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Generalised Polygons and their Symmetries

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Abstract

The generalised polygons are a particularly interesting type of point-line incidence geometry, whose symmetry groups are strongly linked to the finite simple groups of Lie type. Working towards an understanding of all of the generalised polygons whose symmetry groups are finite simple groups, one is led to try to classify all of the point-primitive generalised polygons. In this report, we discuss some basic facts about the polygons and the attempt to classify the point-primitive examples, and then present a new method for determining whether a group G can act point-primitively on some generalised quadrangle, and show its utility in ruling out previously open cases.

1 Introduction

Generalised polygons are a type of point-line incidence geometry; they are finite configurations of points and lines obeying a collection of axioms asserting the existence and nonexistence of certain types of cycles within the geometry.

These objects turn out to be extremely rare and highly symmetric. Their symmetry groups are very interesting objects to study, especially if some symmetry conditions are enforced. They are often strongly tied to the finite simple groups of Lie-type.

The focus of this report is on the attempt to understand all of the generalised polygons whose symmetry groups are point-primitive. The basic method of progress is to use the theory of group actions and permutations to understand all of the primitive permutation groups, and then to attempt to restrict which of these can be a group of symmetries of some generalised polygon.

After recounting the basic theory required, we then describe a new method of progress of our own devising: a computational method for determining whether specific groups have any generalised quadrangles which they can act on point-primitively. We use it to partially rule out the case of the almost simple sporadic groups.

I would like to thank Associate Professor John Bamberg and Emilio Pierro for their guidance and effort spent supervising this project, and also the Australian Mathematical Sciences Institute for giving me the opportunity to complete the Vacation Research Scholarship.

2 Generalised Polygons

The following introduction to generalised polygons is largely borrowed from Van Maldeghem 1998. More detail may be found there.

2.1 Incidence Geometry

An *incidence geometry* is a triple $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. Here, $\mathcal{P}$ and $\mathcal{L}$ are arbitrary sets whose elements are called "points" and "lines" respectively and $\mathcal{I}$ is an incidence relation. That is $\mathcal{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. If $(x, y) \in \mathcal{I}$ then we say that the elements x and y (one of which will be a point, the other a line) are incident. This is denoted $x \text{ I } y$. It is assumed that $\mathcal{I}$ is symmetric: if $(x, y) \in \mathcal{I}$ then (y, x) is also.

We will also need the following definitions:

Definition 2.1 (Collinear)**.** Points p and q are collinear, denoted $p \sim q$, if some line is incident with both. Usually we require $p \neq q$, so a point is not collinear with itself.

Definition 2.2 (Subgeometry)**.** $\mathcal{G}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a subgeometry of $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, denoted $\mathcal{G} \leq \mathcal{G}'$, if $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$ and $(p, L) \in \mathcal{I}'$ whenever $p \in \mathcal{P}'$, $L \in \mathcal{L}'$ and $(p, L) \in \mathcal{I}$.

Definition 2.3 (Symmetry, Symmetry group)**.** A symmetry of an incidence geometry is a pair (ϕ, ψ), where $\phi : \mathcal{P} \to \mathcal{P}$ and $\psi : \mathcal{L} \to \mathcal{L}$ are permutations which obey: $(p, L) \in \mathcal{I} \iff (\phi(p), \psi(L) \in \mathcal{I}$. That is, a symmetry is a pair of shuffles on the points and the lines which preserves incidence. Usually, the geometry is such that a line is uniquely specified by the set of points incident with it, and so the symmetry is entirely specified by the permutation on the point set.

The symmetry group is then the set of all symmetries of the geometry, together with the operation of function composition.

2.2 The axioms of Generalised Polygons

A (finite) *ordinary n-gon* is an incidence geometry of the following form: $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$, $\mathcal{L} = \{L_1, L_2, \ldots, L_n\}$, $(p_i, L_j) \in \mathcal{I} \iff i \equiv j \text{ or } j + 1 \mod n$. In words, it is n points and lines, each point on two lines, each line containing two points, arranged to form a closed chain.

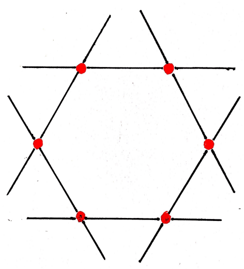

Figure 1: An ordinary hexagon or 6-gon

We can get an ordinary n-gon for each $n \geq 2$ and collectively these are called the ordinary polygons.

Similarly, we can get a generalised n-gon for each $n \geq 2$, and collectively these are the generalised polygons.

A *generalised n-gon* is an incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ obeying the following axioms:

1. For all k<n, there is no ordinary k-gon as a subgeometry.
2. Any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are contained in some ordinary n-gon (as a subgeometry).
3. Thickness: Every point is incident with ≥ 3 lines, every line is incident with ≥ 3 points.

For example consider a generalised 6-gon (or hexagon). Axiom 1 implies that there are no ordinary triangles, squares or pentagons in the geometry. Axiom 2 implies that there are many ordinary hexagons everywhere. The thickness condition ensures non-degeneracy and eliminates trivial examples.

It is an exercise to verify that these axioms hold for the following, which is a generalised hexagon.

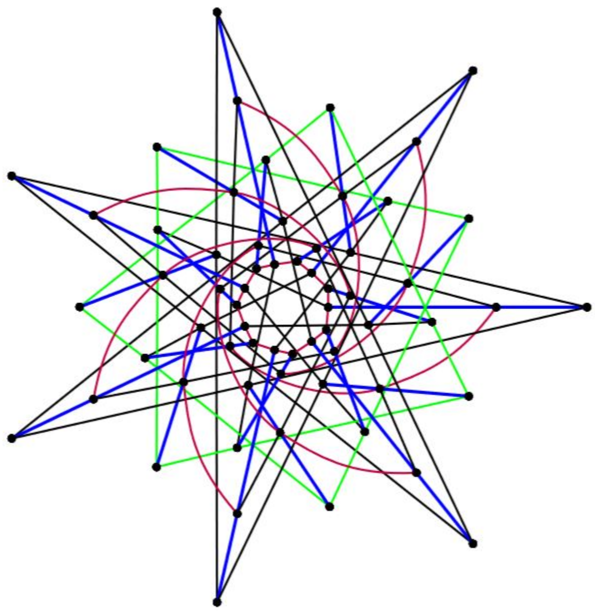

Figure 2: The Split Cayley Hexagon, H(2)

2.3 The basic properties

From now on, we assume that the point and line sets of all polygons are finite.

Despite their apparent simplicity, the axioms are extremely powerful, as evidenced by the following theorems. Proofs of the following results may be found in Van Maldeghem 1998.

Theorem 2.1 (Feit and Higman 1964)**.** *Finite generalised n-gons exist only for n = 2,3,4,6,8.*

This theorem is the foundational result for the study of generalised polygons. It is perhaps the best indication of just how strong the restrictions on the possible structures of the generalised polygons are, and how rare they will be.

The 2-gons are uninteresting: they consist of any number of points and lines, with every point lying on every line and vice versa. They are ignored from here on out. Generalised 3-gons are exactly the projective planes. The fact that n is odd and small makes the n=3 case very different from the others, and so it is mostly neglected in this report.

The polygons are forced by the axioms to be locally regular and symmetric:

Theorem 2.2 (Existence of an order)**.** *Every generalised polygon has an order: a pair of number* $s, t \geq 2$ *such that every line is incident with* $s+1$ *points and each point is incident with* $t+1$ *lines.*

That is, the axioms force there to be the same number of points on every line and the same number of lines on every point.

There are also severe restrictions on the possible values of s and t. We state these for quadrangles only. Similar conditions hold in all of the cases, but only the quadrangle case is need in this report. The full statement for all cases is contained in Appendix A.

Theorem 2.3. *Let* $\mathcal{G}$ *be some finite generalised quadrangle with order* (s, t)*. Then:*

- $\frac{st(st+1)}{s+t}$ *is an integer,*
- $s \leq t^2$, $t \leq s^2$,
- $|\mathcal{P}| = (s+1)(st+1)$, $|\mathcal{L}| = (t+1)(st+1)$.

The message to takeaway from these results is that the generalised quadrangles (and polygons in general) must be highly regular and structured, but that the possible structures are tightly constrained. The numerical restrictions on the number of points and lines in terms of s and t alone guarantees that they will be quite rare.

3 The Classical Polygons and their Symmetries

In 1959, Jacques Tits wrote a paper (Tits 1959) discussing the problem of finding geometric models for the finite simple groups of Lie type (a brief summary of information about these is contained in appendix B). One of the concepts he developed was that of the generalised polygons.

After Tits came up with his definition for the generalised polygons, he and his contemporaries found numerous examples whose symmetry groups were finite simple groups. These examples came to be known as the classical polygons.

The classical polygons consists of several infinite collections of polygons. Each collection consists entirely of n-gons for the same n, and is associated to a family of finite simple groups which are the symmetry groups of the members of that collection.

The following lists the families of finite simple groups associated to some collection of generalised n-gons:

- $n = 3 : PSL(3, q)$
- $n = 4$: $PSp(4, q)$, $PSU(4, q)$, $PSU(5, q)$
- $n = 6$: $G_2(q)$, ${}^3D_4(q)$
- $n = 8$: ${}^2F_4(q)$

It is not known for certain if the classical polygons are the only examples of generalised polygons whose symmetry groups are finite simple groups (which we shall call classical-like polygons). It is strongly suspected that they are, and no others have ever been found. The importance of the finite simple groups, and the fact that they are the reason generalised polygons were invented, means that there is a strong desire amongst mathematicians to completely classify and understand all of the examples of classical-like polygons.

These efforts are aided by the fact that the classical polygons are extremely symmetric. They have huge numbers of symmetries compared to their sizes, and their symmetry groups obey some very strong conditions. In particular, all of the classical polygons are point-primitive, line-primitive, flag-transitive and distance-transitive.

Briefly, point (resp. line) primitive means that the point (resp. line) set cannot be partitioned into sets which are not mixed together by some element of the symmetry group. A flag is an incident point-line pair, and flag-transitive means that the symmetry group has an element mapping any such pair to any other. Finally, distance-transitive means that the symmetry group can take any pair of elements at some distance d from each other to any other such pair (for some suitable definition of distance in a geometry).

These are all extremely strong symmetry conditions, showing that in various ways, the classical polygons look pretty much the same in any direction when observing from any viewpoint.

The desire to classify all of the classical-like polygons has lead mathematicians to attempt to classify all of the polygons which obeys some or all of the above symmetry conditions. The hope is that understanding those is easier, but that they are similar enough to tell us something about the classical-like polygons, allow us to find more of them or rule out their existence.

Much work has already been done, and the current focus of research is on the classification of all point-primitive generalised polygons.

4 Classifying Point-Primitive Generalised Polygons

The strategy employed in the classification of point-primitive generalised polygons is one which is used throughout the area of finite geometry. In general it is used to understand all examples of some type of geometry obeying a particular symmetry condition. The way it works is rather unusual. Instead of constructing the geometries and then figuring out their symmetry groups, it instead goes the other way. It starts by asking what the possible symmetry groups are, and then only later asks of which geometries are they the symmetry groups.

The method works like this. You start with some class, $\mathcal{X}$, of incidence geometries, and some set of conditions on the symmetry groups of these geometries. Then you:

1. Understand all of the permutation groups obeying the symmetry conditions.
2. Restrict which of those groups can possibly be the symmetry groups of geometries of class $\mathcal{X}$.
3. Use the restrictions on the allowable symmetry groups to understand the geometries of class $\mathcal{X}$ obeying the symmetry conditions.

This is the method applied to the classification of point-primitive generalised polygons.

Prior results in the areas of group theory and group actions have already accomplished the first step: understanding all of the primitive permutation groups. This background on the concept of primitivity is covered in the next section of this report.

Most of the work on this problem so far for generalised polygons centres on step two: restricting which primitive groups can be symmetries of polygons. Section 6 of this report contains a brief summary of the most recent results, and also a brief explanation of why point-primitivity is the current focus.

So far, there has been little work done on step three. Section 7 discusses our contribution to it: a computational method for finding all of the generalised quadrangles which an arbitrary finite group G acts on point-primitively.

5 Primitive permutation groups

In the following, basic knowledge of group theory and group actions is assumed. A brief introduction to this material can be found in Dixon and Mortimer 1996. That reference also contains a more detailed introduction to the concept of primitivity than the one found here.

Definition 5.1 (Block System)**.** Suppose G acts transitively on Ω. A block system for this action is a partition $\Sigma = \{\Delta_1, \dots, \Delta_n\}$ such that:

- All of the Δ_i have the same cardinality.
- For each i, and for all $g \in G$, $\Delta_i^g \cap \Delta_i = \Delta_i$ or $\varnothing$.

(Any set $\Delta \subseteq \Omega$ such that $\Delta^g \cap \Delta = \Delta$ or $\varnothing$ for all $g \in G$ is called a block for the action).

A block system is a division of Ω into equal sized sets, such that the action of G preserves this division. Each block either maps to itself or entirely disjoint from itself. The action only shuffles around the blocks and/or mixes around the points within the blocks, but does not mix them together.

All actions have two trivial block systems: the set of all of the singletons and $\{\Omega\}$. The existence of a non-trivial block system demonstrates that the action does not thoroughly mix the points, by separating them into sets which are not mixed together by any permutation induced by the action.

A group action/permutation group is then *primitive* if it "thoroughly mixes" the points of Ω:

Definition 5.2 (Primitive group action)**.** Suppose G acts transitively on Ω. Then G is called primitive if the action does not have any non-trivial block systems.

A generalised polygon is called *point-primitive* if there is some group acting on the polygon (as a symmetry group), which when we consider the induced action on the points is a primitive group. For future reference, if it is said that a group acts on an incidence geometry, it is implicitly assumed that it does so as a group of symmetries.

The following theorem characterises primitive actions in terms of group theoretic properties:

Theorem 5.1. *Let G be a group which acts transitively on a set Ω (which has at least two points). Then G is primitive $\iff$ each point stabiliser G_ω is a maximal subgroup of G.*

Combined with the fundamental theorem of group actions (explained in Dixon and Mortimer 1996), this result allows us to construct all primitive actions of a group: apply the F.T.G.A. to one member of each conjugacy class of maximal subgroups. Each conjugacy class allows us to construct a primitive action of G, and these account for all of G's primitive actions, up to some notion of equivalence.

However, this method would be impractical for understanding in general what all of the primitive actions/permutation groups look like. Fortunately, we have the following theorem to give us that information.

Theorem 5.2. *O'Nan-Scott Theorem: Suppose G is a primitive group of permutations on some finite set Ω (so G is a finite group). Then G is one of the following types:*

- *Holomorph Affine, Holomorph Simple, Holomorph Compound, Twisted Wreath, Product Action, Simple Diagonal, Compound Diagonal, Almost Simple*

The above theorem is one of the most important in permutation group theory. The unparalleled understanding of the primitive permutation groups that it provides is the entire reason that the method discussed in Section 4 of this report works at all. It single-handedly completes most of the first step.

We don't need it in its full detail and power here. The only relevant detail for this report is that it splits the finite primitive groups into eight different types, and gives us a lot of information about these types: their group structure, numerical limitations on their sizes, etc. These facts are the fundamental basis for all of the results discussed in the next section. Combined with some basic knowledge of the generalised polygons, they are used to rule out various O'Nan-Scott types as being able to act point-primitively on any generalised polygon.

We comment briefly on the Almost Simple type. This type turns out to be the most important for the study of generalised polygons. In this type, G is what is called an almost simple group, which means that $T \leq G \leq \mathrm{Aut}(T)$ for some non-abelian finite simple group T (T can be identified with the subgroup of inner automorphisms of $\mathrm{Aut}(T)$, so this makes sense).

6 Results so far

With this background in place, we can now discuss the progress so far on this problem. The initiating result for this whole program is found in Buekenhout and Van Maldeghem 1994.

Theorem 6.1. *A finite distance-transitive generalised polygon is a classical polygon, with two exceptions.*

So distance-transitivity (almost) characterises the classical polygons. The main goal with all of this, as discussed in the above cited paper, is to see how weak the symmetry conditions can be made and still get (mostly only) the classical or classical-like polygons. This would capture the widest possible net of polygons while still being reasonably likely that the polygons found would have finite simple groups as their symmetries. There would then be a chance to find more classical-like polygons, or to conclusively rule them out. Point-primitivity is currently the limit of these efforts.

The distance-transitive case having been completed, the next effort made was to classify all of the polygons obeying the other three conditions: point-primitive, line-primitive and flag-transitive. It is strongly suspected that flag-transitivity also (with a few exceptions) characterises the classical polygons. In this vein we get these two results:

Theorem 6.2. *Schneider and Van Maldeghem 2008: Suppose a group G acts point-primitively, line -primitively and flag-transitively on some generalised hexagon or octagon. Then G is almost simple of Lie type.*

Theorem 6.3. *Bamberg, Giudici, et al. 2012: Suppose G acts point and line-primitively on some generalised quadrangle. Then G is almost simple. Moreover, if G is also flag-transitive, then G is almost simple of Lie type.*

Recall that the classical polygons, having simple (and sometimes almost simple) symmetry groups and being point-primitive, fall into the almost simple O'Nan-Scott type. Moreover, all of their symmetry groups are of Lie type.

What these theorems tell us then, is that if a polygon has sufficient amounts of symmetry, then the symmetry group is forced to be almost simple, and thus the polygon must be classical-like.

This sort of result is exactly what we want. It restricts the possibilities to one O'Nan-Scott type, makes the polygons classical-like and limits which simple groups can be involved. It is also extremely useful. If we were checking for new polygons obeying these conditions (or trying to show none exist), these results tell us that the symmetry groups must be almost simple of Lie type, and so these are the only cases we would have to check.

At this point however, the efforts run out of steam and little progress has been made on ruling out particular classes of Lie type groups (except in very special cases, such as Morgan and Popiel 2016), and so narrowing the search further seems out of reach for now.

All known flag-transitive polygons are also point-primitive or line-primitive. Since point-primitive and line-primitive are essentially the same condition (using the point-line duality present in generalised polygons), it would seem to suffice, or at least be useful, to understand all of the point-primitive polygons. In this direction, these are the two most up to date results, as of the writing of this report:

Theorem 6.4. *Bamberg, Glasby, et al. 2017: Suppose a group G acts point-primitively on some generalised hexagon or octagon. Then G is almost simple of Lie type.*

Theorem 6.5. *Bamberg, Popiel, and Praeger 2018+: If G acts point primitively on a generalised quadrangle, and G is not of Holomorph Affine type, then G is not Holomorph Compound type, and there are some other restrictions for the remaining O'Nan-Scott types.*

The first result is essentially the same as the previous two, but only assumes point-primitivity. It is exactly what we are after: same result but weaker conditions.

The result for quadrangles is far weaker. Only one O'Nan-Scott type is ruled out, and the others have a bunch of condition on them, but other than that, very little is known. It is not expected that these sort of methods can do much better, since we know of examples of point-primitive quadrangles whose symmetry groups do not have almost simple type.

It is in the context of this incomplete analysis of the point primitive quadrangle case that we make our own contribution, discussed in the next section.

7 An algorithm to determine whether specific groups act point-primitively on any generalised quadrangle

7.1 The overall idea

This section presents an overview of an algorithm which, given a group G, finds all generalised quadrangles with G as a point-primitive symmetry group. Since we only need to run it on the groups not yet ruled out, this accomplishes step three of the strategy mentioned in Section 4: turning restrictions on the symmetry groups into information about the quadrangles. It can also rule out G, if the algorithm finds no quadrangles for G.

The algorithm searches for quadrangles in a way reminiscent of a search tree. The initial input into the algorithm is some arbitrary finite group G (though the version we present is optimised to work with almost simple groups). This forms the initial roof vertex of the tree.

Starting at this initial vertex, the algorithm grows the tree of possibilities in a series of steps. At each step, the algorithm takes some structural property of quadrangles (such as number of points, values of s and t, etc) and for each branch of the tree reaching that step, does the following: using mathematical trickery it divines all of the possibilities for that structural property, given the information collected on the possible generalised quadrangles within that branch. All of the newly calculated possibilities split off to become their own branches, all of which proceed to the next step. There may be zero possibilities, in which case the branch terminates and does not proceed.

Thus each branch, or more precisely each path out from the root vertex, represents one set of possibilities for various pieces of structural information about the possible generalised quadrangles G acts on point-primitively. At each step there may be more possibilities created, but each branch now has more information about the possible quadrangles within that branch.

Thus, at each step some branches grow and divide, while others are pruned. More information is gathered at each step, until there is enough information to construct the unique generalised quadrangle obeying the conditions of each branch, or conclusively show that none exists.

The following discusses how the fives steps used in our version of this process function mathematically. The primary program used in their implementation was the G.A.P. computer algebra software (*GAP – Groups, Algorithms, and Programming, Version 4.10.0* 2018), with assistance primarily from the AtlasRep (Wilson et al. 2016) and FinInG (Bamberg, Betten, et al. 2018) packages. Beyond this, there is not time to fully discuss how the steps were implemented.

Step 1:

The first step is to calculate the possible sizes of the sets of points upon which G can act primitively. Since the algorithm is looking for generalised quadrangles that G acts on point-primitively, these are the possible point sets $\mathcal{P}$ of those quadrangles.

In Section 5, it was mentioned that each choice of (conjugacy class of) maximal subgroup(s), M, of G gives a primitive action of G (using the Fundamental Theorem of Group Actions), and that up to equivalence, these are all of G's primitive actions. The maximal subgroups M are accessed using AtlasRep. From there we can calculate the size of the point set, $\left|{}^{G}/_{M}\right|$, for each action.

Thus we get one branch emerging from the root vertex for each conjugacy class of maximal subgroups of G, and the information collected at this step for each branch is M and the size of point sets in the associated primitive action of G. The actual action of G in these cases is not yet calculated: it is computationally expensive and is delayed as long as possible.

Step 2:

Recall, from section one, that all generalised quadrangles have an order (s,t) such that all lines have $s+1$ points and all points touch $t+1$ lines. We also have that the number of points was $|\mathcal{P}| = (s+1)(st+1)$, that $s \le t^2$, $t \le s^2$ and $s+t$ divides $st(st+1)$.

Step 2 finds all possible values of (s,t) such that the size of the point set (found in Step 1) =

$(s+1)(st+1)$, and which obey the restrictions.

The formula for $|\mathcal{P}|$ in terms of s and t implies that $s+1$ divides $|\mathcal{P}|$. The algorithm thus runs $s+1$ over the divisors of $|\mathcal{P}|$ and solves for t:

$$t = \frac{\frac{|\mathcal{P}|}{s+1} - 1}{(s+1) - 1}$$

It checks whether t is an integer, and if s, t obey the necessary restrictions. If they do, the pair (s,t) is added to the list of possibilities. Each (s,t) pair becomes its own branch going into Step 3.

Step 3:

Step 3 and 4 use the concept of point neighbourhoods. In an incidence geometry the *point neighbourhood* N_a of a point a is the set of all points collinear with a. It varies according to convenience as to whether N_a should include a itself. This step excludes a, while Step 4 includes it.

In a generalised quadrangle of order (s,t), the point neighbourhoods all have size $s(t+1)$, or $s(t+1)+1$ if a is included.

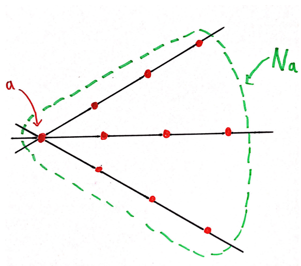

Figure 3: A point neighbourhood

The point neighbourhood N_a is invariant under the subgroup of symmetries, G_a, which fix a: symmetries preserve collinearity, a is fixed, so any point collinear with a remains so under the symmetries. Thus the sets of points which could be point neighbourhoods of a in some generalised quadrangle are those subsets fixed by the action of G_a. These, in turn, must be the union of orbits of G_a.

The algorithm uses G, M and the fundamental theorem of group actions to calculate the action of G on the point set for each possibility. This automatically gives a point stabiliser: M itself turns out to be the stabiliser of a point which is labelled 1 by the algorithm.

Inbuilt G.A.P. functions are used to calculate the orbits of M on the points. The trivial orbit $\{1\}$ is excluded. The problem of finding the possible neighbourhoods of 1 is then a basic version of the knapsack problem: what are all combinations of these orbits whose combined size is $s(t+1)$, and how may they be found efficiently? Solutions to this problem have been well studied, and this is not the place to discuss them. The exact solution used is not particularly important.

However it is done, at the end of this process, all the possible sets which can be neighbourhoods of the point 1 have been calculated. Each of these spawns a new branch going into Step 4.

Step 4:

In any quadrangle of order (s, t), the point neighbourhoods all obey:

- If a and b are collinear points, $N_a \cap N_b$ is the line that both lie on, and thus has size $s + 1$
- If a and b are non collinear points $N_a \cap N_b$ will have size $t + 1$

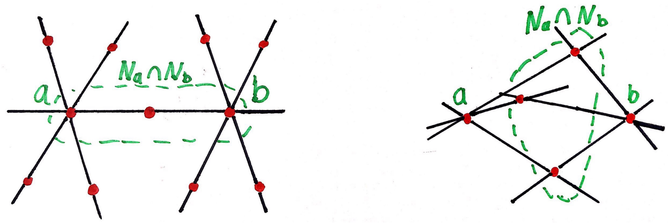

Figure 4: Intersection of the neighbourhoods of collinear points and of non collinear points

Proofs may be found in Payne and Thas 2009. This step requires that a be included in N_a.

At this stage each currently open branch of the search tree contains the following information about any generalised quadrangle within that branch: $\{G, M$, size of point set, action of G, (s, t), $N_1\}$. There is also the assumption that G acts as a symmetry group.

From N_1, N_p can be calculated for any $p \in \mathcal{P}$ by finding an element $g \in G$ such that $1^g = p$, whence $N_p = (N_1)^g$ by the assumption that G acts as a symmetry group. This g can be found using G.A.P. functions, for the action of G on points is known and it is transitive.

The algorithm then tests if this choice of N_1 leads to a set of point neighbourhoods obeying the intersection conditions. It does not need to calculate the intersections for all pairs of point neighbourhoods. The assumption that G is a transitive symmetry group means that the situation is identical around every point. So only the intersections of N_1 with other neighbourhoods needs to be calculated. Moreover, $N_1 \cap N_p$ will have constant size for p in the same orbit of M, since $N_1 \cap N_{p^g} = (N_1 \cap N_p)^g$ for all $g \in M$ (since g fixes 1). So only one intersection per orbit of M is needed.

If any intersection does not obey the conditions, then the branch terminates, since that choice of N_1 cannot give a quadrangle. If everything works out, the branch passes onto the next step.

Step 5:

In Step 4, the intersection of the neighbourhoods of co-linear points must be the line which both points line on, if that branch is to give a quadrangle. Using the calculation of neighbourhood intersections from that step, and by taking images by elements of G under the action, a complete list of what all of the lines must be is obtained. These lines are specified as the list of points incident with them.

All that remains is to test whether the incidence geometry specified by this point-line configuration is a quadrangle. This is done using what is essentially the original axiomatic definition of a quadrangle given in Section 2 in a slightly different form. What it reduces to is checking that there are no triangles and enough squares in the geometry.

If a quadrangle is obtained, by construction, it is one upon which G acts point-primitively. If not, the branch is conclusively terminated. At this point all possible generalised quadrangles which G acts on point-primitively have been found, and so the algorithm concludes.

8 Results

Having designed and implemented the above process, it needs to be verified that it works.

Firstly, we ran the algorithm on groups ($PSp(4, q)$, $PSU(4, q)$, $PSU(5, q)$ for small q) known to act point-primitively on some generalised quadrangle. It successfully found the known quadrangles in those cases.

Next was to see if it could discover something new. The algorithm was used to analyse the case of the 26 simple Sporadic groups (and the associated almost simple groups). This is a case which had not yet been ruled out.

The program successfully managed to check most of these groups, and in all cases showed that they did not act point-primitively on any generalised quadrangle. This is a completely new result.

18 of the 26 sporadic groups were successfully ruled out. The program performed very well in all cases, taking less than a minute to rule out even the largest groups. It was enlightening to see how hard to satisfy all of the properties used in these steps were. The vast majority of all cases failed at each step, and if they didn't fail, they only generated a small number of possibilities. This demonstrates that the generalised quadrangles will be quite rare and special. We already knew that, but this program gave a very good hands on demonstration.

There were a few problems however. The algorithm failed to rule out these eight sporadic groups (and their associated almost simple groups).

$$J_1, Co_2, Fi'_{24}, Suz, O'N, HN, B, M$$

These either failed due to there being no known effective way to calculate with the groups due to their size, or in the J_1 case, due to there being so many branches of the search tree that the computers running the program ran out of memory. That particular problem only emerged for J_1. All the rest has at most around 1000 branches, while J_1 blew up to over 200 million.

9 Conclusion

The algorithm presented here has successfully managed to eliminate previously open cases in the classification of point-primitive generalised quadrangles.

This method can never replace theoretical methods: it checks one group at a time, while there are infinitely many groups to check. However, it can play a role alongside them: its power is that is can analyse any group, thus it can rule out difficult cases which resist theoretical methods, or be used to spot patterns. One way forward would be to reanalyse the results so far and to see how much further they might be pushed with a tool like this.

The lingering sporadic simple groups that this method did not rule out show that there is room for improvement. The next step would be to see what can be done about this. Raw computer science optimisation of the program might help, but the issues primarily seem to be mathematical. One could investigate and see if there are any other steps or conditions or theorems that can be incorporated into the algorithm to allow it to rule out the troublesome cases and limit how often the computationally expensive steps must be used.

Now that we have verified that this method actually works, it is worth trying to see what else can be done with it. One idea would be to try to extend it and create similar algorithms for other generalised n-gons, especially hexagons and octagons. Such a method might work or act quite differently and produce different results due to the different nature of those cases, so it would be an enlightening exercise.

Another possibility is to weaken point-primitivity to point-transitivity. The same algorithm could be used, but instead of taking maximal subgroups in Step 1, it would take any subgroup. This would result in many more cases to check, but it might reveal something interesting.

References

Bamberg, John, Michael Giudici, et al. (2012). "Generalised quadrangles with a group of automorphisms acting primitively on points and lines". In: *J. Combin. Theory Ser. A* 119.7, pp. 1479–1499. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2012.04.005. URL: http://dx.doi.org/10.1016/j.jcta.2012.04.005.

Bamberg, John, S. P. Glasby, et al. (2017). "Point-primitive generalised hexagons and octagons". In: *J. Combin. Theory Ser. A* 147, pp. 186–204. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2016.11.008. URL: https://doi.org/10.1016/j.jcta.2016.11.008.

Bamberg, John, Tomasz Popiel, and Cheryl E. Praeger (2018+). "Simple groups, product actions, and generalized quadrangles". In: *Nagoya Mathematical Journal* to appear, https://doi.org/10.1017/nmj.2017.35.

Buekenhout, F. and Hendrik Van Maldeghem (1994). "Finite distance-transitive generalized polygons". In: *Geom. Dedicata* 52.1, pp. 41–51. ISSN: 0046-5755. DOI: 10.1007/BF01263523. URL: http://dx.doi.org/10.1007/BF01263523.

Dixon, John D. and Brian Mortimer (1996). *Permutation groups*. Vol. 163. Graduate Texts in Mathematics. Springer-Verlag, New York, pp. xii+346. ISBN: 0-387-94599-7. DOI: 10.1007/978-1-4612-0731-3. URL: http://dx.doi.org/10.1007/978-1-4612-0731-3.

Feit, Walter and Graham Higman (1964). "The nonexistence of certain generalized polygons". In: *J. Algebra* 1, pp. 114–131. ISSN: 0021-8693.

Bamberg, John, Anton Betten, et al. (2018). *FinInG – Finite Incidence Geometry, Version 1.4.1*. URL: %5Cverb+(http://www.fining.org)+.

GAP – Groups, Algorithms, and Programming, Version 4.10.0 (2018). The GAP Group. URL: %5Cverb+(https://www.gap-system.org)+.

Morgan, Luke and Tomasz Popiel (2016). "Generalised polygons admitting a point-primitive almost simple group of Suzuki or Ree type". In: *Electron. J. Combin.* 23.1, Paper 1.34, 12. ISSN: 1077-8926.

Payne, Stanley E. and Joseph A. Thas (2009). *Finite Generalised Quadrangles*. second. European Mathematical Society.

Schneider, Csaba and Hendrik Van Maldeghem (2008). "Primitive flag-transitive generalized hexagons and octagons". In: *J. Combin. Theory Ser. A* 115.8, pp. 1436–1455. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2008.02.004. URL: http://dx.doi.org/10.1016/j.jcta.2008.02.004.

Tits, Jacques (1959). "Sur la trialité et certains groupes qui s'en déduisent". In: *Inst. Hautes Études Sci. Publ. Math.* 2, pp. 13–60. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1959__2__13_0.

Van Maldeghem, Hendrik (1998). *Generalized polygons*. Modern Birkhäuser Classics. [2011 reprint of the 1998 original] [MR1725957]. Birkhäuser/Springer Basel AG, Basel, pp. xvi+502. ISBN: 978-3-0348-0270-3; 978-3-0348-0271-0. DOI: 10.1007/978-3-0348-0271-0. URL: http://dx.doi.org/10.1007/978-3-0348-0271-0.

Wilson, R. A. et al. (2016). *AtlasRep, A GAP Interface to the Atlas of Group Representations, Version 1.5.1*. http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep. GAP package.

A Restrictions on the parameters of Generalised Polygons

Theorem A.1. *Feit and Higman 1964: Let $\mathcal{G}$ be some finite generalised n-gon with order (s,t) and $n \geq 3$. Then:*

- *n=3: $s = t$ and $\mathcal{G}$ is a projective plane*
- *n=4: $\frac{st(st+1)}{s+t}$ is an integer*
- *n=6: st is a square number. If $u = \sqrt{st}$ and $w = s+t$ then $\frac{u^2(1+w+u)(1 \pm u+u^2)}{2(w \pm u)}$ is an integer for both choices of sign of the $\pm$ (either $++$ or $--$)*
- *n=8: $2st$ is a square number, so $s \neq t$. If $u = \sqrt{\frac{st}{2}}$ and $w = s+t$ then $\frac{u^2(1+w+2u^2)(1+2u^2)(1 \pm 2u+2u^2)}{2(w \pm 2u)}$ is an integer for both choices of sign of the $\pm$ (either $++$ or $--$).*

Theorem A.2. *Van Maldeghem 1998: Let $\mathcal{G}$ be some finite generalised n-gon with order (s,t) and $n \geq 4$. Then:*

- *n=4: $s \leq t^2$, $t \leq s^2$*
- *n=6: $s \leq t^3$, $t \leq s^3$*
- *n=8: $s \leq t^2$, $t \leq s^2$*

Theorem A.3. *Van Maldeghem 1998: Let $\mathcal{G}$ be some finite generalised n-gon with order (s,t) and $n \geq 3$. Then:*

- *n=3: $\mathcal{P} = s^2+s+1$, $\mathcal{L} = s^2+s+1$ (since $s = t$)*
- *n=4: $\mathcal{P} = (s+1)(st+1)$, $\mathcal{L} = (t+1)(st+1)$*
- *n=6: $\mathcal{P} = (s+1)((st)^2+st+1)$, $\mathcal{L} = (t+1)((st)^2+st+1)$*
- *n=8: $\mathcal{P} = (s+1)((st)^3+(st)^2+st+1)$, $\mathcal{L} = (t+1)((st)^3+(st)^2+st+1)$*

B The Finite Simple Groups

Definition B.1. A finite group G is a finite simple group if G has no non-trivial normal subgroups. That is, the only subgroups N of G such that $g^{-1} \cdot N \cdot g = N$ for all $g \in G$ are the trivial subgroups $\{1\}$ and G.

The finite simple groups are important because they are in some sense the building blocks of other finite groups. With a normal subgroup N of G, you can form the quotient group ${}^{G}/_{N}$, and thus in some sense decompose G into two "factors", N and ${}^{G}/_{N}$. This "factorisation" is only non-trivial if $N \neq \{1\}, G$.

The Jordan-Holder Theorem makes this explicit (for finite groups). Essentially it tells us that we can keep decomposing G and its subsequent "factors" until every factor is a simple group whence the process terminates, for there are no more non-trivial normal subgroups to factor out. The theorem guarantees that the list of simple group factors at the end of this process is uniquely determined by the group (no matter how we decompose G, we always end up with the same list of simple groups at the end, perhaps in a different order). So this "factorisation" into simple groups is similar as the factorisation of integers into primes. The study of simple groups thus play a similarly monumental role in group theory and the history of mathematics as do the primes in number theory.

Though questions remain about the Finite Simple Groups, their study was essentially completed in 1983 with the announcement of the Classification of the Finite Simple Groups (though the proof took until 2004 to be properly completed, and still pretty much no one understands the whole thing). This is one of mathematics' greatest achievements, with the complete proof taking hundreds of mathematicians many decades and perhaps hundreds of thousands of pages of publications.

Theorem B.1. *The Classification of Finite Simple Groups: A finite simple group is isomorphic to a member of one of the following families:*

1. *Cyclic groups $\mathcal{C}_p$ of prime order*
2. *Alternating groups A_n for $n \geq 5$*
3. *The many families of groups of Lie type: (here, q denotes any prime power, n any positive integer, except for the noted restrictions)*
   (a) *Chevalley Groups:*
      i. $A_n(q) \cong PSL(n+1, q)$ *except n=1, p=2,3*
      ii. $B_n(q) \cong O(2n+1, q)$ *except* $B_2(2)$
      iii. $C_n(q) \cong PSp(2n, q)$, *n>2*
      iv. $D_n(q) \cong P\Omega^+(2n, q)$, *n>3*
   (b) *Exceptional Chevalley Groups:*
      i. $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$, *except for* $G_2(2)$

(c) *Steinberg/Twisted Chevalley Groups:*

i. ${}^2A_2(q^2)$, *except q=2*

ii. ${}^2D_n(q^2)$, *n>3*

iii. ${}^2E_6(q^2)$

iv. ${}^3D_4(q^3)$

(d) *Suzuki Groups:* ${}^2B_2(2^{2n+1})$

(e) *Ree Groups i):* ${}^2F_4(2^{2n+1})$

(f) *Ree Groups ii):* ${}^2G_2(3^{2n+1})$ *except n=1*

4. *The tits Group:* ${}^2F_4(2)'$

5. *Sporadic Simple Groups:*

(a) *Mathieu Groups:* $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$

(b) *Janko Groups:* J_1, J_2, J_3, J_4

(c) *Conway Groups:* Co_1, Co_2, Co_3

(d) *Fischer Groups:* $Fi_{22}, Fi_{23}, Fi'_{24}$

(e) *Higman Sims group:* HS

(f) *McLaughlin Group:* McL

(g) *Held Group:* He

(h) *Rudvalis Group: Ru*

(i) *Suzuki Group:* Suz

(j) *O'Nan Group:* $O'N$

(k) *Harada-Norton Group:* HN

(l) *Lyons Group: Ly*

(m) *Thompson Group:* Th

(n) *Baby Monster Group:* B

(o) *Monster Group:* M