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# Why the increasing surface area law for black holes is an open problem


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**Abstract**

This work was completed in the summer of 2018/19 as part of an AMSI vacation research scholarship supervised by Ben Whale and Adam Rennie. The aim of the project was to study a paper by Domenico Giulini and come to an understanding of the open question he poses: is there a general area theorem for black holes? We introduce the necessary differential geometry for a proof of the area theorem for a black hole that has a $C^2$ surface. We compute the surface area of the event horizon in a Schwarzschild metric, and determine the evolution of a spherical "event horizon" in a Minkowski metric. After giving the proof for a $C^2$ event horizon, we consider why the $C^2$ assumption rules out many realistic black hole behaviours.

## 1 Introduction

This report develops the required mathematical material needed to understand the question: is there a general area theorem for black holes? The area theorem for black holes states that the surface area of a black hole can never decrease over time (Giulini, 1998). The surface of a black hole is the event horizon. It has been proven that for a $C^2$ event horizon that the surface area will never decrease. This proof comes down to the positive divergence of the curves that describe the movement of each point on the event horizon. We will go through this proof in Section 7 of the report. Prior to the proof of the area theorem for a $C^2$ surface we will introduce the mathematics involved in the proof with examples along the way. Section 2 introduces the mathematics used in example calculations and the proof of the area theorem for a $C^2$ event horizon. Section 3 introduces the Schwarzschild spacetime. Section 4 gives the definition of the event horizon and Section 5 calculates the surface area of the event horizon in a Schwarzschild metric. Section 6 of the report gives an example calculation of the evolution of the surface of a pretend "spherical event horizon" in Minkowski spacetime. The example will calculate the curves that describe the motion of each point on the event horizon. The last section of this report, Section 8, demonstrates the shortcomings of a $C^2$ event horizon in physical systems and introduces the case of event horizons that are $C^{1-}$ i.e. Lipschitz. The open question is: does the area theorem apply to a $C^{1-}$ event horizon?

This report looks at previously proved theorems, none of the final results are an outcome of my own work.

*Einstein's summation convention** is used throughout the report.

# 2 Background

## 2.1 Manifolds and submanifolds

We will now define manifolds and submanifolds. We will only introduce smooth (differentiable) manifolds since all our applications in this report deal with smooth manifolds. A smooth manifold is a space that locally appears like Euclidean space (Darling, 1994, p.98). Most importantly smooth manifolds have a structure which allows us to use calculus.

**Definition 2.1** (Smooth manifold)**.** For sets $M$ and $U \subseteq M$, a chart $(U, \psi)$ is a one-to-one map onto an open set $\psi(U) \subseteq \mathbb{R}^n$. If we have two charts $(U, \psi)$ and $(V, \phi)$ they are called compatible if

- $U \cap V = \emptyset$ or
- $U \cap V \neq \emptyset$, $\psi(U \cap V)$ and $\phi(U \cap V)$ are open, and $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism (a differentiable bijection with differentiable inverse)

A collection of compatible charts that has domain $M$ is called a smooth atlas. If the union of two atlases is another atlas then the two atlases are called equivalent i.e. the charts in each atlas are compatible to one another. A smooth differentiable structure on $M$ is an equivalence class of atlases on $M$. A set $M$ with a smooth differentiable structure is a smooth (differentible) manifold of dimension $n$.

**Definition 2.2** (Immersions and submersions)**.** Where $W$ and $U$ are open subsets, immersions and submersions are defined in Table 1.

| | Immersion | Submersion |
|---|---|---|
| Domain and Range | $\Phi : W \subseteq \mathbb{R}^n \to \mathbb{R}^{n+k}$ | $f : U \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^k$ |
| Defining property | $d\Phi(u)$ is 1-1 $\forall w \in W$ | $df(x)$ is onto $\forall u \in U$ |
| Derivative matrix is | $(n + k) \times k$ | $k \times (n + k)$ |
| ... whose rank should be | rank $= n$ | rank $= k$ |
| ... or in other words | columns are independent | rows are independent |

Table 1: Immersions and submersions (Darling, 1994, p.55)

**Definition 2.3** (Submanifold)**.** A submanifold is an $n$-dimensional manifold contained within a $(n+k)$-dimensional manifold. The result of an immersion or a submersion is a submanifold. We can define submanifolds in terms of a submersion. $M \subset \mathbb{R}^{n+k}$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+k}$ if, for each $x \in M$ there is a neighbourhood U of $x$ in $\mathbb{R}^{n+k}$ and a submersion $f : U \to \mathbb{R}^k$ such that

$$U \cap M = f^{-1}(\{0\}) \tag{1}$$

For a submersion $f$, the kernal of $df(x)$ is $n$-dimensional.

**Example.** Sphere
The sphere is a set $M \subset \mathbb{R}^3$. We define the function $f : \mathbb{R}^3 \to \mathbb{R}$ as $f(x, y, z) = x^2 + y^2 + z^2 - r^2$ where $r \in \mathbb{R}^+$ is the radius of the sphere. Taking the derivatives of $f$ with respect to $x$, $y$ and $z$ we get the derivative matrix,

$$\begin{bmatrix} 2x, & 2y, & 2z \end{bmatrix}$$

The only point $v = (x, y, z)$ where the derivative matrix is zero is $(0, 0, 0)$. The inverse image of $f$ at 0 is $f^{-1}(\{0\}) = \{(x, y, z) : x^2 + y^2 + z^2 - r^2 = 0\} = M$. For any open neighbourhood $U$ of $(x, y, z) \in M$ not containing $(0, 0, 0)$ the restriction $f|_U$ is a submersion and $U \cap M = f|_U^{-1}(\{0\})$.

All the manifolds and submanifolds considered in this report are orientable (Darling, 1994, p.165), so integration will be defined over all our manifolds (Darling, 1994, p.174).

### 2.2 Geodesics

To understand the evolution of the event horizon we need to understand geodesics. Geodesics are the paths in an $n$-dimensional submanifold where the tangential acceleration is zero. Geodesics with length zero are called null geodesics. Each point on the event horizon moves along null geodesics, giving the total evolution of the event horizon. Geodesics in spacetime depend on curvature and the metric tensor. We will introduce the metric tensor before deriving the geodesic equations.

#### 2.2.1 The metric tensor

The metric of a submanifold is a way to keep track of the measure of distances at each point. The metric accounts for any stretching or compressing of volumes compared to Euclidean space. A nice example of this is the way a rectangular map of the world enlarges countries near the north pole, Greenland being greatly enlarged.
The metric, denoted $g$, is an inner product on tangent vectors. The metric takes in two vectors and calculates the length squared $g(v, v)$ or the scalar product $g(u, v)$ for two different vectors. The

symmetry of $g$ arises from symmetry of the inner product. For a parameterised submanifold of Euclidean space $\Psi : W \subset \mathbb{R}^n \to U \subset \mathbb{R}^{n+k}$, the metric tensor is a symmetric $(0,2)$ tensor that can be presented as a matrix with elements

$$g_{ij} = \frac{\partial \Psi}{\partial x^i} \cdot \frac{\partial \Psi}{\partial x^j} = \partial_i \Psi \cdot \partial_j \Psi. \tag{2}$$

For a curve $\gamma : [0,1] \to M$ to have length zero its tangent vector $\gamma'$ must satisfy $g(\gamma', \gamma') = 0$. The possibility for a curve to have length zero is evident in the following example.

**Example.** Scalar product

The metric for Minkowski space $-dt^2 + dx^2 + dy^2 + dz^2$ can be written in the coordinates $(u, v, \phi, \theta)$ which has the metric $g = -dudv + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$, where $r = r(u, v)$. The coordinates $(u, v, \phi, \theta)$ are the result of the spherical coordinate change for $(x, y, z) \to (r, \phi, \theta)$ and then a coordinate change from $(r, t) \to (u, v)$ given by $u = t - r$ and $v = t + r$. The inner product of two vectors $x = (x_u, x_v, x_\theta, x_\phi)$ and $y = (y_u, y_v, y_\theta, y_\phi)$ is given by

$$g(x, y) = \sum_{i,j} x_i y_j g_{ij} = x_u y_v (-1) + x_v y_u (-1) + x_\theta y_\theta (r^2) + x_\phi y_\phi \left(r^2 \sin^2 \theta\right).$$

For the matrix representation this can be written as,

$$g(x, y) = \begin{pmatrix} x_u & x_v & x_\theta & x_\phi \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} y_u \\ y_v \\ y_\theta \\ y_\phi \end{pmatrix}$$

For the inner product $g(x, y)$ we can see that if $x = y$ was the tangent vector to a curve then the curve would have length zero if

$$-2x_u x_v + r^2 x_\theta^2 + \left(r^2 \sin^2 \theta\right) x_\phi^2 = 0.$$

The metric tensor is invertible and the elements of the inverse metric tensor are denoted $g^{ij}$. Since $gg^{-1} = \mathrm{Id}$ we find the components satisfy

$$g_{ij} g^{jm} = \delta_i^m.$$

### 2.2.2 The geodesic equation

To derive the geodesic equation we will consider a parametrised path $\gamma : [0,1] \to M$ mapped into a submanifold $M \subset N$. This is demonstrated visually for a 2-dimensional submanifold of $\mathbb{R}^3$ in figure 1.

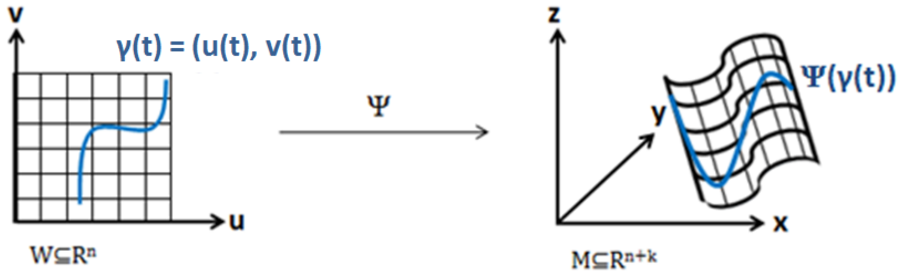


Figure 1: 2-dimensional submanifold

We will begin with the lower dimensional case to build towards the generalised $n$-dimensional equation for a geodesic. Referring to Figure 1, where $\Psi : W \to M \subset N$, the curve on the surface in $\mathbb{R}^3$ is given by

$$\Psi(\gamma(t)) = \Psi(u(t), v(t)) = (x(u(t), v(t)),\ y(u(t), v(t)),\ z(u(t), v(t))) = R(t).$$

As defined, a geodesic has zero tangential acceleration. So we need to find the equation for the acceleration of our curve on the surface. Taking the first and second derivative of the curve $t \mapsto R(t)$ with respect to $t$

$$\frac{dR}{dt} = \frac{\partial R}{\partial u}\frac{du}{dt} + \frac{\partial R}{\partial v}\frac{dv}{dt}$$

$$\frac{d^2R}{dt^2} = \left(\frac{\partial^2 R}{\partial u^2}\frac{du}{dt} + \frac{\partial^2 R}{\partial u \partial v}\frac{dv}{dt}\right)\frac{du}{dt} + \frac{\partial R}{\partial u}\frac{d^2u}{dt^2} + \left(\frac{\partial^2 R}{\partial u \partial v}\frac{du}{dt} + \frac{\partial^2 R}{\partial v^2}\frac{dv}{dt}\right)\frac{dv}{dt} + \frac{\partial R}{\partial v}\frac{d^2v}{dt^2}$$

For an $n$-dimensional parameterisation, $\{u^1, \dots, u^n\}$, the equation for the second derivative will be given by

$$\frac{d^2R}{dt^2} = \sum_i \frac{d^2u^i}{dt^2}\frac{\partial R}{\partial u^i} + \sum_i \sum_j \frac{du^i}{dt}\frac{du^j}{dt}\frac{\partial^2 R}{\partial u^i \partial u^j} \tag{3}$$

The last term in this equation, $\frac{\partial^2 R}{\partial u^i \partial u^j}$, can be expressed as a linear combination of tangent vectors of the ambient manifold $N$. The partial derivatives of our map $\Psi$ span the tangent plane of $M \subset N$. Thus our basis of the tangent space to $N$ is composed of tangent vectors to $M$ and normal components. The coefficients of our tangential basis vectors, $\Gamma^k_{ij}$, are called the Christoffel symbols.

Where $\bar{n}$ is the normal component of $\frac{\partial^2 R}{\partial u^i \partial u^j}$ given by the projection of $\frac{\partial^2 R}{\partial u^i \partial u^j}$ onto the normal bundle, $\bar{n} = \frac{\partial^2 R}{\partial u^i \partial u^j} - \sum_k \Gamma^k_{ij}\frac{\partial R}{\partial u^k}$, we have

$$\frac{\partial^2 R}{\partial u^i \partial u^j} = \sum_k \Gamma^k_{ij}\frac{\partial R}{\partial u^k} + \bar{n} \tag{4}$$

Also note that since $\bar{n}$ is normal to the tangent space of $M \subset N$, $TM$, the inner product of $\bar{n}$ and $\frac{\partial R}{\partial u^l}$ where $\frac{\partial R}{\partial u^l} \in TM$ will be zero. So taking the inner product of both sides by an element of the tangent space of $M \subset N$, $\frac{\partial R}{\partial u^l}$, we have

$$\frac{\partial^2 R}{\partial u^i \partial u^j} \cdot \frac{\partial R}{\partial u^l} = \sum_k \Gamma^k_{ij} \frac{\partial R}{\partial u^k} \cdot \frac{\partial R}{\partial u^l} \quad or \quad \frac{\partial^2 R}{\partial u^i \partial u^j} \cdot \frac{\partial R}{\partial u^l} = \sum_k \Gamma^k_{ij} g_{kl}$$

With the metric inverse $g^{lm}$ such that $g_{kl} g^{lm} = \delta^m_k$, we multiply both sides by $g^{lm}$,

$$\sum_k \Gamma^k_{ij} \delta^m_k = \frac{\partial^2 R}{\partial u^i \partial u^j} \cdot \frac{\partial R}{\partial u^l} g^{lm} \quad or \quad \Gamma^m_{ij} = \frac{\partial^2 R}{\partial u^i \partial u^j} \cdot \frac{\partial R}{\partial u^l} g^{lm}$$

The Christoffel symbols can also be given in terms of the metric $g$ as

$$\Gamma^m_{ij} = \frac{1}{2} g^{lm} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Now that we have broken down the second order differential term we substitute Equation (4) back into the expression for acceleration found in Equation (3).

$$\frac{d^2 R}{dt^2} = \sum_i \left( \frac{d^2 u^i}{dt^2} \frac{\partial Rv}{\partial u^i} \right) + \sum_i \sum_j \left( \frac{du^i}{dt} \frac{du^j}{dt} \sum_k \left( \Gamma^k_{ij} \frac{\partial R}{\partial u^k} + \bar{n} \right) \right)$$

We now wish to separate the tangential and normal acceleration components. We will change the first summation to be over $k$ and expand the second term. We will use Einstein's summation convention to simplify the equation.

$$\frac{d^2 R}{dt^2} = \frac{d^2 u^k}{dt^2} \frac{\partial Rv}{\partial u^k} + \frac{du^i}{dt} \frac{du^j}{dt} \Gamma^k_{ij} \frac{\partial R}{\partial u^k} + \frac{du^i}{dt} \frac{du^j}{dt} \bar{n}$$

Pairing the tangential and normal components we have,

$$\frac{d^2 R}{dt^2} = \left( \frac{d^2 u^k}{dt^2} + \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right) \frac{\partial R}{\partial u^k} + \frac{du^i}{dt} \frac{du^j}{dt} \bar{n}$$

The geodesic equation is given by setting the tangential acceleration to zero,

$$\frac{d^2 u^k}{dt^2} + \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0. \tag{5}$$

### 2.3 Raychaudhuri equation

The event horizon of a black hole is a surface that flows along a velocity vector field. Taking the gradient of the velocity vector field results in a (0,2) tensor. The gradient, as a (0,2) tensor, can be separated into three components; the trace, a symmetric trace free component and an anti-symmetric component. These three components of the gradient of a velocity vector field are called the expansion ($\theta_{ab}$), shear ($\sigma_{ab}$) and vorticity ($\omega_{ab}$) respectively. Physically in a 3-dimensional space, expansion refers to the stretching on a small ball that causes a change in volume, shear refers to the stretching on a small ball that causes the ball to become elliptical, and vorticity refers to the rotation of the small ball. In $n$-dimensions the gradient of the velocity vector field is decomposed as,

$$\nabla_b v_a = \sigma_{ab} + \omega_{ab} + \frac{1}{n-1} h_{ab}\theta,$$

where the metric $h_{ab}$ is the projection of $g_{ab}$ to the normal space of the curve, and

$$\sigma_{ab} = \frac{1}{2}\left(\nabla_b v_a + \nabla_a v_b\right) - \frac{1}{n-1} h_{ab}\theta, \quad \omega_{ab} = \frac{1}{2}\left(\nabla_b v_a - \nabla_a v_b\right), \quad \theta = \nabla_a v^a.$$

Defining $B_{ab} = \nabla_b v_a$ then evaluating $v^c \nabla_c B^a{}_a$ will derive the Raychaudhuri equation (Wald, 1984, p.217-218)

$$\dot{\theta} = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{ab}v^a v^b \tag{6}$$

Note that when $v_a$ is normal to a surface $\nabla_b v_a$ is the negative of the second fundamental form of the surface. We denote this below by $L_v$. In this case $\theta = -\operatorname{tr}(L_v)$.

## 3 The original black hole

We have now introduced the tools we will need to use to understand the evolution of the surface area of a black hole. Before considering the evolution of the black hole surface through time we will look at an example calculation of the surface area of the event horizon at $t = 0$. The example we will consider is the Schwarzschild black hole.

The Schwarzschild black hole has no rotation or charge. It is the simplest form of a black hole; isolated and stationary. The Schwarzschild spacetime when first discovered was presented in coordinates, $r \in (0, 2m) \cup (2m, \infty)$, $t \in \mathbb{R}$, $\theta \in (0, \pi)$, $\phi \in (-\pi, \pi)$, is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{7}$$

Note that for points with $r > 2m$ the $r$ coordinate is radial and the $t$ coordinate is timelike. In contrast for points with $r < 2m$ the $t$ coordinate is radial and the $r$ coordinate is timelike.

The radius of the Schwarzschild black hole, with centre $r = 0$, is $r_s = 2m$. Notice that at the values $r = r_s$ and $r = 0$ the coordinate components of the $dt^2$ and $dr^2$ terms of the metric are either 0 or $\infty$. Hence the metric is not defined on the $r = 0$ point and the $r = 2m$ coordinate surface. This behaviour at the $r = 2m$ surface, the event horizon, is caused by a poor choice of coordinates. This behaviour can be removed with an appropriate coordinate transformation (Hawking and Ellis, 1973, p.149).

Ignoring the spherical part of the metric and considering the equation that a null vector must satisfy we derive the equation

$$-\left(1-\frac{2m}{r}\right)dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2=0,$$

which implies that

$$\left(\frac{dt}{dr}\right)^2=\left(1-\frac{2m}{r}\right)^{-2}.$$

This shows that the $t$ and $r$ components of a curve with null tangent vector must satisfy

$$t=\pm(r+2m\log(r-2m)+k),\quad k\in\mathbb{R}.$$

Now let

$$r^*=r+2m\log(r-2m)$$

and define $v=t+r^*$. Since the level surfaces of $v$ are null, the argument above implies that the function $v$ is well defined on the $r=2m$ surface. Then in the $v,r,\theta,\phi$ coordinates the metric is given by

$$ds^2=-\left(1-\frac{2m}{r}\right)dv^2+2dvdr+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right).$$

The components of this coordinate representation of the metric are non-degenerate at the $r=2m$ surface. This demonstrates that the divergent behaviour was due to a poor choice of coordinates.
In order to understand the Schwarzschild metric new coordinates where found, (Hawking and Ellis, 1973, p.155). These represented the metric in a conformally compactified form, which made it easy to represent the global structure of the manifold as a two dimensional diagram, see Figure 2. Such diagrams are call Penrose diagrams and are an excellent tool for understand black holes. A typical Penrose diagram will have surfaces, represented as $\mathscr{I}^+$ in Figure 2, through which every complete null geodesic must travel. This surface is called future null infinity. The degenerate behaviour at the $r=0$ surface is a not a coordinate dependent singularity and cannot be removed. This can be seen by calculating the Ricci scalar and taking the limit of it as $r$ goes to 0. Specifically

$$R=\frac{48m^2}{r^6}\to\infty,\qquad \text{as}\quad r\to 0.$$

This allows us to interpret the $r=0$ surface as a gravitational singularity.

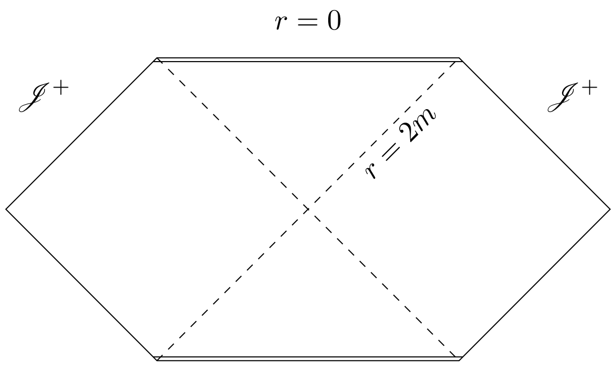


Figure 2: A diagrammatic representation of conformally compactified Schwarzschild spacetime with the $\theta$ and $\phi$ coordinate suppressed. The symbol $\mathscr{I}^+$ is the label for the future null infinity of the manifold. This is the surface that every complete null geodesic must intersect. The double lines show the final singularity, a black hole, and the initial singularity, a "white" hole, both of which are on the $r = 0$ surface. The dashed lines, on the upper half of the diagram, give the event horizon of the black and the dashed lines on the lower half give the even horizon of the white holes. Such a diagram is usually called a Penrose diagram, see (Hawking and Ellis, 1973, p.154).

## 4 The definition of an event horizon

The Penrose diagram, Figure 2, visually demonstrates that a light cone with $r$ coordinate less than $2m$ contains a subset of the $r = 0$ surface. All future timelike curves from such a point must hit the $r = 0$ surface. Hence once someone has crossed the $r = 2m$ surface they can never leave. Because of this the $r = 2m$ surface is call the event horizon of the singularity at $r = 0$. The set of points for which $r < 2m$ is called the black hole.

This definition is specific to the Schwarzschild spacetime. The generalisation of the definition has only been agreed for a type of spacetime for which a conformal compactification, similar to that described in Figure 2 exists.

**Definition 4.1** (Event horizon)**.** If $M$ is a spacetime with a conformal compactification, and therefore there exists a well-defined future null infinity $\mathscr{I}^+$, then the event horizon is the boundary $\partial\big(M \setminus I^-(\mathscr{I}^+)\big)$.

Let $F = I^+\Big(\partial\big(M \setminus I^-(\mathscr{I}^+)\big)\Big)$. This is called a future set as $F = I^+(F)$. It is known that the boundary of a future set is Lipschitz, $C^{1-}$, (Hawking and Ellis, 1973, Proposition 6.3.1). There is also an example of an event horizon which is $C^{1-}$ (Chruściel and Galloway, 1998). **Therefore we can only assume that event horizons are Lipschitz submanifolds.**

We defined a submanifold as the set of points that satisfy an equation, Definition 2.3. In contrast the definition of an event horizon gives the set of points directly. This makes it hard to perform calculations on event horizons. In the case of the Schwarzschild spacetime, however, we know that the event horizon is the $r = 2m$ surface.

Consider a point on an event horizon, $p \in \partial\big(M \setminus I^-(\mathscr{I}^+)\big)$. By definition, any timelike curve starting from $p$ does not reach $\mathscr{I}^+$, and so enters the black hole. Since timelike futures of points $p$ are open this implies that the null curves starting from $p$ must either enter the black hole or lie on its surface. The boundary $\partial\big(M \setminus I^-(\mathscr{I}^+)\big)$ is a subset of $\partial F$ which is an achronal boundary. This implies that at least one null curve from $p$ lies on the event horizon. This gives a method to calculate the change in area of an event horizon even if it is impossible to calculate the area directly as there is no defining equation for the event horizon.

It is important to note that null curves can always be reparameterised as null geodesics. This follows directly from the definition of a null curve.

## 5 Calculating the area of the event horizon of Schwarzschild spacetime

In the particular case of the Schwarzschild spacetime we know that that the event horizon is given as the $r = 2m$ surface. This surface is 3-dimensional, thus to calculate an area we must fix a particular time at which to do the calculation. We need coordinates that cover the $r = 2m$ surface to do the calculation so we shall use $(v, r, \theta, \phi)$ coordinates. Thus fixing a time is equivalent to picking a particular value of $v$. The immersion $f : (0, \pi) \times (-\pi, \pi) \to M$ that defines the intersection of the $r = 2m$ and $v = v_0 \in \mathbb{R}$ surface with the event horizon is given by

$$f(\theta, \phi) = (v_0, 2m, \theta, \phi).$$

The pullback of the metric is

$$4m^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Let $S = (0, \pi) \times (-\pi, \pi)$, and let $dA$ be the area form on $f(S)$ and let $g$ be the Schwarzschild metric. Then the area integral over the submanifold is given by,

$$\int_{f(S)} dA = \int_S f^*(dA) = \int_{-\pi}^{\pi}\int_0^{\pi} \sqrt{\det f^* g}\, d\theta d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} (2m)^2 \sin\theta\, d\theta d\phi = 4\pi(2m)^2.$$

## 6 An example of the calculation of the change in surface area of an event horizon in time

Now that we have shown an example of the calculation of the surface area of a black hole we will consider how the surface area changes through time. As proven at the end of Section 4 the surface of a black hole follows null geodesics, the paths of light. So it is possible to calculate the change in area by considering how congruences of null geodesic behave. Below we give an example of the required calculations by doing the calculation explicitly and in terms of a null geodesic congruence.
Our manifold $M$ will be four dimensional Minkowski with the flat metric, $g$, is

$$-dt^2 + dx^2 + dy^2 + dz^2,$$

given in standard coordinates on $\mathbb{R}^4$. Let $S = (0, x, y, z) : x^2 + y^2 + z^2 = 1$ be the sphere at time zero, which has area $4\pi$. The sphere at time zero is parameterised by $\Psi : (0, \pi) \times (-\pi, \pi)$,

$$\Psi(u, v) = (0, \cos(u) \sin(v), \sin(u) \sin(v), \cos(v)) \tag{8}$$

We may pretend that $S$ is the surface of an event horizon at time 0. In this case $S$ will evolve along one of the two congruences of normal null geodesics, $n$, from the surface. The following must be satisfied for a normal null geodesic:

- $g(n, s) = 0$, where s is a vector in the tangent space of the surface
- $g(n, n) = 0$.

The tangent space to $S$ is spanned $\Psi_*(\partial_u)$ and $\Psi_*(\partial_v)$. Direct calculation shows that

$$\Psi_*(\partial_u) = (0, -\sin u \sin v, \cos u \sin v, 0)$$
$$\Psi_*(\partial_v) = (0, \cos u \cos v, \sin u \cos v, -\sin v)$$

Thus if $n = (n_0, n_1, n_2, n_3)$ is normal null vector the equations that need to be satisfied are

$$g(\Psi_*\partial_u, n) = 0, \quad g(\Psi_*\partial_v, n) = 0, \quad g(n, n) = 0.$$

These conditions give three simultaneous equations. This leaves one free parameter when finding the null normal. Hence we can solve for $n_0, n_1, n_3$ in terms of $n_2$. It is convenient to set $(n_2)^2 = \sin^2 u \sin^2 v$ in which case there are two solutions

$$n = (1, \cos u \sin v, \sin u \sin v, \cos v)$$
$$n = (1, -\cos u \sin v, -\sin u \sin v, -\cos v)$$

Hence the null normal geodesics, $\gamma$, with tangent $n$ are found to be

$$\gamma(\tau) = (0, \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)) + \tau(1, \cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$
$$\gamma(\tau) = (0, \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)) - \tau(1, \cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

We want the surface to evolve along these null normal geodesics. Hence a parameterisation of the surface at time $\tau$ is $\Psi(u, v) = (t, (1 \pm t)\cos u \sin v, (1 \pm t)\sin u \sin v, (1 \pm t)\cos v)$. Performing the calculation of area, just as shown for Schwarzschild, shows that the area of the ball at time $t$ is $A(t) = 4\pi(1 \pm t)^2$. So $\frac{dA}{dt} = \pm 8\pi(1 \pm t)$.

Note that for one choice of the null normal geodesics the area increases and the other it decreases. For an abstract event horizon we will also have two null normal geodesic congruences, but only one will define the evolution. The correct choice is the "outward" point null normal. What it means to be "inward" pointing is difficult to define.

Recall that the definition of the black hole is $M \setminus I^-(\mathscr{I}^+)$. Our intuition says that one of the two null normal geodesic congruences will enter this set. This congruence is the inward point congruence. This definition, however, assumes that the surface is $C^2$ (Giulini, 1998). In the case of merging black holes there can exist points on the event horizon for which both null normal congruences enter the black hole (Chruściel and Galloway, 1998). Since the result we have been working towards assumes that the event horizon is $C^2$ we ignore this technical issue. This is a serious problem. We are assuming that the surface is $C^2$ when we know that it is only $C^{1-}$.

## 7 Area theorem for black holes

The area theorem for black holes states that the surface area of the event horizon can only increase. This can be proven for an event horizon that is $C^2$ in a spacetime that satisfies the weak energy condition. The weak energy condition places a restriction on the stress-energy-momentum tensor, $T_{\mu\nu}$, ensuring that a non-negative energy density will be observed. Assuming the weak energy condition holds, we can show that the divergence of the null normal congruence of the event horizon (called the generators of the event horizon) is positive. We will show that positive divergence implies increase of the area of the event horizon.

To show that the divergence of the generators is strictly positive we introduce the weak energy condition and apply it to the Raychaudhuri equation, Equation (6). The weak energy condition states that for a non-spacelike vector $v^a$, $T_{ab}v^a v^b \geq 0$. Einstein's field equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4}T_{ab}.$$

Hence if these equations hold then $T_{ab}v^a v^b \geq 0 \Leftrightarrow R_{ab}v^a v^b \geq 0$ (Hawking and Ellis, 1973, p.95). Recalling the Raychaudhuri equation, Equation (6), if the vorticity is zero $\omega = 0$ then as $\sigma^2 \geq 0$ we have that

$$\frac{d}{dt}\theta \leq -R(v,v) - \frac{1}{3}\theta^2.$$

Hence if the weak energy condition holds

$$\frac{d}{dt}\theta \leq -\frac{1}{3}\theta^2. \tag{9}$$

Thus, if $\theta < 0$ at some point on some null normal geodesic then $\theta \to -\infty$ in finite time. This can be seen by solving the deferential inequality (Wald, 1984, Equation 9.2.24). This will give a contradiction which implies that $\theta \geq 0$ on each null normal geodesic.

Before continuing on to show the positive divergence of the generators, we will introduce three properties of event horizons (Hawking and Ellis, 1973):

- *Achronicity property.* No two points of the event horizon can be connected by a timelike curve.
- The null geodesic generators of the event horizon may have joined the event horizon in the past
- The generators of the event horizon may never leave the horizon in the future

We now prove the claimed contradiction. Assume that the $\theta < 0$. Then Equation (9) draws the conclusion that $\dot{\theta} \to -\infty$ which means a caustic (intersection) of generators will occur at a point $p$ in the future. With at least two generators, $g_1$ and $g_2$, involved in the intersection it can be shown that a timelike curve can connect points $r$ and $s$ on the generators (Figure 3) (Wald, 1984, Theorem 9.3.3). This contradicts the achronicity property of en event horizon. Thus $\theta \geq 0$ as claimed.

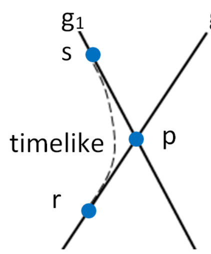


Figure 3: Timelike curve connecting surface generators

Let $U \subset \mathbb{R}^2$, $V \subset \mathbb{R}$ and $\phi : U \times V \to M$ be a parametrisation of the event horizon. We assume that for fixed $t \in V$ $\phi(U,t)$ is a spacelike 2-surface. For convenience define $\phi_t(U) = \phi(U,t)$. Let the time dependent area be

$$A(t) = \int_U \sqrt{\det{(\phi_t)^*\, g}}dx,$$

where $dx$ is the standard Lebesgue measure on $\mathbb{R}^2$. We claim that

$$\frac{d}{dt}A(t) = \int_U \frac{d}{dt}\sqrt{\det{(\phi_t)^* g}} dx = \int_U \theta\sqrt{\det{(\phi_t)^* g}} dx.$$

Hence as $\theta \geq 0$, $\frac{d}{dt}A(t) \geq 0$ which proves the area theorem.

We now prove this claim. Since the surface $\phi(U,t)$ is assumed to be $C^2$ the surface area of the event horizon flows along a $C^1$ velocity vector field, $n$, described by the null geodesics normal to the event horizon surface. For a small change in time the metric $(\phi_t)^* g$ is a $C^1$ of $t$ on $U$. Thus

$$\frac{d}{dt}\int_U \sqrt{\det{(\phi_t)^* g}} dx = \int_U \frac{d}{dt}\sqrt{\det{(\phi_t)^* g}} dx. \tag{10}$$

Applying Jacobi's formula (product rule) on $\frac{d}{dt}\det{(\phi_t)^* g}$ we get

$$\frac{d}{dt}\det{(\phi_t)^* g} = \det{(\phi_t)^* g}\,\mathrm{tr}\left(\frac{1}{(\phi_t)^* g}\frac{d}{dt}(\phi_t)^* g\right). \tag{11}$$

Letting $h = \phi^* g(t)$ we analyse the term $\frac{d}{dt}(\phi_t)^* g = \frac{d}{dt}h$. The derivative of the $ij$ component of $h$ is

$$\frac{d}{dt}h(\partial_i, \partial_j) = h(\nabla_{\gamma'}\partial_i, \partial_j) + h(\partial_i, \nabla_{\gamma'}\partial_j). \tag{12}$$

where $\gamma'$ is the vector field of the time derivative of the normal null geodesics paths.

We are free to assume that $[\partial_i, \gamma'] = 0$ so that $\nabla_{\gamma'}\partial_i = \nabla_{\partial_i}\gamma'$. Letting $L_{\gamma'}$ be the pullback of the second fundamental form of of $\phi_t(U)$ we get that

$$\begin{aligned}\frac{d}{dt}h(\partial_i, \partial_j) &= -h(L_{\gamma'}(\partial_i), \partial_j) - h(\partial_i, L_{\gamma'}(\partial_j))\\ &= -h(L_{\gamma'}(\partial_i), \partial_j) - h(L_{\gamma'}(\partial_j), \partial_i)\\ &= -2h(L_{\gamma'}(\partial_i).\partial_j)\end{aligned}$$

Taking $h^{-1}$ and then the trace on both sides gives

$$\mathrm{tr}\left(h^{-1}\frac{d}{dt}h\right) = -2\,\mathrm{tr}\left(L_{\gamma'}\right) = 2\theta. \tag{13}$$

Substituting this result into Equation 10 gives

$$\frac{d}{dt}A(t) = 2\int_U \theta\sqrt{\det{(\phi_t)^* g}} dU.$$

Since $\theta \geq 0$ we see that

$$\frac{d}{dt}A(t) \geq 0.$$

Thus the area theorem for $C^2$ black holes has been proven.

## 8 Assumptions in the area theorem proof

The proof of the area theorem given in Section 7 of the report contained two assumptions; the weak energy condition and a $C^2$ event horizon. The weak energy condition is a valid assumption when describing physical systems as it removes the possibility for an observer to measure a negative energy density. The $C^2$ assumption on the other hand is very limiting on physical applications. Most event horizons are not $C^2$ due to the physical processes they undertake; for example, merging with other black holes, eating stars and rotating. The Kerr model of black holes accounts for the rotation of the black hole. Cusps are located at its poles (similar to an apple) due to its rotation. More generally the event horizon is distorted. In an even more generalised construction of the black hole (Chruściel and Galloway, 1998), the event horizon contains an infinite number of cusps. This causes problems with taking the covariant derivative of the normal and hence with defining the second fundamental form.

## 9 Conclusion

The area theorem for black holes is an open question. The area theorem has been proven for a $C^2$ event horizon however not for a $C^{1-}$ event horizon. The class of $C^2$ event horizons restricts most practical behaviours of black holes, whereas $C^{1-}$ event horizons is inclusive of all the classical types of black holes. To prove the area theorem for an event horizon that is $C^{1-}$, we lose the ability to use the divergence of the surface generators. This makes the question extremely difficult and leads to the need for a alternative approaches.

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