AMSI VACATION RESEARCH SCHOLARSHIPS 2018-2019

Nonuniqueness in Geometric Partial Differential Equations

Jacquie Omnet

SUPERVISED BY DR. CHRIS VAN DER HEIDE

UNIVERSITY OF QUEENSLAND

2019

Vacation Research Scholarships are funded jointly by the Department of Education and Training and the Australian Mathematical Sciences Institute.

1 Abstract

We consider a particular geometric partial differential equation with Dirichlet boundary conditions, conformal solutions to which arise as surfaces of constant mean curvature. We look at how variational techniques can be used to show existence, nonuniqueness and regularity of solutions to the PDE.

2 Introduction

Let $\Omega = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ be the open unit disk. We seek functions $u \in H^1(\Omega; \mathbb{R}^3)$ satisfying

$$\Delta u = 2H u_x \times u_y \qquad \text{in } \Omega \tag{1}$$

$$u = \gamma \qquad \text{on } \partial\Omega \tag{2}$$

where we are given the constant $H > 0$ and the function $\gamma \in C^1(\Omega; \mathbb{R})$. Finally, set $R = \sup_{\partial\Omega} |\gamma|$. The aim of this project was to understand the proof of the following theorem as given by Brézis and Coron [2].

Theorem 1 Suppose $\gamma \in C^1(\Omega; \mathbb{R})$ is not constant on $\partial\Omega$ and that $HR < 1$. Then there exists at least two distinct solutions of the Dirichlet problem $(1) - (2)$.

Conformal solutions to the problem $(1) - (2)$ can be realised as surfaces with boundary γ and constant mean curvature H. We refer the reader to [3] for an introduction to surfaces and curvature. In this report we will discuss and provide a proof of **Theorem 1** following the methods of Brézis and Coron [2], which we have expanded upon to include calculations omitted in the original paper. We look at key elements of the proof and explore how they work together to imply **Theorem 1**. The proof relies largely on variational techniques, and the interested reader should consult [4],[9].

3 Background and Notation

The integrals in this report are taken in the sense of Lebesgue using the appropriate Lebesgue measure on $\mathbb{R}^2$ or $\mathbb{R}$. As usual, the space $C_c^\infty(\Omega)$ consists of all functions with compact support on Ω. In particular, a function v is in $C_c^\infty(\Omega)$ if the closure of the support set $\{x \in \Omega \ : \ v(x) \neq 0\}$ is contained within Ω. We say a function f is *harmonic* on a set Ω if $\Delta f = 0$ on Ω. For the measurable set $\Omega = \{(x,y) \ : \ x^2 + y^2 < 1\}$, we work primarily in the Sobolev space $H^1(\Omega; \mathbb{R}^3)$. This is the space of weakly differentiable functions $f : \Omega \to \mathbb{R}^3$, such that both $|f|^2$ and $|\nabla f|^2$ have finite integrals over Ω. An introduction to Sobolev spaces can be found in [4].

In §2, we introduced the problem of looking for functions in $H^1(\Omega;\mathbb{R}^3)$ satisfying (1) − (2). Notice that while (1) involves second derivatives, these may not be defined for functions in $H^1(\Omega;\mathbb{R}^3)$. This necessitates the notion of a *weak solution*. In particular, we say u weakly satisfies (1) if for all $v \in C_c^\infty(\Omega)$ there holds

$$-\int_\Omega \nabla v \cdot \nabla u = 2H \int_\Omega v \cdot u_x \times u_y. \tag{3}$$

To see how this equation relates to (1), we first suppose u is a smooth solution. That is, $\Delta u = 2Hu_x \times u_y$. Multiplying through by any $v \in C_c^\infty(\Omega)$ and integrating over Ω, we obtain

$$\int_\Omega v \cdot \Delta u = 2H \int_\Omega v \cdot u_x \times u_y.$$

Integrating by parts on the left hand side gives

$$-\int_\Omega \nabla v \cdot \nabla u = 2H \int_\Omega v \cdot u_x \times u_y,$$

which is the same as (3). We call (1) the *Euler Lagrange equation* of (3), and (3) the *weak formulation* of (1). This same method can be applied to obtain weak formulations of other partial differential equations.

By a similar token, we say a sequence $\{u^n\} \subset L^2(\Omega)$ converges *weakly* in L^2 to some function $u \in L^2(\Omega)$ if for all $v \in L^2(\Omega)$, there holds

$$\int_\Omega v \cdot u^n \to \int_\Omega v \cdot u$$

as $n \to \infty$. If in addition, u and the u^n are in $H^1(\Omega)$ and

$$\int_\Omega v \cdot \nabla u^n \to \int_\Omega v \cdot \nabla u$$

for all $v \in H^1(\Omega)$, then we say $u^n \to u$ weakly in $H^1(\Omega)$. More general definitions of weak convergence can be found in the appendix of [4], however, for the purpose of this report it suffices to think about weak convergence in the above manner.

The calculus of variations plays a large role in the proof of **Theorem 1**. We will often find ourselves presented with some functional F for which we wish to find a minimiser over some space K. The *direct method in the calculus of variations* is the method of taking a minimising sequence $\{u^n\} \subset K$, showing it converges weakly to some $u \in K$ and then showing the inequality

$$F(u) \le \liminf_{v \in K} F(v)$$

is satisfied. Note this requires F to be bounded below. If the above inequality holds, we say F is *sequentially lower semicontinuous* on K.

4 Regularity

Before we prove **Theorem 1**, it will be useful to note that weak solutions of $(1)-(2)$ are in $C^\infty(\Omega)\cap C(\overline{\Omega})$. To see this, suppose $u=(u^1,\ u^2,\ u^3)$ is a weak solution of $(1)-(2)$. Then u must weakly satisfy $\Delta u = 2Hu_x\times u_y$ in Ω. Expanding this out, we get coupled system of semilinear PDE

$$\begin{aligned}\Delta u^1 &= 2H(u_x^2u_y^3-u_x^3u_y^2)\\ \Delta u^2 &= 2H(u_x^3u_y^1-u_x^1u_y^3)\qquad (*)\\ \Delta u^3 &= 2H(u_x^1u_y^2-u_x^2u_y^1).\end{aligned}$$

If $\gamma=(\gamma_1,\ \gamma_2,\ \gamma_3)$ is the continuously differentiable Dirichlet data, we aditionally have $u^i=\gamma^i$ $(i=1,2,3)$ on $\partial\Omega$. Then by **Lemma B.5**, there exists some harmonic $f=(f^1,f^2,f^3)\in C^\infty(\Omega)\cap C(\overline{\Omega})$ satisfying $f=\gamma$ on $\partial\Omega$. It follows that $v=u-f$ weakly satisfies

$$\begin{aligned}\Delta v^1 &= 2H(u_x^2u_y^3-u_x^3u_y^2)\\ \Delta v^2 &= 2H(u_x^3u_y^1-u_x^1u_y^3)\qquad (*)'\\ \Delta v^3 &= 2H(u_x^1u_y^2-u_x^2u_y^1)\end{aligned}$$

and vanishes on the boundary. Applying **Lemma A.1** on each of the above three equations $(*)'$ shows $v\in C(\overline{\Omega})\cap H_0^1(\Omega)$. Given the regularity of f, this can only be possible if $u\in C(\overline{\Omega})\cap H^1(\Omega)$ too. Moroever, Wente proved in [10] that any function u satisfying $\Delta u = 2Hu_x\times u_y$ on Ω is necessarily in $C^\infty(\Omega)$. Together these show that any solution to $(1)-(2)$ is in $C^\infty(\Omega)\cap C(\overline{\Omega})$. In particular, we note solutions to $(1)-(2)$ are in $L^\infty(\Omega)$. These observations will be useful throughout the report.

5 Solving $(1)-(2)$.

5.1 The Small Solution (Existence)

5.1.1 $(1)-(2)$ as an Euler Lagrange equation

It turns out that $(1)-(2)$ has variational structure. Indeed, consider the energy functional

$$E(v)=\int_\Omega |\nabla v|^2+\frac{4}{3}H\int_\Omega v\cdot v_x\times v_y$$

on the set $G=\{v\in H^1(\Omega;\mathbb{R}^3)\cap L^\infty(\Omega;\mathbb{R}^3):\ \ v=\gamma \text{ on } \partial\Omega\}$. The critical points of E in G are solutions to $(1)-(2)$. To see this, fix $v\in C_c^\infty(\Omega;\mathbb{R}^3)$ and define $i:\mathbb{R}\to\mathbb{R}$ by $i(t):=E(\tilde{u}+tv)$ for some critical point $\tilde{u}\in G$ of E. Note $\tilde{u}+tv\in G$, so

$$i(0)=E(\tilde{u})\qquad\text{and hence}\qquad i'(0)=0.$$

Using the definition of E, we compute

$$i'(t) = 2\int_\Omega \nabla v\cdot\nabla\tilde{u} + \frac{4}{3}H\Big(\int_\Omega \tilde{u}\cdot\tilde{u}_x\times v_y + \int_\Omega \tilde{u}\cdot v_x\times\tilde{u}_y + \int_\Omega v\cdot\tilde{u}_x\times\tilde{u}_y\Big) + \text{terms involving t}$$

so that

$$\begin{aligned} 0 = i'(0) &= 2\int_\Omega \nabla v\cdot\nabla\tilde{u} + \frac{4}{3}H\Big(\int_\Omega \tilde{u}\cdot\tilde{u}_x\times v_y + \int_\Omega \tilde{u}\cdot v_x\times\tilde{u}_y + \int_\Omega v\cdot\tilde{u}_x\times\tilde{u}_y\Big) \\ &= 2\int_\Omega \nabla v\cdot\nabla\tilde{u} + \frac{4}{3}H\Big(\int_\Omega u\cdot(u_x\times v_y + v_x\times u_y) + v\cdot u_x\times u_y\Big) \\ &= 2\int_\Omega \nabla v\cdot\nabla u + 4H\int_\Omega v\cdot u_x\times u_y \end{aligned} \tag{4}$$

where in the final equality we used **Lemma A.3**. Recalling our discussion from §3, it follows that $\tilde{u}$ is a weak solution to (1). Since $\tilde{u}\in G$, we moreover have $\tilde{u}=\gamma$ on $\partial\Omega$. That is, $\tilde{u}$ satisfies both (1) and (2). Hence if we can show that a critical point of E on G exists, we have found a solution to (1) – (2).

5.1.2 Existence of a minimiser

Define for some $R' > R$ the set

$$K = \{v\in G \;:\; \|v\|_{L^\infty}\le R'\}.$$

From §4 we know any critical point $\tilde{u}$ of E must be smooth on Ω and continuous up to the boundary. The maximum principle then gives $\|\tilde{u}\|_{L^\infty}\le R$, so we are not losing much by restricting E to K. In particular, fix R' so that $HR' < 1$. This allows us to find a lower bound for E. To find this bound, we estimate for $v\in K$

$$|v_x\times v_y| \le |v_x||v_y| \le \frac{|v_x|^2}{2} + \frac{|v_y|^2}{2} = \frac{1}{2}|\nabla v|^2 \tag{5}$$

via Young's inequality, so that

$$\Big|\int_\Omega v\cdot v_x\times v_y\Big| \le \int_\Omega |v||v_x\times v_y| \le \frac{1}{2}\|v\|_{L^\infty}\int_\Omega |\nabla v|^2.$$

Then using the definition of E, it follows that

$$\begin{aligned} E(v) &\ge \int_\Omega |\nabla v|^2 - \frac{4}{3}H\Big|\int_\Omega v\cdot v_x\times v_y\Big| \\ &\ge \int_\Omega |\nabla v|^2 - \frac{2}{3}H\|v\|_{L^\infty}\int_\Omega |\nabla v|^2 \\ &= (1-\frac{2}{3}H\|v\|_{L^\infty})\int_\Omega |\nabla v|^2 \\ &\ge \frac{1}{3}\int_\Omega |\nabla v|^2 \end{aligned} \tag{6}$$

where we noted $H\|v\|_{L^\infty}\le HR' < 1$. We are now in a position to implement the direct method in the calculus of variations. Take a minimising sequence $\{u^n\}\subset K$. That is,

$$\lim_{n\to\infty} E(u^n) = \inf_{v\in K} E(v).$$

Compactness results show there exists some $\tilde{u}\in K$ such that $u^n\to\tilde{u}$ weakly in H^1, strongly in L^2

and almost everywhere on Ω. These results allow us to show that E is lower semicontinuous. This is discussed more explicitly in the proof of **Lemma C.1**. Lower semicontinuity says $\inf_{v\in K} E(v) \geq E(\tilde{u})$. But since $\tilde{u} \in K$, we get $\inf_{v\in K} E(v) \leq E(\tilde{u})$. This gives

$$\inf_{v\in K} E(v) = E(\tilde{u}),$$

so $\tilde{u} \in K$ minimises E over K. As we will see in the next section, however, this does not immediately mean $\tilde{u}$ is a critical point for E.

5.1.3 A minimiser of E is a critical point for E

Fix a nonzero function $v \in C_c^\infty(\Omega;\mathbb{R})$. Because it is possible that $\|\tilde{u}\|_{L^\infty} = R'$, the function $\tilde{u} + tv$ might not be in K, even for small t. Instead we try the function $(1-tv)\tilde{u}$ in the case where $t \geq 0$ and v is nonnegative. Using the compact support of v, we have $(1-tv)\tilde{u} = \tilde{u} = \gamma$ on $\partial\Omega$. Recalling $\tilde{u} \in K$, there holds

$$\|(1-tv)\tilde{u}\|_{L^\infty(\Omega;\mathbb{R}^3)} \leq \|1-tv\|_{L^\infty}\|\tilde{u}\|_{L^\infty} \leq R'$$

since $\|1-tv\|_{L^\infty} \leq 1$ for $t < \frac{1}{\|v\|_{L^\infty}}$. It follows that $(1-tv)\tilde{u} \in K$ for such t. We don't know if $\tilde{u}$ is a critical point for E, but what we do know is that $\tilde{u}$ minimises E on K. Hence

$$E(\tilde{u}) - E((1-tv)\tilde{u}) \leq 0.$$

Analysis on this inequality is the content of **Lemma C.2** in the appendix, and leads to the bound

$$-\Delta|\tilde{u}|^2 \leq 0 \qquad \text{which gives} \qquad \|\tilde{u}\|_{L^\infty} \leq R$$

by the weak maximum principle. We now turn our attention back to the function $\tilde{u} + tv$ for any fixed $v \in C_c^\infty(\Omega;\mathbb{R})$ and $t \in \mathbb{R}$. Define $i : \mathbb{R} \to \mathbb{R}$ by $i(t) = E(\tilde{u} + tv)$. Because $\|u\|_{L^\infty} < R'$, the function $\tilde{u} + tv$ is in K for sufficiently small t. Hence i attains a local minimum at $t = 0$ for every smooth v. It follows that $i'(0) = 0$, so the calculations from §5.1.1 yield

$$-\Delta\tilde{u} + 2H\tilde{u}_x \times \tilde{u}_y = 0$$

in the weak sense, which shows $\tilde{u}$ solves (1) – (2). In particular, there exists a solution to (1) – (2).

5.2 The Large Solution (Nonuniqueness)

In §5.1, we found a solution $\tilde{u}$ to (1) – (2). We would now like to show this solution is not unique. Notice that if some $\overline{u}$ solves (1) – (2), we can write $\overline{u} = \tilde{u} + v$ for some $v \in H_0^1(\Omega;\mathbb{R}^3)$. In this case,

$$\Delta(\tilde{u} + v) = 2H(\tilde{u} + v)_x \times (\tilde{u} + v)_y \qquad \text{in } \Omega.$$

Recalling $2H\tilde{u}_x \times \tilde{u}_y = \Delta\tilde{u}$, using distributivity of the cross product and rearranging the above equality gives constraints on v

$$2H(\tilde{u}_x \times v_y + v_x \times \tilde{u}_y + v_x \times v_y) - \Delta v = 0 \qquad \text{in } \Omega. \tag{7}$$

This new partial differential equation is the Euler Lagrange equation of the functional

$$F(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y + \frac{4}{3} H \int_\Omega v \cdot v_x \times v_y.$$

That is, the nontrivial critical points of F in $H_0^1(\Omega)$ are the nonzero solutions of (7) in $H_0^1(\Omega)$. Hence the problem of finding another solution of (1) − (2) distinct from $\tilde{u}$ amounts to finding a nontrivial critical point of F in $H_0^1(\Omega)$. We consider the components of F separately by defining

$$L(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y \quad \text{and} \quad Q(v) = \int_\Omega v \cdot v_x \times v_y$$

for $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ so that $F(v) = L(v) + \frac{4}{3} H Q(v)$ for such v. This is beneficial because L and Q both have valuable properties we can exploit later to help prove **Theorem 1**. Namely, Q is continuous when we extend it to $H_0^1(\Omega)$ with an important bound (see **Lemma A.5**), and L corresponds to a symmetric bilinear form which we will show is positive definite in §5.2.2.

We digress for a moment to talk about the geometric interpretation of Q; let S be a surface parametrised by u. Recall the area element for such a surface is $|u_x \times u_y|$ (see [3]). Then the area integral of S is given by

$$\int_S 1 \, dxdy = \int_\Omega |u_x \times u_y| \, dudv.$$

We call this the algebraic area of S. If S is additionally a closed surface, let V denote the region enclosed by S. Gauss' divergence theorem then gives

$$\begin{aligned} \int_V 1 \, dxdydz &= \frac{1}{3} \int_V \operatorname{div}(x, y, z) \, dxdydz \\ &= \frac{1}{3} \int_S (x, y, z) \cdot n \, dxdy \\ &= \frac{1}{3} \int_S u \cdot \frac{u_x \times u_y}{|u_x \times u_y|} \, dxdy \\ &= \frac{1}{3} \int_\Omega u \cdot u_x \times u_y \, dudv. \end{aligned}$$

This is the algebraic volume enclosed by the closed surface S. If S is not closed but instead has some boundary curve, there is no well defined notion of enclosed volume. However, we assign such a surface an enclosed volume by *defining* it to be $\frac{1}{3} \int_\Omega u \cdot u_x \times u_y$. It turns out this definition behaves in much the same way as an actual volume. Indeed **Lemma A.5** shows that for $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ there holds

$$|Q(v)| = \left| \int_\Omega v \cdot v_x \times v_y \right| \leq C \|\nabla v\|_{L^2}^3 \quad \text{so} \quad |Q(v)|^{2/3} \leq C \int_\Omega |\nabla v|^2.$$

This is a slightly weaker version of the isoperimetric inequality; an inequality which provides an upperbound for the volume enclosed by a surface in terms of its area. Wente [10] showed in particular that this inequality holds for $C = \frac{1}{S}$ where $S = (32\pi)^{1/3}$. That is,

$$|Q(v)|^{2/3} \leq \frac{1}{S} \int_\Omega |\nabla v|^2. \tag{8}$$

Brézis and Coron [2] extended (8) to hold for functions $v \in C^\infty(\mathbb{R}^2; \mathbb{R}^3)$. This estimate will be a key element in the proof of **Theorem 1**. We will also frequently make use of the following calculation: Let $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. By the definition of E,

$$\begin{aligned} E(\tilde{u}+v) &= \int_\Omega |\nabla(\tilde{u}+v)|^2 + \frac{4}{3}H \int_\Omega (\tilde{u}+v)\cdot(\tilde{u}+v)_x \times (\tilde{u}+v)_y \\ &= \int_\Omega |\nabla \tilde{u}|^2 + \int_\Omega |\nabla v|^2 + 2\int_\Omega \nabla\tilde{u}\cdot\nabla v + \frac{4}{3}H\int_\Omega \tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y + \frac{4}{3}H\int_\Omega v\cdot v_x\times v_y \\ &\quad + \frac{4}{3}H\int_\Omega \tilde{u}\cdot v_x\times v_y + v\cdot\tilde{u}_x\times\tilde{u}_y + \frac{4}{3}H\int_\Omega (\tilde{u}+v)\cdot(\tilde{u}_x\times v_y + v_x\times\tilde{u}_y) \\ &= E(\tilde{u}) + E(v) + 2\int_\Omega \nabla\tilde{u}\cdot\nabla v + 4H\int_\Omega \tilde{u}\cdot v_x\times v_y + v\cdot\tilde{u}_x\times\tilde{u}_y \end{aligned}$$

where we have used **Lemma A.3** in the final equality. Now $\tilde{u}$ satisfies (1), so

$$4H\int_\Omega v\cdot\tilde{u}_x\times\tilde{u}_y = 2\int_\Omega v\cdot\Delta\tilde{u} = -2\int_\Omega \nabla v\cdot\nabla\tilde{u}$$

when we have integrated by parts. Hence whenever $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, it follows that

$$E(\tilde{u}+v) = E(\tilde{u}) + E(v) + 4H\int_\Omega \tilde{u}\cdot v_x\times v_y. \tag{9}$$

5.2.1 The Quantity J

In this section we will show that a second solution to (1) – (2) exists, provided certain other technical conditions are satisfied. We start by recalling the functional $Q(v) = \int_\Omega v\cdot v_x\times v_y$ for $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$. **Lemma A.5** allows us to continuously extend Q to $H_0^1(\Omega)$. Then for $v \in H_0^1(\Omega)$ with $Q(v) \neq 0$, we introduce

$$J_v = \frac{L(v)}{|Q(v)|^{2/3}} \qquad \text{and} \qquad J = \inf_{\substack{v\in H_0^1 \\ Q(v)\neq 0}} J_v.$$

Since $H_0^1 \cap L^\infty$ is dense in H_0^1 and because Q is a continuous function of v, there holds

$$\inf_{\substack{v\in H_0^1 \\ Q(v)\neq 0}} \frac{L(v)}{|Q(v)|^{2/3}} = \inf_{\substack{v\in H_0^1\cap L^\infty \\ Q(v)\neq 0}} \frac{L(v)}{|Q(v)|^{2/3}}.$$

Notice that if we fix the value of Q to 1, the above infimum is unchanged. Indeed, we can take any $v \in H_0^1$ with $Q(v) \neq 0$ and define the normalisation $\tilde{v} = \frac{1}{|Q(v)|^{1/3}}v$. Then $\tilde{v} \in H_0^1$ with $Q(\tilde{v}) = 1$ and

$$L(\tilde{v}) = \int_\Omega |\nabla\tilde{v}|^2 + 4H\int_\Omega \tilde{u}\cdot\tilde{v}_x\times\tilde{v}_y = \frac{1}{|Q(v)|^{2/3}}\left(\int_\Omega |\nabla v|^2 + 4H\int_\Omega \tilde{u}\cdot v_x\times v_y\right) = \frac{L(v)}{|Q(v)|^{2/3}}.$$

It follows that

$$J = \inf_{\substack{v\in H_0^1 \\ Q(v)\neq 0}} \frac{L(v)}{|Q(v)|^{2/3}} = \inf_{\substack{v\in H_0^1 \\ Q(v)=1}} L(v).$$

We claim that if J is achieved by some $v^0 \in H_0^1$ with $Q(v^0) = 1$, then we can find some $k \in \mathbb{R}\backslash\{0\}$ such that $\overline{u} = \tilde{u} + kv^0$ solves $(1) - (2)$. Indeed, suppose $J = L(v^0)$ for such a v^0. Then for fixed $w \in H_0^1 \cap L^\infty$, define

$$\varphi(t) = \frac{1}{\mu(t)}(v^0 + tw) \quad \text{where} \quad \mu(t) = |Q(v^0 + tw)|^{1/3} \qquad t \in \mathbb{R},$$

so that $Q(\varphi(t)) = 1$ and $\varphi \in H_0^1$. Then defining $i : \mathbb{R} \to \mathbb{R}$ by $i(t) = J_{\varphi(t)}$, we have that $t = 0$ is a local minimum for i over such functions. That is, $i'(0) = 0$. Note that

$$i(t) = \frac{L(v^0 + tw)}{\mu(t)^2} \qquad \text{so} \qquad i'(0) = L(v^0)\frac{d}{dt}\left(\frac{1}{\mu(t)^2}\right)\Big|_{t=0} + \frac{1}{\mu(0)^2}\frac{d}{dt}L(v^0 + tw)\Big|_{t=0}.$$

Now, $\mu(0) = |Q(v^0)|^{1/3} = 1$ and $L(v^0) = J$. We show that $\frac{d}{dt}\frac{1}{\mu(t)^2}\big|_{t=0} = -2\int_\Omega w \cdot v_x^0 \times v_y^0$ in **Lemma C.7**. Finally,

$$\begin{aligned} L(v^0 + tw) &= \int_\Omega |\nabla(v^0 + tw)|^2 + 4H\int_\Omega \tilde{u}\cdot(v^0 + tw)_x \times (v^0 + tw)_y \\ &= 2t\int_\Omega \nabla v^0 \cdot \nabla w + 4Ht\int_\Omega \tilde{u}\cdot(v_x^0 \times w_y + w_x \times v_y^0) + \text{constant terms} + \text{terms involving } t^2 \end{aligned}$$

and so

$$\frac{d}{dt}L(v^0 + tw)\Big|_{t=0} = 2\int_\Omega \nabla v^0 \cdot \nabla w + 4H\int_\Omega \tilde{u}\cdot(v_x^0 \times w_y + w_x \times v_y^0).$$

Putting this all together, we get

$$i'(0) = -2J\int_\Omega w \cdot v_x^0 \times v_y^0 + 2\int_\Omega \nabla v^0 \cdot \nabla w + 4H\int_\Omega \tilde{u}\cdot(v_x^0 \times w_y + w_x \times v_y^0) = 0. \tag{10}$$

Let v be a smooth function satisfying

$$2(-J(v_x \times v_y) - \Delta v + 2H(\tilde{u}_x \times v_y + v_x \times \tilde{u}_y)) = 0.$$

We claim that v^0 satisfies the weak formulation of this equation. Indeed, multiplying the above equation through by any test function $w \in C_c^\infty(\Omega)$ and integrating over Ω gives

$$-2J\int_\Omega w \cdot v_x \times v_y - 2\int_\Omega w \cdot \Delta v + 4H\int_\Omega w \cdot (\tilde{u}_x \times v_y + v_x \times \tilde{u}_y) = 0.$$

Integrating by parts on the second integral and using **Lemma A.2** on the third integral yields

$$-2J\int_\Omega w \cdot v_x \times v_y + 2\int_\Omega \nabla v \cdot \nabla w + 4H\int_\Omega \tilde{u}\cdot(v_x \times w_y + w_x \times v_y) = 0.$$

In view of (10), we get that v^0 weakly satisfies

$$\Delta v^0 = 2H(\tilde{u}_x \times v_y^0 + v_x^0 \times \tilde{u}_y) - J(v_x^0 \times v_y^0). \tag{11}$$

Next, suppose $\tilde{u} + kv^0$ satisfies (1) − (2) in the weak sense for some $k \in \mathbb{R}$. That is,

$$\Delta(\tilde{u} + kv^0) = 2H(\tilde{u} + kv^0)_x \times (\tilde{u} + kv^0)_y$$

weakly. Expanding both sides and using the fact that $\Delta\tilde{u} = 2H\tilde{u}_x \times \tilde{u}_y$, this becomes

$$k\Delta v^0 = 2Hk(\tilde{u}_x \times v^0_y + v^0_x \times \tilde{u}_y) + 2Hk^2 v^0_x \times v^0_y. \tag{12}$$

Comparing this with (11), we see $\tilde{u} + kv^0$ solves (1) − (2) if and only if $k = -\frac{J}{2H}$. It follows that $\overline{u} = \tilde{u} - \frac{J}{2H}v^0$ is a solution to the Dirichlet problem (1) − (2). This is distinct from $\tilde{u}$ if and only if J is nonzero.

All of this was done under the assumption that $J = L(v^0)$ for some $v^0 \in H^1_0$ with $Q(v^0) = 1$. If we can can show this is true, and moreover show that J is nonzero, we have found a large solution to (1) − (2). We will first focus on showing the infimum of L is attained by such a v^0.

5.2.2 A bound on L

In order to show that the infimum of L is attained, we require lower bounds on L. In this section we first show that L is bounded below by zero, and then go on to improve this bound as it will be useful in later calculations.

Now $\tilde{u} + tv \in K$ for $v \in H^1_0(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$ and t sufficiently small. Using that $\tilde{u}$ minimises E, the definition of E, and (9), there holds

$$\begin{aligned} 0 &\le E(\tilde{u} + tv) - E(\tilde{u}) \\ &= E(tv) + 4Ht^2 \int_\Omega \tilde{u} \cdot v_x \times v_y \\ &= t^2 \int_\Omega |\nabla v|^2 + \frac{4}{3}Ht^3 \int_\Omega v \cdot v_x \times v_y + 4Ht^2 \int_\Omega \tilde{u} \cdot v_x \times v_y \end{aligned} \tag{13}$$

Dividing through by t^2 and taking $t \to 0$, we get

$$\int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y \ge 0 \tag{14}$$

for all $v \in H^1_0(\Omega; \mathbb{R}^3) \cap L^\infty(\Omega; \mathbb{R}^3)$. Since $H^1_0 \cap L^\infty$ is dense in H^1_0, the above inequality holds for all $v \in H^1_0(\Omega; \mathbb{R}^3)$. We claim that (14) is in fact strict. To see this, we suppose

$$\int_\Omega |\nabla \tilde{v}|^2 + 4H \int_\Omega \tilde{u} \cdot \tilde{v}_x \times \tilde{v}_y = 0. \tag{15}$$

for some $\tilde{v} \in H^1_0(\Omega; \mathbb{R}^3)$. Define the symmetric bilinear form on $H^1_0(\Omega)$

$$B(v, w) = \int_\Omega \nabla v \cdot \nabla w + 2H \int_\Omega \tilde{u} \cdot (v_x \times w_y + w_x \times v_y).$$

The functional L has nice structure; notice $L(v) = B(v, v)$. In view of (14), we see that B is positive semidefinite, that is, $B(v, v) \geq 0$ for all $v \in H_0^1$. It turns out that B is in fact positive definite, as shown in **Lemma C.3**. Then since $B(\tilde{v}, \tilde{v}) = 0$ by (15), we get $\tilde{v} = 0$. This shows that the only zeros of L are trivial.

We can further improve this bound on L. In fact, we now claim that there exists some $\delta > 0$ such that for all $v \in H_0^1$

$$L(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y \geq \delta \int_\Omega |\nabla v|^2. \tag{16}$$

For the sake of contradiction, assume for all $\delta > 0$ there exists some $v^\delta \in H_0^1$ such that

$$\int_\Omega |\nabla v^\delta|^2 + 4H \int_\Omega \tilde{u} \cdot v_x^\delta \times v_y^\delta < \delta \int_\Omega |\nabla v^\delta|^2.$$

Clearly $\nabla v^\delta \neq 0$. Consider the sequence $\delta_n = \frac{1}{n}$. Then the corresponding sequence of H_0^1 functions v^{δ_n} satisfies

$$1 + \frac{4H \int_\Omega \tilde{u} \cdot \ v_x^{\delta_n} \times v_y^{\delta_n}}{\int_\Omega |\nabla v|^2} < \frac{1}{n} \tag{17}$$

Set $u^{\delta_n} = \frac{v^{\delta_n}}{\|\nabla v^{\delta_n}\|_{L^2}}$. Then u^{δ_n} is a sequence of H_0^1 functions with $\int_\Omega |\nabla u^{\delta_n}|^2 = 1$. Hence we can rewrite (17) as

$$B(u^{\delta_n}, u^{\delta_n}) = \int_\Omega |\nabla u^{\delta_n}|^2 + 4H \int_\Omega \tilde{u} \cdot u_x^{\delta_n} \times u_y^{\delta_n} < \frac{1}{n}.$$

Then as $n \to \infty$, we get

$$B(u^{\delta_n}, u^{\delta_n}) \to 0. \tag{18}$$

Since $\{\nabla u^{\delta_n}\}$ is a bounded sequence in L^2, Poincaré's inequality (see **Lemma B.4**) tells us that $\{u^{\delta_n}\}$ is a bounded sequence in H_0^1. Then by **Lemma B.2**, we can assume u^{δ_n} converges to some u in H_0^1 weakly and hence

$$\int_\Omega |\nabla u|^2 \leq \liminf_{n\to\infty} \int_\Omega |\nabla u^{\delta_n}|^2.$$

Moreover, by **Lemma A.4**, we get

$$\lim_{n\to\infty} \int_\Omega \tilde{u} \cdot u_x^{\delta_n} \times u_y^{\delta_n} = \int_\Omega \tilde{u} \cdot u_x \times u_y. \tag{19}$$

This implies the lower semicontinuity of B

$$B(u, u) \leq \liminf B(u^{\delta_n}, u^{\delta_n}).$$

Combining this with (18) gives $B(u, u) \leq 0$. But B is positive definite, so $u = 0$. Recalling $L(u) = B(u, u)$, we get that $u = 0$. That is, $u^{\delta_n} \to 0$ weakly in H_0^1. Thus via (19)

$$\int_\Omega \tilde{u} \cdot (u_x^{\delta_n} \times u_y^{\delta_n}) \to 0.$$

Recalling $\int_\Omega |\nabla u^{\delta_n}| = 1$, we then get $B(u^{\delta_n}, u^{\delta_n}) \to 1$, contradicting (18). This contradiction establishes the bound

$$L(v) \geq \delta \int_\Omega |\nabla v|^2, \qquad \text{for all } \ v \in H^1_0(\Omega;\mathbb{R}^3). \tag{16}$$

Remark 2

Again suppose there is some $v^0 \in H^1_0(\Omega;\mathbb{R}^3)$ with $Q(v^0) = 1$ and $L(v^0) = J$. The assumption $Q(v^0) = 1$ can only hold if $v^0, \nabla v^0$ are nonzero. This and the above results then show

$$J = L(v^0) \geq \delta \int_\Omega |\nabla v^0|^2 > 0.$$

It follows that $\overline{u} = \tilde{u} - \frac{J}{2H} v^0$ is distinct from $\tilde{u}$. Hence to prove **Theorem 1**, it suffices to show that such a v^0 exists.

Remark 3

While multiple solutions to (1) – (2) might exist, it is straightforward to show that minimisers of E are unqiue. Indeed, suppose $\tilde{u}, u \in K$ both satisfy

$$E(\tilde{u}) = \inf_{v\in K} E(v) = E(u).$$

Since $K \subset H^1 \cap L^\infty$ and $\tilde{u}, u = \gamma$ on $\partial\Omega$, it follows that $v = u - \tilde{u}$ is in $H^1_0 \cap L^\infty$. Then by (9), we have

$$E(u) = E(\tilde{u} + v) = E(\tilde{u}) + E(v) + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y \tag{20}$$

$$E(\tilde{u}) = E(u - v) = E(u) + E(-v) + 4H \int_\Omega \tilde{u} \cdot \tilde{v}_x \times \tilde{v}_y \tag{21}$$

Now $E(u) = E(\tilde{u})$ by assumption, so subtracting (21) from (20) and using the definition of E gives

$$0 = E(u) - E(\tilde{u}) + E(v) - E(-v) = \frac{8}{3} H \int_\Omega v \cdot v_x \times v_y$$

That is,

$$\int_\Omega v \cdot v_x \times v_y = 0.$$

Again since $E(\tilde{u}) = E(u)$, subtracting $E(\tilde{u})$ from both sides of (20) and noting the above equality gives

$$0 = E(v) + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y = \int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y.$$

Then the results of §5.2.2 imply $v = 0$. In particular $\tilde{u} = u$, so there is a *unique* $\tilde{u} \in K$ which minimises E. This means that for any other function u solving (1) – (2), there holds

$$E(\tilde{u}) \leq E(u).$$

For this reason, we call $\tilde{u}$ the *small solution*, and naturally any other solution u a *large solution*.

5.2.3 Minimiser for L

We now claim there does exist some $v^0 \in H^1_0(\Omega;\mathbb{R}^3)$ with $Q(v^0) = 1$ and $J = L(v^0)$. Take some minimising sequence $v^n \in H^1_0(\Omega;\mathbb{R}^3)$ with

$$Q(v^n) = 1, \text{ and } L(v^n) \to J. \tag{22}$$

Using the bound (16) from §5.2.2, we show in **Lemma C.7** that there exists some $v^0 \in H^1_0(\Omega;\mathbb{R}^3)$ with $v^n \to v^0$ weakly. Defining $w^n = v^n - v^0$, we get $w^n \to 0$ weakly in H^1_0. Since $\tilde{u} \in H^1 \cap L^\infty$ and $v^n \in H^1_0$, **Lemma A.4** tells us that

$$\int_\Omega \tilde{u} \cdot v^n_x \times v^n_y \to \int_\Omega \tilde{u} \cdot v^0_x \times v^0_y. \tag{23}$$

Furthermore, since $w^n \to 0$ weakly in H^1_0, we have

$$\int_\Omega \nabla v^0 \cdot \nabla w^n \to 0.$$

Together with (22), (23) and keeping in mind $v^n = w^n + v^0$, this gives

$$L(v^n) = \int_\Omega |\nabla v^0|^2 + \int_\Omega |\nabla w^n|^2 + 4H \int_\Omega \tilde{u} \cdot v^0_x \times v^0_y + o(1) = J + o(1) \tag{24}$$

as $n \to \infty$. Since $v^0 \in H^1_0(\Omega)$, so is $\tilde{v}^0 = \frac{v^0}{Q(v^0)^{1/3}}$. Moreover, $Q(\tilde{v}^0) = 1$. Hence by the definition of J,

$$L(\tilde{v}^0) = \frac{1}{|Q(v^0)|^{2/3}} \int_\Omega |\nabla v^0|^2 + \frac{4H}{|Q(v^0)|^{2/3}} \int_\Omega \tilde{u} \cdot v^0_x \times v^0_y \geq J,$$

and in particular

$$\int_\Omega |\nabla v^0|^2 + 4H \int_\Omega \tilde{u} \cdot v^0_x \times v^0_y \geq J|Q(v^0)|^{2/3}.$$

Together with (24), this implies

$$J|Q(v^0)|^{2/3} + \int_\Omega |\nabla w^n|^2 \leq J + o(1). \tag{25}$$

In **Lemma C.6**, we show

$$|Q(v^0)|^{2/3} + |Q(w^n)|^{2/3} + o(1) \geq 1,$$

as $n \to \infty$, which combined with (25), and the isoperimetric bound (8) gives

$$\begin{aligned} \int_\Omega |\nabla w^n|^2 &\leq J + o(1) - J|Q(v^0)|^{2/3} \\ &\leq J + o(1) - J(1 - |Q(w^n)|^{2/3} + o(1)) \\ &= J|Q(w^n)|^{2/3} + o(1) \\ &\leq \frac{J}{S} \int_\Omega |\nabla w^n|^2 + o(1). \end{aligned} \tag{26}$$

Rearranging (26) we obtain

$$(1-\frac{J}{S})\int_\Omega |\nabla w^n|^2 \leq o(1).$$

Note that if $J < S$, then $1-\frac{J}{S} > 0$ is a positive number. In this case,

$$\int_\Omega |\nabla w^n|^2 \leq o(1), \qquad \text{that is,} \qquad \int_\Omega |\nabla w^n|^2 \to 0.$$

Hence $\nabla v^n \to \nabla v^0$ in L^2. From **Lemma B.4**, we then get $v^n \to v^0$ strongly in H_0^1. Hence by the continuity of Q (see **Lemma A.5**), sending $n \to \infty$ in (22) and (24) gives

$$Q(v^0) = 1 \qquad \text{and} \qquad \int_\Omega |\nabla v^0|^2 + 4H\int_\Omega \tilde{u}\cdot v_x^0 \times v_y^0 = J$$

as required.

5.2.4 Showing $J < S$

From **Remark 1** we know that $\tilde{u}$ is smooth, so $\nabla\tilde{u}$ is well defined. Fix a point $(x_0, y_0) \in \Omega$ such that $\nabla\tilde{u}(x_0, y_0) \neq 0$, set $\mathbf{a} = \tilde{u}_x(x_0, y_0)$, and $\mathbf{b} = \tilde{u}_y(x_0, y_0)$. If $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, take $\bar{i} = -\frac{\mathbf{a}}{|\mathbf{a}|}$, $\bar{j} = \bar{i} \times \frac{\mathbf{b}}{|\mathbf{b}|}$ and $\bar{k} = \bar{i} \times \bar{j}$. If $\mathbf{a}, \mathbf{b}$ are linearly dependent, assume without loss of generality that $\mathbf{a} \neq 0$. Define $\bar{i} = -\frac{\mathbf{a}}{|\mathbf{a}|}$, let $\bar{j}$ be any unit vector orthogonal to $\mathbf{a}$, and take $\bar{k} = \bar{i} \times \bar{j}$. Then $\{\bar{i},\ \bar{j},\ \bar{k}\}$ is an orthonormal basis of $\mathbb{R}^3$ with the same orientation as the standard basis in $\mathbb{R}^3$. Notice with this choice of basis, we have

$$\mathbf{a}\cdot\bar{i} + \mathbf{b}\cdot\bar{j} < 0. \tag{27}$$

Recall for $v \in H_0^1 \cap L^\infty$ with $Q(v) \neq 0$ the function $J_v = L(v)/|Q(v)|^{2/3}$. Note $J_v \geq J$ by the definition of J as an infimum. Hence if we can show that $J_v < S$ for an appropriate choice of v, we will have $J < S$ and the proof of **Theorem 1** will follow. In order to choose such a v, we require a bit of machinery. Define $f_\varepsilon(r) = (\varepsilon^2 + r^2)^{-1}$ where $r^2 = (x-x_0)^2 + (y-y_0)^2$, and set

$$\varphi^\varepsilon(x, y) = f_\varepsilon(r)(x - x_0, y - y_0, \varepsilon)$$

in the basis $\{\bar{i},\ \bar{j},\ \bar{k}\}$. We fix a function $\xi \in C_c^\infty(\Omega; \mathbb{R})$ which satisfies $\xi \equiv 1$ near (x_0, y_0). For $\varepsilon > 0$ and such a ξ, define

$$v^\varepsilon = \xi\varphi^\varepsilon.$$

In **Lemma C.4** we perform the straightforward calculations to show

$$\varphi_x^\varepsilon = -2f_\varepsilon\varphi^\varepsilon(x-x_0) + f_\varepsilon\bar{i}, \qquad \varphi_y^\varepsilon = -2f_\varepsilon\varphi^\varepsilon(y-y_0) + f_\varepsilon\bar{j}, \qquad |\nabla\varphi^\varepsilon|^2 = 2f_\varepsilon^2 \tag{28}$$

Now for all $\varepsilon > 0$, f_ε is bounded above by $\frac{1}{r^2}$. This function has a finite integral on $\mathbb{R}^2\backslash\Omega$. Hence

$$\int_\Omega f_\varepsilon^2 = \int_{\mathbb{R}^2} f_\varepsilon^2 - \int_{\mathbb{R}^2\backslash\Omega} f_\varepsilon^2 = \int_{\mathbb{R}^2} f_\varepsilon^2 + O(1) = 2\pi\int_0^\infty \frac{r}{(\varepsilon^2+r^2)^2}dr + O(1) = \frac{\pi}{\varepsilon^2} + O(1) \tag{29}$$

as $\varepsilon \to 0$. Similarly

$$\int_\Omega f_\varepsilon^3 = \int_{\mathbb{R}^2} f_\varepsilon^3 + O(1) = 2\pi \int_0^\infty \frac{r}{(\varepsilon^2+r^2)^3} + O(1) = \frac{\pi}{2\varepsilon^4} + O(1). \tag{30}$$

Recalling $\xi = 1$ and hence $\nabla\xi = 0$ in a neighbourhood of (x_0, y_0), we get that $(\xi_x^2+\xi_y^2)|\varphi^\varepsilon|^2$, $(\xi^2 - 1)|\nabla\varphi^\varepsilon|^2$ and $2\xi\varphi^\varepsilon\cdot(\xi_y\varphi_y^\varepsilon + \xi_x\varphi_x^\varepsilon)$ are bounded for each $\varepsilon > 0$ on Ω. Then using (28) and (29);

$$\begin{aligned}\int_\Omega |\nabla v^\varepsilon|^2 = \int_\Omega |v_x^\varepsilon|^2 + |v_y^\varepsilon|^2 &= \int_\Omega (\xi_x^2+\xi_y^2)|\varphi^\varepsilon|^2 + 2\xi\varphi^\varepsilon\cdot(\xi_y\varphi_y^\varepsilon+\xi_x\varphi_x^\varepsilon) + \xi^2|\nabla\varphi^\varepsilon|^2 \\ &= \int_\Omega \xi^2|\nabla\varphi^\varepsilon|^2 + O(1) \\ &= \int_\Omega |\nabla\varphi^\varepsilon|^2 + \int_\Omega(\xi^2-1)|\nabla\varphi^\varepsilon|^2 + O(1) \\ &= \int_\Omega 2f_\varepsilon^2 + O(1) \\ &= \frac{2\pi}{\varepsilon^2} + O(1)\end{aligned} \tag{31}$$

as $\varepsilon \to 0$. In **Lemma C.4** we also show $v^\varepsilon\cdot(v_x^\varepsilon \times v_y^\varepsilon) = \varepsilon\xi^3 f_\varepsilon^3$. Along with (30), it follows that

$$Q(v^\varepsilon) = \int_\Omega v^\varepsilon\cdot v_x^\varepsilon\times v_y^\varepsilon = \int_\Omega \varepsilon\xi^3 f_\varepsilon^3 = \varepsilon\int_\Omega f_\varepsilon^3 + \varepsilon\int_\Omega(\xi^3-1)f_\varepsilon^3 = \frac{\pi}{2\varepsilon^3} + O(\varepsilon) = \frac{\pi}{2\varepsilon^3}(1+O(\varepsilon^4)) \tag{32}$$

as $\varepsilon \to 0$. In particular, we can use the extended binomial theorem

$$|Q(v^\varepsilon)|^{2/3} = \left(\frac{\pi}{2}\right)^{2/3}\frac{1}{\varepsilon^2}(1+O(\varepsilon^4)). \tag{33}$$

In view of **Lemma C.4**, we have

$$\int_\Omega \tilde{u}\cdot v_x^\varepsilon \times v_y^\varepsilon = (\mathbf{a}\cdot\bar{i} + \mathbf{b}\cdot\bar{j})\frac{\pi}{2\varepsilon} + O(|\log\varepsilon|).$$

The definition of J_{v^ε}, (31), and (33) let us compute

$$\begin{aligned} J_{v^\varepsilon} &= \frac{\int_\Omega|\nabla v^\varepsilon|^2 + 4H\int_\Omega \tilde{u}\cdot v_x^\varepsilon\times v_y^\varepsilon}{|Q(v^\varepsilon)|^{2/3}} \\ &= \frac{\frac{2\pi}{\varepsilon^2} + O(1) + 4H(\mathbf{a}\cdot\bar{i}+\mathbf{b}\cdot\bar{j})\frac{\pi}{2\varepsilon} + O(|\log\varepsilon|)}{(1/2\pi)^{2/3}\frac{1}{\varepsilon^2}(1+O(\varepsilon^4))} \\ &= \frac{1}{1+O(\varepsilon^4)}\Big(S + SH(\mathbf{a}\cdot\bar{i}+\mathbf{b}\cdot\bar{j})\varepsilon + \varepsilon^2(O(1)+O(|\log\varepsilon|))\Big)\end{aligned}$$

as $\varepsilon \to 0$. Keeping in mind that $\mathbf{a}\cdot\bar{i}+\mathbf{b}\cdot\bar{j} < 0$ by the choice of basis, we have

$$J_{v^\varepsilon} < S \qquad \text{for sufficiently small } \varepsilon > 0.$$

That is, there exists some $\varepsilon > 0$ such that $J_{v^\varepsilon} < S$. But by the definition of J, we have $J_{v^\varepsilon} \geq J$ so

$$J < S,$$

which proves **Theorem 1**.

6 Appendices

6.1 Appendix A

The following lemmata are essential for the proof of **Theorem 1**, and can be found in the appendix of Brézis and Coron [2] along with their proofs. Take Ω to be the open unit disk, and take all functions u, v, w to be mapping into $\mathbb{R}^3$ unless otherwise stated.

Lemma A.1 ([2], page 178) Assume $u, v \in H^1(\Omega; \mathbb{R})$ and let $\varphi \in W_0^{1,1}(\Omega)$ be a solution of

$$\begin{cases} -\Delta\varphi = u_x v_y - u_y v_x & \text{on } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $\varphi \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ and $\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^2} \leq C\|\nabla u\|_{L^1}\|\nabla v\|_{L^2}$.

Lemma A.2 ([2], page 181) Suppose $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$, and $w \in H^1(\Omega)$ *or* $u \in C^1(\overline{\Omega})$, and $v, w \in H^1(\Omega)$. If either $u \times v = 0$ on $\partial\Omega$ or $w = 0$ on $\partial\Omega$, then

$$\int_\Omega u \cdot [(v_x \times w_y) + (w_x \times v_y)] = \int_\Omega v \cdot [(u_x \times w_y) + (w_x \times u_y)].$$

Lemma A.3 ([2], page 182) Let $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ *or* $u \in C^1(\overline{\Omega})$ and $v \in H^1(\Omega)$. Suppose $u \times v = 0$ on $\partial\Omega$. Then

$$2\int u \cdot v_x \times v_y = \int_\Omega v \cdot (u_x \times v_y + v_x \times u_y).$$

Lemma A.4 ([2], page 184) Suppose $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and let $\{v^n\}$ be a sequence with $v^n \in H_0^1(\Omega)$ and $v^n \to v$ in $H_0^1(\Omega)$ weakly. Then

$$\int_\Omega u \cdot v_x^n \times v_y^n \to \int_\Omega u \cdot v_x \times v_y.$$

Lemma A.5 ([2], page 184) There is a unique continuous map $R : H_0^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ such that

$$R(u, v) = \int_\Omega u \cdot v_x \times v_y$$

for all $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and for all $v \in H^1(\Omega)$. Moreover, $|R(u, v)| \leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}^2$ for all $u \in H_0^1(\Omega)$ and $v \in H^1(\Omega)$.

Lemma A.6 ([2], page 186) For $Q(v) = R(v, v)$ with R defined above, we have for all $v, w \in H_0^1(\Omega)$

$$Q(v + w) = Q(v) + Q(w) + 3R(v, w) + 3R(w, v).$$

Lemma A.7 ([2], page 186) Assume $v \in H_0^1(\Omega)$ and let w^n be a sequence in $H_0^1(\Omega)$ such that $w^n \to 0$ weakly in $H_0^1(\Omega)$. Then $|Q(v + w^n) - Q(v) - Q(w^n)| \to 0$.

6.2 Appendix B

As usual, take Ω to be the open unit disk. The following lemmata are assumed knowledge in the paper [2], but we state them here for the convenience of the less experienced reader.

Lemma B.1 (Rellich Kondrachov, [1] page 285)
We have $H^1(\Omega) \subset\subset L^2(\Omega)$. That is, there exists a constant C such that $\|u\|_{L^2} \leq C\|u\|_{H^1}$ for all $u \in H^1(\Omega)$, and each bounded sequence in $H^1(\Omega)$ has a subsequence converging in $L^2(\Omega)$.

Lemma B.2 (Weak compactness, [4] page 639)
Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then passing to some subsequence, there holds $u_k \to u$ weakly in X for some $u \in X$. Moreover, $\|u\| \leq \liminf \|u_k\|$.

Lemma B.3 ([8] page 165)
Suppose $f^n \to f$ strongly in $L^2(\Omega)$ and $g^n \to g$ weakly in $L^2(\Omega)$. Then

$$\lim_{n\to\infty} \int_\Omega f_n \cdot g_n = \int_\Omega f \cdot g.$$

Lemma B.4 (Poincaré's Inequality for $H_0^1(\Omega)$, [9] page 266)
For a bounded open set Ω and $u \in H_0^1(\Omega)$, there holds $\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$ for some constant C depending only on Ω.

Lemma B.5 (Solving Laplace's equation with boundary data, [5] page 12)
Suppose $\Omega = \{(x, y) \,:\, x^2+y^2 < 1\}$ and $\gamma \in C(\partial\Omega; \mathbb{R})$. Then there exists some $f \in C^\infty(\Omega; \mathbb{R}) \cap C(\overline{\Omega}; \mathbb{R})$ solving the PDE

$$\begin{cases} \Delta f = 0 \text{ in } \Omega, \qquad f = \gamma \text{ on } \partial\Omega. \end{cases}$$

Lemma B.6 (Weak maximum principle, [6] page 179) For an elliptic operator L, let $u \in H^1$ satisfy $Lu \leq 0$ (≥ 0) in Ω. Then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ \qquad (\inf_\Omega u \geq \inf \partial\Omega u^-).$$

Finally, we will note some fundamental properties of the dot product and cross product that are used frequently throughout the paper.

Lemma B.7 (Properties of scalar triple product and cross product)
The cross product is antisymmetric and the scalar triple product has the even permutation property. That is, for $u, v, w : \Omega \to \mathbb{R}^3$, there holds

$$u \times v = -v \times u, \quad \text{and} \quad w \cdot u \times v = u \cdot v \times w = v \cdot w \times u = -u \cdot w \times v = -v \cdot u \times w.$$

6.3 Appendix C

Recall for $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $u \in K = \{u \in H^1(\Omega) \ : u = \gamma \text{ on } \partial\Omega,\ \|u\|_{L^\infty} \leq R'\}$ the functions

$$Q(v) = \int_\Omega v \cdot v_x \times v_y, \quad L(v) = \int_\Omega |\nabla v|^2 + 4H \int_\Omega \tilde{u} \cdot v_x \times v_y, \quad E(u) = \int_\Omega |\nabla u|^2 + \frac{4}{3} H \int_\Omega u \cdot u_x \times u_y.$$

Lemma C.1 The functional E is sequentially lower semicontinuous on K.

Proof In §5.1 we showed the lower bound

$$E(v) \geq \frac{1}{3} \int_\Omega |\nabla v|^2 \qquad v \in K. \tag{6}$$

In particular, we see $\{E(v) : v \in K\} \subset \mathbb{R}$ is bounded below by zero. Since K is nonempty, $\inf_{v \in K} E(v)$ exists. This allows us to take a minimising sequence $\{u^n\} \subset K$ with

$$\lim_{n\to\infty} E(u^n) = \inf_{v\in K} E(v) \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} E(u^n) < \infty.$$

Combining this with (6) and recalling $u^n \in K$, we get that $\{u^n\}$ is bounded in H^1. Passing to a subsequence as necessary, the Rellich Kondrachov theorem and weak compactness (see **Lemma B.1-B.2**) tell us that $u^n \to \tilde{u}$ strongly in L^2 and weakly in H^1 for some $\tilde{u} \in H^1$. Passing to a further subsequence, L^2 convergence implies $u^n \to \tilde{u}$ almost everywhere.

We will now show $\tilde{u} \in K$. Since the trace is a bounded linear operator on H^1, we have weak convergence of $u^n \to u$ H^1 implying weak convergence of $u^n \to u$ in the trace sense. Lower semicontinuity of the norm gives

$$\|T\tilde{u} - \gamma\|_{L^2(\partial\Omega)} \leq \liminf \|Tu^n - \gamma\|_{L^2(\partial\Omega)}.$$

The right hand side is zero since $u^n \in K$ for each n. It follows that $\tilde{u} = \gamma$ on $\partial\Omega$ in the trace sense. Moreover, $\tilde{u} \in H^1(\Omega; \mathbb{R}^3)$ by the definition of weak convergence, and the bound $\|\tilde{u}\|_{L^\infty} \leq R'$ is clear by taking limits on $\|u^n\|_{L^\infty} \leq R'$. Hence $\tilde{u} \in K$. Then for $\theta^n = u^n - \tilde{u}$, we get

$$\|\theta^n\|_{L^\infty} \leq \|u^n\|_{L^\infty} + \|\tilde{u}\|_{L^\infty} \leq 2R'.$$

The above modes of convergence for u^n then imply

$$\theta^n \to 0 \ \text{ weakly in } H^1, \text{ strongly in } L^2 \text{ and a.e.}$$

Next, we expand

$$\begin{aligned} \int_\Omega u^n \cdot (\theta^n + \tilde{u})_x \times (\theta^n + \tilde{u})_y &= \int_\Omega u^n \cdot \theta^n_x \times \theta^n_y + \int_\Omega u^n \cdot \theta^n_x \times \tilde{u}_y + \int_\Omega u^n \cdot \tilde{u}_x \times \theta^n_y + \int_\Omega u^n \cdot \tilde{u}_x \times \tilde{u}_y \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{34}$$

Note that

$$\|u^n \times \tilde{u}_y - \tilde{u} \times \tilde{u}_y\|_{L^2}^2 = \int_\Omega |(u^n - \tilde{u}) \times \tilde{u}_y|^2 \leq \int_\Omega |u^n - \tilde{u}|^2 |\tilde{u}_y|^2 \leq \|u^n - \tilde{u}\|_{L^\infty}^2 \|\tilde{u}_y\|_{L^2}^2 \to 0$$

since $u^n \to \tilde{u}$ almost everywhere. It follows that $u^n \times \tilde{u}_y \to \tilde{u} \times \tilde{u}_y$ strongly in L^2. This allows us to apply **Lemma B.7** followed by **Lemma B.3** to get

$$I_2 = \int_\Omega u^n \cdot \theta_x^n \times \tilde{u}_y = -\int_\Omega \theta_x^n \cdot u^n \times \tilde{u}_y \to 0.$$

Similarly,

$$I_3 = \int_\Omega u^n \cdot \tilde{u}_x \times \theta_y^n \to 0.$$

For I_4, we use dominated convergence

$$I_4 = \int_\Omega u^n \cdot \tilde{u}_x \times \tilde{u}_y \to \int_\Omega \tilde{u} \cdot \tilde{u}_x \times \tilde{u}_y.$$

Finally for I_1, an application of Young's inequality (c.f. (5)) allows us to estimate

$$\frac{4}{3}HI_1 = \frac{4}{3}H\int_\Omega u^n \cdot \theta_x^n \times \theta_y^n \leq \frac{4}{3}H\int_\Omega |u^n||\theta_x^n||\theta_y^n| \leq \frac{2}{3}H\|u^n\|_{L^\infty}\int_\Omega |\nabla\theta^n|^2 \leq \frac{2}{3}\int_\Omega |\nabla\theta^n|^2$$

since $\|u^n\|_{L^\infty} \leq R'$ and $HR' < 1$. Putting this all together in (34) and noting $u^n = \theta^n + \tilde{u}$, we can use the definition of E to compute

$$\begin{aligned}
E(u^n) &= \int_\Omega |\nabla(\theta^n + \tilde{u})|^2 + \frac{4}{3}H\int_\Omega u^n \cdot (\theta^n + \tilde{u})_x \times (\theta^n + \tilde{u})_y \\
&= \int_\Omega |\nabla\theta^n|^2 + \int_\Omega |\nabla\tilde{u}|^2 + 2\int_\Omega \nabla\theta^n \cdot \nabla\tilde{u} + \frac{4}{3}H\Big(\int_\Omega \tilde{u} \cdot \tilde{u}_x \times \tilde{u}_y + \int_\Omega u^n \cdot \theta_x^n \times \theta_y^n\Big) + o(1) \\
&= \Big(\int_\Omega |\nabla\tilde{u}|^2 + \frac{4}{3}H\int_\Omega \tilde{u} \cdot \tilde{u}_x \times \tilde{u}_y\Big) + \int_\Omega |\nabla\theta^n|^2 + \frac{4}{3}H\int_\Omega u^n \cdot \theta_x^n \times \theta_y^n + o(1) \\
&\geq E(\tilde{u}) + \frac{1}{3}\int_\Omega |\nabla\theta^n|^2 + o(1)
\end{aligned} \tag{35}$$

where we used $\nabla\theta^n \to 0$ weakly in L^2. In particular since $\inf\limits_{v\in K} E(v) = E(u^n) + o(1)$, there holds

$$\inf_{v\in K} E(v) \geq E(\tilde{u}) + \frac{1}{3}\int_\Omega |\nabla\theta^n|^2 + o(1). \tag{36}$$

Sending $n \to \infty$, this becomes

$$\inf_{v\in K} E(v) \geq E(\tilde{u}) + \frac{1}{3}\lim_{n\to\infty}\int_\Omega |\nabla\theta^n|^2 \geq \inf_{v\in K} E(v) + \frac{1}{3}\lim_{n\to\infty}\int_\Omega |\nabla\theta^n|^2$$

where the final inequality followed because $E(\tilde{u}) \geq \inf_{v\in K} E(v)$. Hence the above inequality can only hold if

$$\int_\Omega |\nabla u^n - \nabla\tilde{u}|^2 = \int_\Omega |\nabla\theta^n|^2 \to 0.$$

It follows that $u^n \to \tilde{u}$ strongly in H^1. The above limit and (36) also show that $\inf_{v\in K} E(v) \geq E(\tilde{u})$. That is, E is sequentially lower semicontinuous on K. ■

Lemma C.2 If $\tilde{u}$ minimises E on K, then $\|\tilde{u}\|_{L^\infty} \leq R$.

Proof Fix a nonnegative function $v \in C_c^\infty(\Omega)$. We saw in §5.1.3 that for $t > 0$ sufficiently small, there holds

$$E(\tilde{u}) - E((1-tv)\tilde{u}) \leq 0.$$

We wish to expand this. To do so, we note the calculations

$$|\nabla((1-tv)\tilde{u})|^2 = |\nabla\tilde{u}|^2 + t^2|\nabla(v\tilde{u})|^2 - 2t\nabla\tilde{u}\cdot\nabla(v\tilde{u})$$

and

$$\begin{aligned}(1-tv)\tilde{u}\cdot((1-tv)\tilde{u})_x \times ((1-tv)\tilde{u})_y &= (1-tv)\tilde{u}\cdot((1-tv)\tilde{u}_x - tv_x\tilde{u}) \times ((1-tv)\tilde{u}_y - tv_y\tilde{u}) \\ &= (1-tv)^3\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y \\ &= (1+t^3v^3+3t^2v^2-3tv)\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y.\end{aligned}$$

Then this and the definition of E gives

$$\begin{aligned}&0 \geq E(\tilde{u}) - E((1-tv)\tilde{u}) \\ &= \int_\Omega |\nabla\tilde{u}|^2 + \frac{4}{3}H\int_\Omega \tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y - \left(\int_\Omega |\nabla((1-tv)\tilde{u})|^2 + \frac{4}{3}H\int_\Omega (1-tv)\tilde{u}\cdot((1-tv)\tilde{u})_x\times((1-tv)\tilde{u})_y\right) \\ &= 2t\int_\Omega \nabla\tilde{u}\cdot\nabla(v\tilde{u}) - t^2\int_\Omega |\nabla(v\tilde{u})|^2 - \frac{4}{3}Ht^2\int_\Omega (tv^3+3v^2)\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y + 4Ht\int_\Omega v\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y.\end{aligned}$$

Dividing through by $2|t|$ and taking $t \to 0$, we can integrate by parts noting v is compactly supported to obtain

$$\begin{aligned}0 &\geq \int_\Omega \nabla\tilde{u}\cdot\nabla(v\tilde{u}) + 2H\int_\Omega v\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y \\ &= \int_\Omega v|\nabla\tilde{u}|^2 + \int_\Omega \nabla\tilde{u}\cdot(v_x\tilde{u},\ v_y\tilde{u}) + 2H\int_\Omega v\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y \\ &= \int_\Omega v|\nabla\tilde{u}|^2 - \int_\Omega v\nabla(\nabla u\cdot u)) + 2H\int_\Omega v\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y \\ &= \int_\Omega v(|\nabla\tilde{u}|^2 - \frac{1}{2}\Delta|\tilde{u}|^2 + 2H\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y).\end{aligned}$$

Noting $HR' < 1$, together with Young's inequality, the above bound allows us to estimate in the distributional sense

$$\begin{aligned}0 &\geq |\nabla\tilde{u}|^2 - \frac{1}{2}\Delta|\tilde{u}|^2 + 2H\tilde{u}\cdot\tilde{u}_x\times\tilde{u}_y \\ &\geq |\nabla\tilde{u}|^2 - \frac{1}{2}\Delta|\tilde{u}|^2 - 2H\|\tilde{u}\|_{L^\infty}|\tilde{u}_x||\tilde{u}_y| \\ &\geq |\nabla\tilde{u}|^2 - \frac{1}{2}\Delta|\tilde{u}|^2 - (|\tilde{u}_x|^2 + |\tilde{u}_y|^2) \\ &= -\frac{1}{2}\Delta|\tilde{u}|^2.\end{aligned}$$

In particular

$$-\Delta|\tilde{u}|^2 \leq 0.$$

Hence the weak maximum principle (**Lemma B.6**) tells us that

$$\sup_\Omega|\tilde{u}| = \sup_{\partial\Omega}|\tilde{u}| = R,$$

which implies $\|\tilde{u}\|_{L^\infty} \leq R < R'$. This is what we wanted to show. ■

Lemma C.3 For $v, w \in H_0^1(\Omega;\mathbb{R}^3)$, define

$$B(v,w) := \int_\Omega \nabla v \cdot \nabla w + 2H \int_\Omega \tilde{u} \cdot (v_x \times w_y + w_x \times v_y).$$

Then B is positive definite.

Proof As discussed in §5.2.2, B is a symmetric, bilinear, positive semidefinite form on H_0^1 with $B(v,v) = L(v)$. We want to show that B is in fact positive definite; let $\tilde{v} \in H_0^1(\Omega)$ be such that $B(\tilde{v},\tilde{v}) = 0$. We would like to conclude that $\tilde{v} = 0$. We start by showing $B(\tilde{v}, w) = 0$ for all $w \in H_0^1$. We proceed by contradiction:

Suppose there exists $w \in H_0^1$ such that $B(\tilde{v}, w) \neq 0$. Semidefiniteness implies $B(t\tilde{v}+w, t\tilde{v}+w) \geq 0$ for $t \in \mathbb{R}$. Using the assumption $B(\tilde{v},\tilde{v}) = 0$ along with the bilinearity and symmetry of B, we see

$$\begin{aligned} 0 \geq B(t\tilde{v}+w, t\tilde{v}+w) &= t^2 B(\tilde{v},\tilde{v}) + 2tB(\tilde{v},w) + B(w,w) \\ &= 2tB(\tilde{v},w) + B(w,w). \end{aligned}$$

By assumption $B(\tilde{v},w) \neq 0$, so one can always choose t to satisfy $2tB(\tilde{v},w) + B(w,w) < 0$. This contradicts the above inequality, and hence establishes that $B(\tilde{v},w) = 0$ for all $w \in H_0^1(\Omega)$.

Next, take $w \in C_c^\infty(\Omega)$. If we continuously extend w to $\overline{\Omega}$ so that $w = 0$ on $\partial\Omega$, then **Lemma A.2** allows us to compute

$$0 = B(\tilde{v}, w) = \int_\Omega \nabla\tilde{v} \cdot \nabla w + 2H \int_\Omega w \cdot (\tilde{u}_x \times \tilde{v}_y + \tilde{v}_x \times \tilde{u}_y).$$

It follows that $\tilde{v}$ is a weak solution to

$$\Delta\tilde{v} = 2H(\tilde{u}_x \times \tilde{v}_y + \tilde{v}_x \times \tilde{u}_y). \tag{37}$$

Then **Lemma A.1** allows us to conclude that $\tilde{v} \in L^\infty$, so $\tilde{u} + t\tilde{v} \in K$ for $|t|$ small enough. Hence we again get (c.f. (13) from §5.2.2)

$$t^2 B(\tilde{v},\tilde{v}) + \frac{4}{3}Ht^3 Q(\tilde{v}) = t^2 \int_\Omega |\nabla\tilde{v}|^2 + 4Ht^2 \int_\Omega \tilde{u} \cdot \tilde{v}_x \times \tilde{v}_y + \frac{4}{3}Ht^3 \int_\Omega \tilde{v} \cdot \tilde{v}_x \times \tilde{v}_y \geq 0$$

Since $B(\tilde{v},\tilde{v}) = 0$, the above can only hold if $\frac{4}{3}Ht^3 Q(\tilde{v}) \geq 0$. But t was of arbitrary sign, so

$$\int_\Omega \tilde{v} \cdot \tilde{v}_x \times \tilde{v}_y = 0.$$

This, (9), and $B(\tilde{v},\tilde{v}) = 0$ then give

$$\begin{aligned} E(\tilde{u}+t\tilde{v}) &= E(\tilde{u}) + E(t\tilde{v}) + 4Ht^2 \int_\Omega \tilde{u} \cdot \tilde{v}_x \times \tilde{v}_y \\ &= E(\tilde{u}) + t^2 \int_\Omega |\nabla\tilde{v}|^2 + \frac{4}{3}Ht^3 \int_\Omega \tilde{v} \cdot \tilde{v}_x \times \tilde{v}_y + 4Ht^2 \int_\Omega \tilde{u} \cdot \tilde{v}_x \times \tilde{v}_y \\ &= E(\tilde{u}). \end{aligned}$$

It follows that $\tilde{u} + t\tilde{v} \in K$ is a minimiser for E on K. Hence by the results from §5.1, $\tilde{u} + t\tilde{v}$ satisfies (1)

$$\Delta(\tilde{u}+t\tilde{v}) = 2H(\tilde{u}+t\tilde{v})_x \times (\tilde{u}+t\tilde{v})_y.$$

Expanding out the left hand side, noting $\tilde{u}$ solves (1) and recalling (37) gives

$$\begin{aligned}\Delta(\tilde{u}+t\tilde{v}) &= 2H(\tilde{u}_x\times\tilde{u}_y + t^2\tilde{v}_x\times\tilde{v}_y + t(\tilde{u}_x\times\tilde{v}_y+\tilde{v}_x\times\tilde{u}_y))\\ &= \Delta\tilde{u}+t\Delta\tilde{v}+2Ht^2\tilde{v}_x\times\tilde{v}_y\\ &= \Delta(\tilde{u}+t\tilde{v})+2Ht^2\tilde{v}_x\times\tilde{v}_y.\end{aligned}$$

This shows $\tilde{v}_x\times\tilde{v}_y = 0$. Recalling the definition of B, it is clear that $B(\tilde{v},\tilde{v}) = 0$ can only hold if $\int_\Omega |\nabla\tilde{v}|^2 = 0$. That is, $\nabla\tilde{v} = 0$ a.e. However the only constant function in $H^1_0(\Omega)$ is the zero function. It follows that $\tilde{v} = 0$. This shows B is positive definite, as required. ■

Lemma C.4 For the functions $f_\varepsilon, \varphi^\varepsilon, v^\varepsilon$ defined in 5.2.4, there holds

$$|\nabla\varphi^\varepsilon|^2 = 2f_\varepsilon^2 \qquad \text{and} \qquad v^\varepsilon\cdot v^\varepsilon_x\times v^\varepsilon_y = \varepsilon\xi^3 f_\varepsilon^3.$$

Proof We work in the basis $\{\bar{i},\bar{j},\bar{k}\}$ as defined in §5.2.4. Through a straightforward application of the Leibniz and chain rule, we compute

$$\begin{aligned}
&v^\varepsilon_x = \xi_x\varphi^\varepsilon+\xi\varphi^\varepsilon_x &&\varphi^\varepsilon_x = (f_\varepsilon)_x(x-x_0,y-y_0,\varepsilon)+f_\varepsilon\bar{i} &&(f_\varepsilon)_x = (f_\varepsilon)_r r_x\\
&v^\varepsilon_y = \xi_y\varphi^\varepsilon+\xi\varphi^\varepsilon_y &&\varphi^\varepsilon_y = (f_\varepsilon)_y(x-x_0,y-y_0,\varepsilon)+f_\varepsilon\bar{j} &&(f_\varepsilon)_y = (f_\varepsilon)_r r_y\\
&r_x = \frac{(x-x_0)}{r} &&r_y = \frac{(y-y_0)}{r} &&(f_\varepsilon)_r = -\frac{2r}{(\varepsilon^2+r^2)^2} = -2rf_\varepsilon^2.
\end{aligned}$$

The above expressions imply

$$\varphi^\varepsilon_x = -2f_\varepsilon\varphi^\varepsilon(x-x_0)+f_\varepsilon\bar{i}, \qquad \varphi^\varepsilon_y = -2f_\varepsilon\varphi^\varepsilon(y-y_0)+f_\varepsilon\bar{j}. \tag{38}$$

Now since $|\varphi^\varepsilon|^2 = f_\varepsilon$, we have

$$\begin{aligned}|\nabla\varphi^\varepsilon|^2 &= |\varphi_x|^2+|\varphi_y|^2\\ &= 4f_\varepsilon^2(\varphi^\varepsilon)^2((x-x_0)^2+(y-y_0)^2)+2f_\varepsilon^2-4f_\varepsilon^3((x-x_0)^2+(y-y_0)^2)\\ &= 2f_\varepsilon^2\end{aligned}$$

as desired. Next, (38) and the expressions for v^ε_x, v^ε_y, and φ^ε allow us to expand

$$\begin{aligned}v^\varepsilon\cdot(v^\varepsilon_x\times v^\varepsilon_y) &= \xi\varphi^\varepsilon\cdot(\xi_x\xi_y\varphi^\varepsilon\times\varphi^\varepsilon+\xi_x\xi\varphi^\varepsilon\times\varphi^\varepsilon_y+\xi_y\xi\varphi^\varepsilon_x\times\varphi^\varepsilon+\xi^2\varphi^\varepsilon_x\times\varphi^\varepsilon_y)\\ &= \xi^3\varphi^\varepsilon\cdot(\varphi^\varepsilon_x\times\varphi^\varepsilon_y)\\ &= \xi^3\varphi^\varepsilon\cdot(g_1(\varphi^\varepsilon\times\varphi^\varepsilon)+g_2(\varphi^\varepsilon\times\bar{j})+g_3(\bar{i}\times\varphi^\varepsilon)+f_\varepsilon^2(\bar{i}\times\bar{j}))\\ &= \xi^3\varphi^\varepsilon\cdot(f_\varepsilon^2\bar{k})\\ &= \varepsilon\xi^3 f_\varepsilon^3\end{aligned}$$

where the g_i are scalar functions. This is what we wanted to show. ■

Lemma C.5 For v^ε as defined in 5.2.4, there holds

$$\int_\Omega \tilde{u}\cdot v_x^\varepsilon \times v_y^\varepsilon = (\mathbf{a}\cdot\bar{i}+\mathbf{b}\cdot\bar{j})\frac{\pi}{2\varepsilon}+O(|\log\varepsilon|).$$

Proof In §5.2.4 we used **Lemma C.4** to show

$$\int_\Omega f_\varepsilon^2 = \frac{\pi}{\varepsilon^2}+O(1) \qquad \text{and} \qquad \int_\Omega |\nabla v^\varepsilon|^2 = \frac{2\pi}{\varepsilon^2}+O(1) \tag{39}$$

as $\varepsilon\to 0$. Now recalling $\mathbf{a}=\tilde{u}_x(x_0,y_0)$, $\mathbf{b}=\tilde{u}_y(x_0,y_0)$, we use the Taylor expansion of $\tilde{u}$ around (x_0,y_0) to express

$$\tilde{u}=\tilde{u}(x,y)=\tilde{u}(x_0,y_0)+\mathbf{a}(x-x_0)+\mathbf{b}(y-y_0)+O(r^2)$$

as $r\to 0$. Define

$$I=\int_\Omega \tilde{u}(x_0,y_0)\cdot v_x^\varepsilon\times v_y^\varepsilon \qquad II=\int_\Omega [\mathbf{a}(x-x_0)+\mathbf{b}(y-y_0)]\cdot v_x^\varepsilon\times v_y^\varepsilon \qquad III=\int_\Omega O(r^2)\cdot v_x^\varepsilon\times v_y^\varepsilon$$

so that

$$\int_\Omega \tilde{u}\cdot v_x^\varepsilon\times v_y^\varepsilon = I+II+III.$$

For each $\varepsilon>0$, v^ε is a smooth compactly supported function, so $v^\varepsilon\in H_0^1(\Omega)\cap L^\infty(\Omega)$ and $\tilde{u}\times v^\varepsilon=0$ on $\partial\Omega$. Moreover, $\tilde{u}(x_0,y_0)$ is constant so $\tilde{u}(x_0,y_0)\in H^1(\Omega)\cap L^\infty(\Omega)$. Hence **Lemma A.3** says

$$I=\int_\Omega \tilde{u}(x_0,y_0)\cdot v_x^\varepsilon\times v_y^\varepsilon=\frac{1}{2}\int_\Omega v^\varepsilon\cdot(0\times v_y^\varepsilon+v_x^\varepsilon\times 0)=0.$$

Since $(\mathbf{a}(x-x_0)+\mathbf{b}(y-y_0))_x=\mathbf{a}$ and $(\mathbf{a}(x-x_0)+\mathbf{b}(y-y_0))_y=\mathbf{b}$, we can use (38) and **Lemma A.3** to show

$$\begin{aligned} II &= \frac{1}{2}\int_\Omega v^\varepsilon\cdot(\mathbf{a}\times v_y^\varepsilon+v_x^\varepsilon\times\mathbf{b}) \\ &= \frac{1}{2}\int_\Omega \xi^2\varphi^\varepsilon\cdot(\mathbf{a}\times\varphi_y^\varepsilon+\varphi_x^\varepsilon\times\mathbf{b}) \\ &= \frac{1}{2}\int_\Omega \xi^2\varphi^\varepsilon\cdot(\mathbf{a}\times(-2f_\varepsilon(y-y_0)\varphi^\varepsilon+f^\varepsilon\bar{j})+(-2f_\varepsilon(x-x_0)\varphi^\varepsilon+f^\varepsilon\bar{i})\times\mathbf{b}) \\ &= \frac{1}{2}\int_\Omega \xi^2 f_\varepsilon\varphi^\varepsilon\cdot(\mathbf{a}\times\bar{j}+\bar{i}\times\mathbf{b}). \end{aligned}$$

Letting $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$, we note

$$\mathbf{a}\times\bar{j}=(-a_3,0,a_1) \qquad \bar{i}\times\mathbf{b}=(0,-b_3,b_2)$$

by the definition of $\bar{i},\bar{j}$ as basis vectors. Recalling $\varphi^\varepsilon=f_\varepsilon(x-x_0,y-y_0,\varepsilon)$ it follows that

$$II=\frac{1}{2}\int_\Omega \xi^2 f_\varepsilon^2(\ (a_1+b_2)\varepsilon-a_3(x-x_0)-b_3(y-y_0)\ ).$$

To treat the first term, we recall (39)

$$\frac{1}{2}\int_\Omega \xi^2 f_\varepsilon^2(a_1+b_2)\varepsilon=(a_1+b_2)\frac{\pi}{2\varepsilon}+O(\varepsilon).$$

For the remaining terms, suppose B is a small ball around (x_0, y_0) inside the neighbourhood where $\xi \equiv 1$, then

$$\frac{1}{2}\int_\Omega \xi^2\ f_\varepsilon^2(a_3(x-x_0)+b_3(y-y_0)) = \frac{1}{2}\int_B f_\varepsilon^2(a_3(x-x_0)+b_3(y-y_0)) + O(1).$$

Since f_ε^2 has radial symmetry about a line parallel to the $z-$axis centred at $(x_0,\ y_0)$, and because $(x-x_0)$ and $(y-y_0)$ are odd functions about the point (x_0, y_0), we get

$$\int_B f_\varepsilon^2(x-x_0) = 0 \qquad \text{and} \qquad \int_B f_\varepsilon^2(y-y_0) = 0.$$

It follows that

$$II = (a_1+b_2)\frac{\pi}{2\varepsilon} + O(1) = (\mathbf{a}\cdot\bar{i} + \mathbf{b}\cdot\bar{j})\frac{\pi}{2\varepsilon} + O(1).$$

Finally, we estimate III. Using (39) and keeping in mind $f_\varepsilon = f_\varepsilon^2 f_\varepsilon^{-1} \geq f_\varepsilon^2 r^2$, we get

$$\begin{aligned}
|III| \leq \int_\Omega |O(r^2)||v_x^\varepsilon||v_y^\varepsilon| &\leq C\int_\Omega r^2|\nabla v^\varepsilon|^2 \\
&\leq C\int_\Omega r^2 f_\varepsilon^2 + O(1) \\
&\leq C\int_\Omega f_\varepsilon + O(1) \\
&= C\int_0^{2\pi}\int_0^1 \frac{r}{r^2+\varepsilon^2}drd\theta = C(\pi\ln(1+\varepsilon^2) - \pi\ln(\varepsilon^2)) \\
&= O(|\log\varepsilon|)
\end{aligned}$$

as $\varepsilon \to 0$. It follows that

$$III = O(|\log\varepsilon|) \qquad \text{and hence} \qquad \int_\Omega \tilde{u}\cdot v_x^\varepsilon \times v_y^\varepsilon = (\mathbf{a}\cdot\bar{i} + \mathbf{b}\cdot\bar{j})\frac{\pi}{2\varepsilon} + O(|\log\varepsilon|).$$

■

Lemma C.6 For v^n, v^0 and w^n as defined in 5.2.3, there holds

$$|Q(v^0)|^{2/3} + |Q(w^n)|^{2/3} + o(1) \geq 1.$$

Proof For the minimising sequence $v^n \in H_0^1$, we have $Q(v^n) = 1$ and $L(v^n) \to J$. Now $L(v^n)$ is a convergent sequence in $\mathbb{R}$, and is hence bounded above by some constant $C < \infty$. Then using our bound on L from §**Lemma C.3**, it follows that for any n there holds

$$\int_\Omega |\nabla v^n|^2 \leq \frac{1}{\delta}L(v^n) \leq C < \infty.$$

In particular, ∇v^n is a bounded sequence in L^2, and Poincaré's inequality (**Lemma B.4**) implies v^n is bounded in H_0^1. Weak compactness (see **Lemma B.2**) then implies $v^n \to v^0$ weakly in H_0^1 for some $v^0 \in H_0^1$ when passing to an appropriate subsequence. This gives $w^n = v^n - v^0 \to 0$ weakly in H_0^1 as well. We have $Q(v^0 + w^n) = Q(v^n) = 1$. The hypotheses of **Lemma A.7** are hence satisfied, so

$$|Q(v^0 + w^n) - Q(v^0) - Q(w)| = |1 - Q(v^0) - Q(w^n)| \to 0.$$

In particular,

$$Q(v^0) + Q(w^n) = 1 + o(1)$$

Using this with the extended binomial formula shows

$$1 + o(1) = (1 + o(1))^{2/3} \leq |Q(v^0) + Q(w^n)|^{2/3} \leq |Q(v^0)|^{2/3} + |Q(w^n)|^{2/3}$$

and in particular

$$1 \leq |Q(v^0)|^{2/3} + |Q(w^n)|^{2/3} + o(1).$$

■

Lemma C.7 Let v^0 be such that $J = L(v^0)$ and suppose $w \in H_0^1 \cap L^\infty$ is fixed. For $t \in \mathbb{R}$, and $\mu = |Q(v^0 + tw)|^{1/3}$, it follows that

$$\left.\frac{d}{dt}\frac{1}{\mu^2}\right|_{t=0} = -2R(w, v^0).$$

Proof Define $\varphi(t) = \frac{1}{\mu}(v^0 + tw)$ and note

$$Q(\varphi(t)) = \frac{1}{\mu^3} Q(v^0 + tw) = \frac{Q(v^0 + tw)}{|Q(v^0 + tw)|} = \pm 1.$$

The continuity of Q ensures $Q(v^0 + tw) \to 1$ as $t \to 0$, so $Q(\varphi) = 1$ for small enough t. Furthermore, by **Lemma A.6**, we have

$$\begin{aligned}
\mu^3 = |Q(v^0 + tw)| &= |Q(v^0) + t^3 Q(w) + 3t^2 R(v^0, w) + 3tR(w, v^0)| \\
&= |1 + 3tR(w, v^0) + O(t^2)| \\
&= 1 + 3tR(w, v^0) + O(t^2)
\end{aligned}$$

as $t \to 0$, since $Q(v^0 + tw) > 0$ for sufficiently small t. It follows that

$$\frac{1}{\mu^2} = (\mu^3)^{-2/3} = (1 + 3tR(w, v^0) + O(t^2))^{-2/3}$$

so

$$\frac{d}{dt}\frac{1}{\mu^2} = -2R(w, v^0)(1 + 3tR(w, v^0) + O(t^2))^{-5/3}.$$

It follows that $\left.\frac{d}{dt}\frac{1}{\mu^2}\right|_{t=0} = -2R(w, v^0)$. ■

7 Acknowledgements

I would like to thank my supervisor Chris van der Heide for all of the help he has given me and time he has put aside for me over the past few months. I am also grateful to Timothy Buttsworth for helpful discussions and advice. Finally I would like to thank the Australian Mathematical Sciences Institute and the Department of Education and Training for this opportunity.

8 References

[1] Brézis, H. (2011) – *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* New York: Springer.

[2] Brézis, H., & Coron, J.M. (1984) – *Multiple solutions of H-systems and Rellich's conjecture.* Comm. Pure Appl. Math. 37.

[3] Do Carmo, M.P., (2016) – *Differential Geometry Of Curves And Surfaces*, 2nd Edition. New York: Dover.

[4] Evans, L. C. (1998) – *Partial Differential Equations.* Providence, R.I: American Mathematical Society.

[5] Giaquinta, M., & Martinazzi, L. (2012) – *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, 2nd edition. Pisa: Edizioni della Normale.

[6] Gilbarg, D., & Trudinger, N.S. (2001) – *Elliptic Partial Differential Equations.* Berlin: Springer-Verlag.

[7] Pressley, A.N. (2010) – *Elementary Differential Geometry,* 2nd Edition. London: Springer London, Limited.

[8] Royden, H., & Fitzpatrick, P. (2010) – *Real analysis*, 4th Edition. N/A: Pearson Education.

[9] Strüwe, M. (2008) – *Variational Methods.* Berlin: Springer.

[10] Wente, H. (1969) – *An existrence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26.