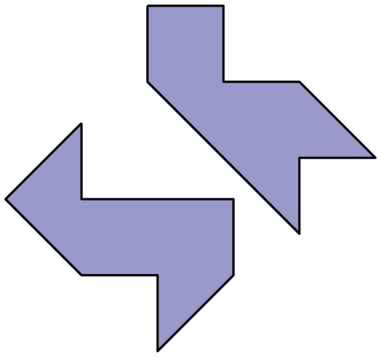

Figure 1: Identical sounding drums (taken from `http://www.math.udel.edu/~driscoll/`)

The main result of this paper extends on their work. We were interested in whether one can hear the number of corners of a drum much as one can hear the number of holes of a drum. We prove that this can be partially done for any simple polygonal shaped drum.

Theorem 1 (One can hear a lower bound on the number of corners of a simple polygon)**.** *The number of corners for a simple polygon is at least* $\frac{1}{6a_0-1}+2$ *where* a_0 *is the corner contribution term in the asymptotic heat trace expansion.*

As far as we are aware, this result is unpublished, or at the very least we proof we provide is original, including its preceding lemma (Lemma 20). It should be noted that unless specified otherwise, every other result or definition in this report is from the literature on this subject. We also comment on our progress towards an open problem: Can one hear the shape of a convex polygon? Moreover, we discuss briefly the problem of whether one can hear cusps.

2. The setup of the problem

Consider a bounded domain $\Omega \subset \mathbb{R}^2$. An idealised drum will vibrate according to the wave equation if tapped, and so we have the following initial boundary value problem:

$$\begin{cases} (\partial_t^2 - c^2\Delta)u = 0, & \\ u(x,0) = f(x) & x \in \Omega, \\ \partial_t^2 u(x,t)|_{t=0} = 0 & x \in \Omega, \\ u(x,t) = 0 & x \in \partial\Omega, \end{cases}$$

where $u : \Omega \times [0,\infty) \to \mathbb{R}$. Note that $u = 0$ on the boundary of the Ω models the fact that the drum must be fixed in place along its border. This problem can be reduced to solving for the eigenfunctions and eigenvalues of the Laplacian which is the following Dirichlet boundary value problem:

$$\begin{cases} \Delta\phi = -\lambda\phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\phi : \Omega \to \mathbb{R}$.

Definition 2. *The **Dirichlet spectrum** of the Laplacian on Ω, which we denote by $\mathrm{Spec}_D(\Omega)$ is the set of eigenvalues λ such that a non-trivial solution u to the boundary value problem above exists.*

Definition 3. *We say that two bounded domains $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are **isospectral** if $\mathrm{Spec}_D(\Omega_1) = \mathrm{Spec}_D(\Omega_2)$.*

Thus, we can phrase the isospectral problem as follows:

Problem: *If $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are two bounded domains which are isospectral, then are Ω_1 and Ω_2 isometric?*

Before we continue, we note that in higher dimensions the isospectral problem is: *If (M,g) and (M',g') are compact Riemannian manifolds that are isospectral then are they isometric?* In this setting, the spectrum of interest is that of the Laplace-Beltrami operator defined by

$$\Delta_g = -\frac{1}{\sqrt{\det g}}\partial_i g^{ij}\sqrt{\det g}\,\partial_j.$$

In Kac's problem, we aren't concerned with differing metrics since we are in flat 2-dimensional space, so we need only worry about what the boundaries of the domains look like.

Definition 4. *A **spectral invariant** is a quantity which is uniquely determined by the Dirichlet spectrum of the Laplacian.*

Using the spectral theorem for compact self-adjoint operators, one can show that the spectrum satisfies

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \uparrow \infty.$$

One of the first important results was due to Weyl in 1912 [14], and is now known as Weyl's law:

Theorem 5 (Weyl's Law)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $\operatorname{Spec}_D(\Omega)$ be its Dirichlet spectrum. Then,*

$$\lim_{k\to\infty} \frac{|\Omega|\lambda_k}{4\pi k} = 1$$

where $|\Omega|$ denotes the area of Ω.

Weyl's law is proved using variational methods which involve minimising the *Rayleigh Quotient*,

$$\rho(u) = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}} = \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega |u|^2\, dx},$$

and utilising the theory of Sobolev spaces and weak derivatives. Weyl's law effectively showed that $|\Omega|$ was the first spectral invariant discovered.

In studying the interplay between the spectrum of the Laplacian and the geometry of a domain, one often employs the *heat trace*.

Definition 6. *We define the **heat trace** as*

$$h(t) = \operatorname{Tr} e^{-t\Delta} = \sum_{k=1}^{\infty} e^{-\lambda_k t}.$$

The operator $e^{-t\Delta}$ can be made rigorous but we do not discuss that here. The motivation behind this name comes from the study of the heat equation:

$$\begin{cases} (\partial_t - \Delta)u = 0, & \\ u(x,0) = f(x) & x \in \Omega, \\ u(x,t) = 0 & x \in \partial\Omega, \end{cases}$$

where f is the initial heat distribution over Ω. Using the technique of separation of variables one finds that $u(x,t) = e^{-\lambda_k t}\phi_k(x)$ where $k \in \mathbb{N}$ is a solution to the heat equation, where the ϕ_k satisfies

$$\Delta\phi_k = -\lambda_k\phi_k; \ \ \phi_k|_{\partial\Omega} = 0$$

and $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\Omega)$. Basic theorems from Hilbert space theory allow one to write

$$f = \sum_{k=1}^{\infty}\langle f, \phi_k\rangle\phi_k$$

where the inner product on $L^2(\Omega)$ is $\langle f, g\rangle := \int_\Omega f(x)g(x)\,dx$. The solution to the heat equation is then given by

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty}\langle f, \phi_k\rangle e^{-\lambda_k t}\phi_k(x) \\ &= \sum_{k=1}^{\infty}\left(\int_\Omega f(y)\phi_k(y)\,dy\right)e^{-\lambda_k t}\phi_k(x) \\ &= \int_\Omega\left(\sum_{k=1}^{\infty}e^{-\lambda_k t}\phi_k(x)\phi_k(y)\right)f(y)\,dy \\ &= \int_\Omega H(x,y,t)f(y)\,dy \end{aligned}$$

where

$$H(x,y,t) := \sum_{k=1}^{\infty}e^{-\lambda_k t}\phi_k(x)\phi_k(y)$$

is the heat kernel. In the finite-dimensional case, to compute the trace of a matrix we sum the diagonal. Now since we are in infinite dimensions, the appropriate thing to do is integrate the heat kernel over the diagonal. That is, we set $x = y$. Thus, the heat trace is given by

$$h(t) = \sum_{k=1}^{\infty}e^{-\lambda_k t} = \int_\Omega H(x,x,t)\,dx.$$

Now, one can show from Weyl's law that the heat trace is asymptotically equal to $\frac{|\Omega|}{4\pi t}$ as $t \to 0$. In 1954, Pleijel [11] improved on this by showing that

$$h(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}}, \ \text{as } t \to 0$$

where $|\partial\Omega|$ denotes the perimeter of Ω. This was the second spectral invariant discovered. For a smooth boundary, the third spectral invariant was discovered independently by McKean and

Singer [9], and Kac [6]. It is

$$\frac{1}{12\pi}\int_{\partial\Omega} k\,ds$$

where k is the geodesic curvature of the boundary and s is arc-length. For domains with corners, another spectral invariant was discovered by Dan Ray (unpublished) and Fedosov [3]. It is given by

$$\frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{\alpha_i}-\frac{\alpha_i}{\pi}\right)$$

where $\{\alpha_i\}_{i=1}^n$ are the angles of the corners. Putting these all together, we have the following asymptotic heat trace expansion:

Theorem 7. *For a bounded planar domain Ω with piecewise smooth boundary and a set of corners $C=\{p_j\}_{j=1}^n$ with interior angles $\{\theta_j\}_{j=1}^n$, the heat trace satisfies*

$$h(t)\sim\frac{|\Omega|}{4\pi t}-\frac{|\partial\Omega|}{8\sqrt{\pi t}}+\frac{1}{12\pi}\int_{\partial\Omega\setminus C} k\,ds+\frac{1}{24}\sum_{j=1}^{n}\left(\frac{\pi}{\alpha_j}-\frac{\alpha_j}{\pi}\right),\ \text{as } t\to 0. \tag{2}$$

In [8], Zu and Rowlett showed that this also holds for what they call *generalised corners* which are roughly points for which the boundary curve need only be asymptotically straight on either side of the corner point. In that same paper, they prove the following:

Theorem 8. *Let Ω be a simply connected planar domain with piecewise smooth Lipschitz boundary. If Ω has at least one corner, then Ω is not isospectral to any bounded planar domain with smooth boundary that has no corners.*

From the proof of the above theorem, with a slight modification for Corollary 10, they conclude the following. Define

$$a_0 := \frac{1}{12}\int_{\partial\Omega\setminus C} k\,ds+\frac{1}{24}\sum_{j=1}^{n}\left(\frac{\pi}{\theta_j}-\frac{\theta_j}{\pi}\right).$$

Corollary 9. *One can hear corners. Specifically, a bounded simply connected planar domain $\Omega\subset\mathbb{R}^2$ with piecewise smooth boundary has at least one corner if and only if*

$$a_0>\frac{1}{6}.$$

Equivalently, a bounded connected planar domain $\Omega\subset\mathbb{R}^2$ has smooth boundary if and only if

$$a_0\leqslant\frac{1}{6}.$$

Corollary 10. *Amongst all planar domains of fixed genus with piecewise smooth Lipschitz boundary, those that have at least one corner are spectrally distinguished.*

3. The heat trace for a rectangle

Before we continue to the main result of this paper, we look at an example which verifies the general theory for the rectangle. Here we note that this example was worked through independently by the author and then later verified by surrounding literature. Let Ω be a rectangle with side lengths a and b in the plane. Upon solving the Dirichlet boundary problem (1) over this domain, we calculate the eigenfunctions and eigenvalues to be

$$u_{mn}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \text{ and } \lambda_{mn} = \left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2$$

where $n, m \in \mathbb{Z}^+$. The heat trace is given by

$$h(t) = \sum_{m,n} e^{-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2 t} = \left(\sum_{n=1}^{\infty} e^{-\frac{n^2}{a^2}\pi^2 t}\right)\left(\sum_{m=1}^{\infty} e^{-\frac{m^2}{b^2}\pi^2 t}\right).$$

This is a rather complicated sum to deal with, so we will use the Poisson summation formula as a tool to deal with it.

Definition 11. *The Schwartz space $\mathcal{S}(\mathbb{R})$ is the set of smooth functions $f \in C^\infty(\mathbb{R})$ such that*

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} |x^\alpha f^{(\beta)}(x)| < \infty$$

where $\alpha, \beta \in \mathbb{N}$.

Note that $\|\cdot\|_{\alpha,\beta}$ defines a norm and we can think of Schwartz space as the space of functions that, together with their derivatives, decay sufficiently rapidly.

Lemma 12. (Poisson summation formula) *For a Schwartz function $f \in \mathcal{S}(\mathbb{R})$, the following holds:*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$$

where $\hat{f}$ is the Fourier transform of f.

It is a known fact that the Fourier transform of the Gaussian $\gamma(x) = e^{-\pi x^2}$ is itself. In what follows, we require the scaling property of the Fourier transform:

Fact 13. *If f, g are integrable functions, $a \in \mathbb{R}\setminus\{0\}$ and $f(x) = g(ax)$ then*

$$\hat{f}(\xi) = \frac{1}{|a|}\hat{g}\left(\frac{\xi}{a}\right).$$

Lemma 14. *Let $a > 0$. Then*

$$\sum_{n\in\mathbb{Z}^+} e^{-\frac{\pi^2}{a^2}n^2t} = -\frac{1}{2} + \frac{a}{2\sqrt{\pi t}} + o(t^k)$$

as $t \to 0^+$ for all $k \in \mathbb{Z}^+$.

Proof. Let $f(x) = e^{-\frac{\pi^2}{a^2}tx^2} = \gamma\left(\frac{\sqrt{\pi t}}{a}x\right)$. Then from the scaling property of the Fourier transform, it is clear that

$$\hat{f}(\xi) = \frac{a}{\sqrt{\pi t}}e^{-a^2\xi^2/t}.$$

From the Poisson summation formula we have

$$\sum_{n\in\mathbb{Z}} e^{-\pi^2n^2t/a^2} = \frac{a}{\sqrt{\pi t}}\sum_{m\in\mathbb{Z}} e^{-a^2m^2/t}.$$

As $t \to 0^+$, the terms $e^{-a^2m^2/t}$ all tend to 0 faster than any power of t unless $m = 0$. Hence

$$\sum_{n\in\mathbb{Z}} e^{-\pi^2n^2t/a^2} = \frac{a}{\sqrt{\pi t}} + o(t^k)$$

as $t \to 0^+$ for all $k \in \mathbb{Z}^+$. Since the sum counts $e^{-\pi^2n^2t/a^2}$ twice unless $n = 0$, it follows easily that

$$\sum_{n\in\mathbb{Z}^+} e^{-\frac{\pi^2}{a^2}n^2t} = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} e^{-\frac{\pi^2}{a^2}n^2t} - 1\right)$$

and so we conclude the proof. □

Theorem 15. *The heat trace for the rectangle Ω above is given by*

$$h(t) = \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + o(t^{k-1/2})$$

as $t \to 0^+$.

Proof. Using Lemma 14 we have

$$\begin{aligned}
h(t) &= \left(\sum_{n=1}^{\infty} e^{-\frac{n^2}{a^2}\pi^2t}\right)\left(\sum_{m=1}^{\infty} e^{-\frac{m^2}{b^2}\pi^2t}\right)\\
&= \left(-\frac{1}{2} + \frac{a}{2\sqrt{\pi t}} + o(t^k)\right)\left(-\frac{1}{2} + \frac{b}{2\sqrt{\pi t}} + o(t^k)\right)\\
&= \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} - o(t^k) + \frac{1}{2\sqrt{\pi t}}(a+b)o(t^k) + o(t^{2k})\\
&= \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + o(t^k) + o(t^{k-1/2}) + o(t^{2k})\\
&= \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + o(t^{k-1/2})
\end{aligned}$$

as $t \to 0^+$ for all $k \in \mathbb{N}$ and so the result follows. □

As we can see above, the heat trace gives us the area and perimeter of the rectangle which agrees with the general heat trace asymptotic expansion. For the constant term, note that the integral term in the heat trace expansion vanishes since a rectangle has straight sides so they each have geodesic curvature zero. If one substitutes in the values for each angle of the rectangle, $\frac{\pi}{2}$, into the corner contribution term, one arrives at $\frac{1}{4}$. So the heat trace expansion for the rectangle agrees with the general theory, as one should expect. Since this constant term is greater than $\frac{1}{6}$, and assuming the domain is bounded and simply connected, we can conclude that Ω has at least one corner using Corollary 9. Of course, we knew this a priori but it demonstrates that the theorems work for this case.

4. Main results and progress

In this section we restrict ourselves to *simple* polygons. A simple polygon is a polygon for which no sides intersect. Recalling the asymptotic heat trace expansion (2), we see that the integral term vanishes since the geodesic curvature of the boundary edges is zero for a polygon. So, the term a_0 reduces to

$$a_0 = \frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right)$$

and we will examine this term more closely.

Lemma 16. *For a regular n-gon, where the interior angles $\{\alpha_i\}_{i=1}^n$ satisfy $0 < \alpha_i \leqslant 2\pi$, the heat trace quantity a_0 is equal to*

$$\frac{1}{6}\left(1 + \frac{1}{n-2}\right).$$

Proof. Recall that the interior angle sum of a simple n-gon is given by $(n-2)\pi$ and each angle of a regular polygon is equal. Then it is easy to see that $\alpha_i = \frac{n-2}{n}\pi$ for every $1 \leqslant i \leqslant n$. Substituting this value into a_0 gives

$$\begin{aligned} a_0 &= \frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{(n-2)\pi/n} - \frac{(n-2)\pi/n}{\pi}\right) \\ &= \frac{n}{24}\left(\frac{n}{n-2} - \frac{n-2}{n}\right) \\ &= \frac{1}{6}\left(1 + \frac{1}{n-2}\right). \end{aligned}$$

□

As the number of sides of a regular polygon goes to infinity, the regular polygon approaches a disk. Moreover, $a_0 \to \frac{1}{6}$ which is exactly the value of a_0 for a disk. Rearranging the above equation, we see that

$$n = \frac{1}{6a_0 - 1} + 2.$$

Thus, if we know a_0 which we can get from the asymptotic heat trace expansion, and if we are dealing with a regular polygon, then we can find the number of sides (number of corners) exactly. In this sense, one can hear the number of corners of a regular polygon. In [7], Lu and Rowlett show that in fact one can hear regular polygons.

For a general simple polygon, what can one say about the number of corners? One might guess that a_0 is minimised over all simple polygons by a regular polygon and this turns out to be true. In order to prove this we introduce some preliminaries.

Fact 17. (Jensen's inequality) *Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, $x_1, \dots, x_n \in \mathbb{R}$ and $a_1, \dots, a_n > 0$. Then the following inequality holds,*

$$\varphi\left(\frac{\sum_i a_i x_i}{\sum_i a_i}\right) \leqslant \frac{\sum_i a_i \varphi(x_i)}{\sum_i a_i},$$

with equality if and only if $x_1 = \cdots = x_n$ or φ is linear.

This is quite a well known inequality so we simply state it here without proof.

Definition 18. *The **arithmetic mean** of $x_1, \dots, x_n \in \mathbb{R}$ is given by*

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

*and the **harmonic mean** of $x_1, \dots, x_n \in \mathbb{R}^+$ is given by*

$$H_n = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}.$$

Lemma 19. *Let $S = \{x_1, \dots, x_n\} \subset \mathbb{R}^+$. Let A_n denote the arithmetic mean for S and let H_n denote the harmonic mean for S. Then $H_n \leqslant A_n$.*

Proof. First note that $\phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\varphi(x) = \frac{1}{x}$ is convex. Using Jensen's inequality, we find

$$H_n = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} = \varphi\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right) \leqslant \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{1}{x_i}\right) = \frac{1}{n}\sum_{i=1}^{n} x_i = A_n.$$

□

Lemma 20. *The corner contribution term*

$$a_0 = \frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right)$$

from the heat trace is minimised over all simple polygons by the a_0 corresponding to that of a regular n-gon. That is,

$$\min_{0<\alpha_i\leqslant 2\pi} \frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) = \frac{1}{6}\left(1 + \frac{1}{n-2}\right).$$

Proof. As in the proof for the calculation of a_0 for a regular n-gon, we have $\sum_i \alpha_i = (n-2)\pi$ for a simple n-gon. By Lemma 19, it is easy to see that

$$\sum_{i=1}^{n}\frac{1}{\alpha_i} \geqslant \frac{n^2}{\sum_{i=1}^{n}\alpha_i} = \frac{n^2}{(n-2)\pi}.$$

Therefore, we have

$$\begin{aligned} a_0 &= \frac{1}{24}\sum_{i=1}^{n}\left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right) \\ &= \frac{1}{24}\left(\sum_{i=1}^{n}\frac{\pi}{\alpha_i} - (n-2)\right) \\ &\geqslant \frac{1}{24}\left(\frac{n^2}{n-2} - (n-2)\right) \\ &= \frac{1}{6}\left(1 + \frac{1}{n-2}\right) \end{aligned}$$

with equality holding if and only if $\alpha_i = \alpha_j$ for every $1 \leqslant i, j \leqslant n$ as per Jensen's inequality. This is exactly the condition for a regular n-gon and so this completes the proof. □

There is a proof for this result using the method of Lagrange multipliers but we have omitted it since is not as simple. We can now proceed with the proof of Theorem 1.

Proof of Theorem 1. Let Ω be a simple polygon and let a_0 be the corner contribution term from the heat trace for this polygon. By the above lemma, we have that

$$a_0 \geqslant \frac{1}{6}\left(1 + \frac{1}{n-2}\right)$$

so rearranging this gives

$$n \geqslant \frac{1}{6a_0 - 1} + 2.$$

□

Now, it is known that one can hear triangles as shown by Durso in her Ph.D thesis [2] and an alternative proof was given in a paper by Grieser and Maronna [5] using only spectral invariants from the heat trace. Furthermore, it is shown in [7] that one can hear parallelograms and acute trapezoids. However, to solve the problem for acute trapezoids, one has to utilise another tool apart from the heat trace which we will come back to. A natural question to ask is whether or not it is possible to hear the shape of any convex polygon? As far as we are aware, it is still an open problem. Let us focus on convex quadrilaterals for now. We show that the heat trace turns out to not be enough to solve this problem in the following theorem. Here we note that this example was found by the author, but has likely been noticed before.

Theorem 21. *Let Ω be a convex quadrilateral. The quantities given by the heat trace: $|\Omega|$, $|\partial\Omega|$, and the corner contribution $\frac{1}{24}\sum_{i=1}^{4}\left(\frac{\pi}{\alpha_i}-\frac{\alpha_i}{\pi}\right)$ (where $0<\alpha_i<\pi$ are the angles of each corner of the quadrilateral), do not uniquely determine a convex quadrilateral.*

Proof. We give a proof by picture. Consider the parallelogram ABCD given below.

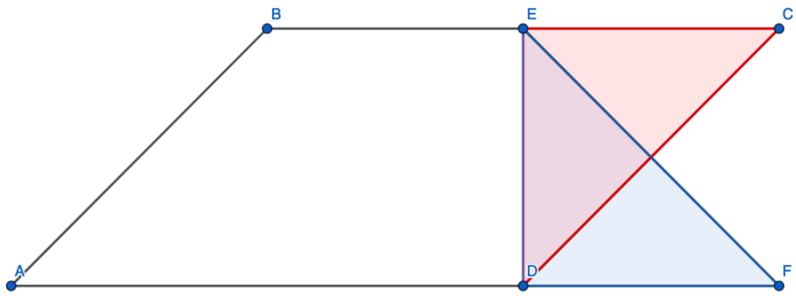

Figure 2: Parallelogram ABCD and trapezoid ABEF

The trapezoid ABEF is obtained by reflecting the red shaded area given by triangle ECD into the blue shaded area given by EFD. Thus, it is easy to see that the areas, perimeters, and angles (while given in a different order) of the parallelogram and the trapezoid coincide. □

5. The Wave Trace

Of course, one could try and extract more quantities from the heat trace but this turns out to not be feasible. In order to make more progress, we introduce a new tool called the wave trace. Some problems that the wave trace was used in include, but are not limited to, Durso's Ph.D thesis and in showing that one can hear acute trapezoids.

Definition 22. *Let λ_k be the eigenvalues of the Dirichlet spectrum. The wave trace is defined by*

$$w(t) := \sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k}t}.$$

The wave trace is motivated in a similar fashion to the heat trace but by studying the wave equation instead of the heat equation. The wave trace is actually what is known as a *tempered distribution*. Recall the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ from section 3[1]. Tempered distributions are elements of the dual space $\mathcal{S}^*(\mathbb{R}^n)$, that is, continuous linear functionals with respect to a topology on $\mathcal{S}(\mathbb{R})$ which we will not go into.

Definition 23. *The singular support of a tempered distribution ϕ,* singsupp ϕ*, is roughly the set of points for which ϕ is not given by a smooth function.*

A more precise definition of the singular support is beyond the scope of this report. In studying the singular support of the wave trace, one can find more spectral invariants. Let us digress a little. Consider a billiard table such as the one below. Suppose we set a billiard ball in motion along this table, and consider the set of paths for which the ball returns to its starting point and continues along the same path again. We call these paths *closed geodesics*.

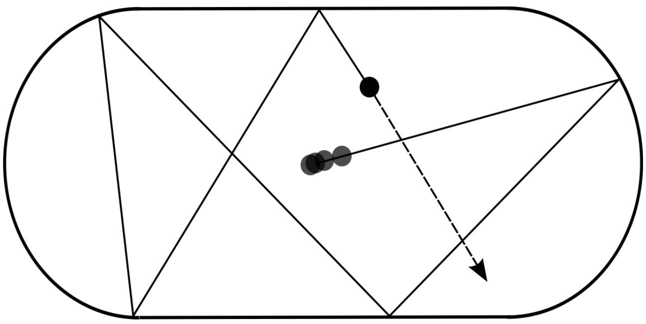

Figure 3: Geodesics on a billiard table (taken from `https://commons.wikimedia.org/wiki/File:BunimovichStadium.svg`)

Definition 24. *The **length spectrum** $\mathcal{L}$ is the set of lengths of these closed geodesics counted with multiplicities.*

[1]We actually defined $\mathcal{S}(\mathbb{R})$ but Schwartz space can be defined similarly for $\mathbb{R}^n$

It was shown by Duistermaat and Guillemin that singsupp $w \subset \mathrm{cl}(\mathcal{L})$ [1] and if there is exactly one path for each length $l \in \mathcal{L}$ (up to reversal of direction), then the inclusion becomes an equality. It is an open problem whether this equality holds in general. Given that the hypothesis is true, the lengths l in the length spectrum are spectral invariants and so this highlights the utility of the wave trace to study the inverse spectral problem. Moreover, this theorem presents a deep connection between dynamical systems and spectral analysis!

To end this section, we briefly discuss the question of whether one can hear the presence of cusps. In [12], Stewartson and Waechter found that the contribution of an inward point cusp is determined from the quantity

$$\frac{1}{24}\left(\frac{\pi}{\alpha} - \frac{\alpha}{\pi}\right),$$

by substituting $\alpha = 2\pi$, in which case it equals $-\frac{1}{16}$. They note that one cannot determine the presence of outward pointing cusps using the same quantity since it diverges as $\alpha \to 0$. They do however show that the presence of an outward pointing cusp is detectable by other means but note that the number of cusps cannot be deduced using the same method.

6. Further questions

All known counterexamples to "Can one hear the shape of a drum?" are non-convex polygons, including of course the first counterexample given by Gordon, Webb and Wolpert in [4]. Naturally, this leaves some questions: "Can one hear the shape of a convex polygon?" and "Can one hear the shape of a planar domain with smooth boundary?". As far as we are aware, these two problems are still unsolved. For the first question, we looked specifically at convex quadrilaterals, and from the example we produced in section 4 we expect that their length spectra should differ, but we were unable to verify this.

There exists more modern techniques in inverse spectral problems that form part of a relatively new field of mathematics called microlocal analysis, in which one not only utilises the singular support for distributions, but also the notion of the wave front set. The wave front set, loosely speaking, takes into account the directions for which their singularities propagate. It is possible that through the study of microlocal analysis, one may find solutions to these problems if older methods don't suffice.

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