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Brownian motion and harmonic functions

Douglas Coulter

Supervised by Laurence Field

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Abstract

In this report we go through a construction of Brownian motion and then use Brownian motion to prove some things which are usually proved by ordinary analysis. We mainly restrict ourselves to $\mathbb{C}$ and $\mathbb{R}^2$ to use some complex analysis. In particular, we look at the Dirichlet problem, the conformal invariance of Brownian motion, harmonic measure, Green's functions, and the Poisson kernel.

1 Introduction

Analysis, especially measure theory, is essential to probability theory. In this project, we looked at the other direction: applying probabilistic techniques to problems in analysis.

In particular, Brownian motion (here usually in $\mathbb{R}^2$ and $\mathbb{C}$) can be used to prove facts from analysis, including properties of solutions to the Dirichlet problem, Green's functions and Poisson kernels.

An important fact is that, in $\mathbb{C}$, the image of a Brownian motion under an onto conformal map is also a Brownian motion in the range, provided that the time is transformed in the right way. We say that Brownian motion is *conformally invariant*. Then using Brownian motion makes it easy to find out how things like the harmonic measure, Green's functions and Poisson kernels transform under conformal maps.

Section 2 concerns the properties and the existence of Brownian motion. The subsection about the construction of Brownian motion is self-contained, so can be safely skipped.

Section 3 is a short introduction to Itô calculus, which is needed to show that Brownian motion is conformally invariant (Lévy's theorem). In particular we need Itô's formula.

In Section 5, Brownian motion is used to solve the Dirichlet problem and in Section 6, Lévy's theorem is proved. Then harmonic measures are introduced in Section 7. In Section 8, Brownian motion in $\mathbb{C}$ is used to construct Green's functions for a domain. The construction is based on ideas by Gregory Lawler. Using Lévy's theorem, it is then easy to show that Green's functions are conformally invariant. The last section is about Poisson kernels and showing how they transform under conformal maps.

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2 Brownian Motion

2.1 Properties of Brownian Motion

Throughout, we'll denote the sample space by Ω, the set of events by $\mathcal{F}$, and the probability measure by $\mathbb{P}$.

A *stochastic process* is a family of random variables indexed by time, or more generally some set T.

A stochastic process $\{B_t(\omega)\}_{t\geq 0}$ is a *Brownian motion* if

- $B_0 = 0$
- for $s < t$, $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$
- for $s < t$, the increment $B_t - B_s$ is independent of the σ-alegbra $\mathcal{F}_s \equiv \sigma\{B_r : r \leq s\}$, which is the smallest σ-algebra generated by all the random variables B_r, $r \leq s$
- and $t \to B_t$ is continuous with probability 1.

The first step though is to show that Brownian motion actually exists. There are several ways of doing so and here we'll follow the approach from [2].

Proposition 1. *Let $X_1, \ldots, X_n$ be jointly normal. That is $X_1, \ldots, X_n$ are a linear combination of i.i.d with distribution $\mathcal{N}(0,1)$ random variables $Z_1, \ldots Z_m$. If Cov$(X_i, X_j) = 0$ for all i and j, then $X_1, \ldots, X_n$ are independent.*

Proof. Without loss of generality suppose that $X_1, \ldots, X_n$ are all normalized. Moreover since $\mathrm{Cov}(X_i, X_j) = 0$ for all i and j, $X_1 \ldots, X_n$ are all orthogonal. Then, by Gram-Schmidt, $X_1, \ldots, X_n$ can be expanded to an orthonormal basis, say $X_1, \ldots, X_m$.

Orthonormal bases are related to each other by orthogonal matrices, so

$$\boldsymbol{X} = O\boldsymbol{Z}$$

where $\boldsymbol{X} = (X_1, \ldots, X_m)$, $\boldsymbol{Z} = (Z_1, \ldots, Z_m)$ and O and $U = O^{-1}$ are orthogonal. The PDF of $\boldsymbol{Z}$ is $(2\pi)^{-\frac{n}{2}} \exp\left[-|\boldsymbol{z}|^2/2\right]$. Since $U^T U = I$, then the joint density for $\boldsymbol{X}$ is,

$$(2\pi)^{-\frac{n}{2}} \exp\left[-\left|U\boldsymbol{x}\right|^2/2\right] = (2\pi)^{-\frac{n}{2}} \exp\left[-\boldsymbol{x}^T U^T U \boldsymbol{x}/2\right] = (2\pi)^{-\frac{n}{2}} \exp\left[-|\boldsymbol{x}|^2/2\right]$$

i.e. $\boldsymbol{X}$ has the same probability density, so the rows of $\boldsymbol{X}$ are independent. □

2.2 Construction of Brownian Motion

First, we'll construct a Brownian motion $B_t(\omega)$ on $t \in [0,1]$. It is enough to define B_t on the *dyadic rationals,* a countable dense subset of $[0,1]$. It will turn out that $t \mapsto B_t$ is uniformly continuous and so B_t can be extended to all of $t \in [0,1]$ by taking limits.

Let
$$\mathcal{D}_k = \left\{ \frac{j}{2^k} : 0 \leq j \leq 2^k \right\}.$$

The *dyadic rationals* are

$$\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k.$$

We'll define B_t inductively, first on $\mathcal{D}_0$, and then on $\mathcal{D}_{k+1}$ given B_t on $\mathcal{D}_k$.

Let $\{Z_d(\omega) : d \in \mathcal{D}\}$ be a countable family of mutually independent $\mathcal{N}(0,1)$ random variables on some suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Base case

On $\mathcal{D}_0$, let $B_0 = 0$ and $B_1 = Z_1$. Clearly $B_1 - B_0 \sim \mathcal{N}(0,1)$.

Inductive step

Let
$$I_j^{(k)} = B_{(j+1)/2^k} - B_{j/2^k}.$$

(I.H.) Suppose that B_t is defined on $\mathcal{D}_k$ and the increments $\left\{ I_j^{(k)} : 0 \leq j \leq 2^k - 1 \right\}$ are mutually independent and each has distribution $\mathcal{N}\left(0, 2^{-k}\right)$. Moreover, assume that for each $t \in \mathcal{D}_k$, B_t is a linear combination of $\{Z_d : d \in \mathcal{D}_k\}$.

For $d \in \mathcal{D}_{k+1} \backslash \mathcal{D}_k$, $d = \left(j + \frac{1}{2}\right)/2^k$ for some j. Then define $B_{\left(j+\frac{1}{2}\right)/2^k}$ by

$$\begin{aligned} I_{2j}^{(k+1)} = B_{\left(j+\frac{1}{2}\right)/2^k} - B_{j/2^k} &= \frac{1}{2} I_j^{(k)} + \frac{1}{2 \cdot 2^{k/2}} Z_{\left(j+\frac{1}{2}\right)/2^k} \\ \therefore B_{\left(j+\frac{1}{2}\right)/2^k} &= \frac{1}{2}\left[B_{(j+1)/2^k} + B_{j/2^k} \right] + \frac{1}{2 \cdot 2^{k/2}} Z_{\left(j+\frac{1}{2}\right)/2^k}. \end{aligned}$$

Therefore
$$I_{2j+1}^{(k+1)} = B_{(j+1)/2^k} - B_{\left(j+\frac{1}{2}\right)/2^k} = \frac{1}{2} I_j^{(k)} - \frac{1}{2 \cdot 2^{k/2}} Z_{\left(j+\frac{1}{2}\right)/2^k}.$$

Note that $a\mathcal{N}(0,1) = \mathcal{N}(0,a^2)$ and the mean and variances of independent Gaussian add up.

By the inductive hypothesis, $I_j^{(k)} \sim \mathcal{N}(0, 2^{-k})$ so $\frac{1}{2} I_j^{(k)} \sim \mathcal{N}(0, 2^{-k-2})$, and since $2^{-k/2-1} Z_{(j+\frac{1}{2})/2^k} \sim \mathcal{N}(0, 2^{-k-2})$, then

$$\begin{aligned} B_{(j+\frac{1}{2})/2^k} - B_{j/2^k} &\sim \mathcal{N}\left(0, 2^{-k-1}\right) \\ \text{and } B_{(j+1)/2^k} - B_{(j+\frac{1}{2})/2^k} &\sim \mathcal{N}\left(0, 2^{-k-1}\right) \end{aligned}$$

i.e. increments have the correct variance. Clearly $B_{(j+\frac{1}{2})/2^k}$ is a linear combination of $\{Z_d : d \in \mathcal{D}_{k+1}\}$ (note $\mathcal{D}_k \subset \mathcal{D}_{k+1}$).

To check independence of increments, since $I_j^{(k)}$ is a linear combination of $\{Z_d : d \in \mathcal{D}_k\}$, $\mathbb{E} I_i^{(k)} Z_{(j+\frac{1}{2})/2^k} = 0$, and so

$$\begin{aligned} \mathbb{E} I_n^{(k+1)} I_m^{(k+1)} =& \mathbb{E}\left(\frac{1}{2} I_j^{(k)} \pm \frac{1}{2 \cdot 2^{k/2}} Z_{(j+\frac{1}{2})/2^k}\right)\left(\frac{1}{2} I_i^{(k)} \pm \frac{1}{2 \cdot 2^{k/2}} Z_{(i+\frac{1}{2})/2^k}\right) \\ =& \frac{1}{4}\left(\mathbb{E} I_j^{(k)} I_i^{(k)} \pm \frac{1}{2^k} \mathbb{E} Z_{(j+\frac{1}{2})/2^k} Z_{(i+\frac{1}{2})/2^k}\right). \end{aligned}$$

If $i \neq j$, then $I_j^{(k)}$ and $I_i^{(k)}$ and $Z_{(j+\frac{1}{2})/2^k}$ and $Z_{(i+\frac{1}{2})/2^k}$ are independent, so

$$\mathbb{E}\left(\frac{1}{2} I_j^{(k)} \pm \frac{1}{2 \cdot 2^{k/2}} Z_{(j+\frac{1}{2})/2^k}\right)\left(\frac{1}{2} I_i^{(k)} \pm \frac{1}{2 \cdot 2^{k/2}} Z_{(i+\frac{1}{2})/2^k}\right) = 0.$$

If $i = j$, and $n \neq m$,

$$\begin{aligned} \mathbb{E} I_n^{(k+1)} I_m^{(k+1)} &= \frac{1}{4}\left(\mathbb{E} I_j^{(k)} I_j^{(k)} - \frac{1}{2^k} \mathbb{E} Z_{(j+\frac{1}{2})/2^k} Z_{(j+\frac{1}{2})/2^k}\right) \\ &= \frac{1}{4}\left(\frac{1}{2^k} - \frac{1}{2^k}\right) = 0. \end{aligned}$$

Then by Proposition 1, the increments $I_i^{(k+1)}$ are independent.

Lemma 2. *If $s, t \in \mathcal{D}$ and $s < t$,*

$$\mathbb{E} B_s (B_t - B_s) = 0.$$

Proof. Choose $\mathcal{D}_k$ so that $s, t \in \mathcal{D}_k$. Then $s = n/2^k$ and $t = m/2^k$ for some $n < m$. Then

$$\begin{aligned} B_{n/2^k} &= \sum_{i=0}^{n-1} I_i^{(k)} \\ \text{and } B_{m/2^k} - B_{n/2^k} &= \sum_{j=n}^{m-1} I_j^{(k)}. \end{aligned}$$

Since increments are independent, then

$$\mathbb{E} B_{n/2^k}\left(B_{m/2^k} - B_{n/2^k}\right) = \mathbb{E}\left[\left(\sum_{i=0}^{n-1} I_i^{(k)}\right)\left(\sum_{j=n}^{m-1} I_j^{(k)}\right)\right] = 0.$$

□

Lemma 3. *For $t \in \mathcal{D}$, $\mathbb{E}B_t^2 = t$.*

Proof. Suppose $t \in \mathcal{D}_k$, i.e. $t = n/2^k$. Then $B_{n/2^k} = \sum_{i=0}^{n-1} I_i^{(k)}$. Since increments are independent and $\mathbb{E}I_i^{(k)} I_i^{(k)} = 2^{-k}$,

$$\mathbb{E}B_{n/2^k}^2 = \mathbb{E}\left[\left(\sum_{i=0}^{n-1} I_i^{(k)}\right)\left(\sum_{j=0}^{n-1} I_j^{(k)}\right)\right] = \sum_{j=0}^{n-1} \mathbb{E}I_j^{(k)} I_j^{(k)} = \sum_{j=0}^{n-1} 2^{-k} = n/2^k = t.$$

□

Also, if $s, t \in \mathcal{D}$ and $s \leq t$,

$$\mathbb{E}\left(B_t - B_s\right) B_s = \mathbb{E}B_t B_s - \mathbb{E}B_s B_s = 0$$

so $\mathbb{E}B_t B_s = s$. If $r \in \mathcal{D}$ and $r \leq s \leq t$,

$$\mathbb{E}\left[\left(B_t - B_r\right) B_r\right] = \mathbb{E}\left[\left(B_t - B_s\right) B_r\right] + \mathbb{E}\left[\left(B_s - B_r\right) B_r\right]$$

$$0 = \mathbb{E}\left[\left(B_t - B_s\right) B_r\right] + 0$$

i.e. $B_t - B_s$ and B_r are uncorrelated and therefore independent.

2.2.1 Uniform Continuity

Theorem 4. Borel-Cantelli Lemma. *If A_n is a sequence of events and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(A_n$ infinitely often$) = 0$.*

Proposition 5. *Suppose $f : \mathcal{D} \to \mathbb{R}$ is uniformly continuous. Then there exists a unique continuous extension $F : [0,1] \to \mathbb{R}$. Moreover F is uniformly continuous.*

Proof. Let $x \in [0,1]$. By uniform continuity, we may choose δ_n such that, for $y, z \in \mathcal{D}$, $|y - z| < \delta_n$ implies $|f(y) - f(z)| < 2^{-n}$.

Choose a sequence $x_n \in \mathcal{D}$ such that $|x_n - x| < \delta_n/2$. Then

$$|x_n - x_m| \leq |x - x_n| + |x - x_m| < \frac{\delta_n}{2} + \frac{\delta_m}{2} \leq \delta_n \vee \delta_m$$

where $\delta_n \vee \delta_m = \min\{\delta_n, \delta_m\}$. Therefore

$$|f(x_n) - f(x_m)| \leq 2^{-n} \vee 2^{-m}$$

i.e. $f(x_n)$ forms a Cauchy sequence. Then let $F(x) = \lim_{n\to\infty} f(x_n)$.

If $y_n \to x$ also, by choosing a subsequence we can assume that $|y_n - x| < \delta_n/2$ and so

$$|x_n - y_n| \leq |x_n - x| + |x - y_n| < \delta_n$$

and so $|f(x_n) - f(y_n)| < 2^{-n}$. Therefore $F(x)$ is independent of the choice of sequence x_n. If $x \in \mathcal{D}$ then $F(x) = f(x)$.

If F was not uniformly continuous, i.e. for some $x, y \in [0,1]$, $|x - y| < \delta_n$ but $|F(x) - F(y)| \geq 2^{-n}$, then f would not be, since x and y can be approximated by points in $\mathcal{D}$. □

Let

$$K_n^* = \sup\left\{|B_s - B_t| : 0 \leq s, t \leq 1, |s - t| \leq 2^{-n}, s, t \in \mathcal{D}\right\}$$

and

$$K_n = \max_{0 \leq k \leq 2^n} \sup\left\{\left|B_s - B_{k/2^n}\right| : \frac{k}{2^n} \leq s \leq \frac{k+1}{2^n}, s \in \mathcal{D}\right\}.$$

Being uniformly continuous is equivalent to $K_n^* \to 0$ as $n \to \infty$. Note that $K_n \leq K_n^*$.
Also, if.... $K_n \leq K_n^* \leq 3K_n$. So it's enough to show that $K_n \to 0$ as $n \to \infty$.
Since $B_{s+t} - B_t$ has the same distribution as B_s, then

$$\begin{aligned}\mathbb{P}(K_n > m) &\leq \sum_{k=0}^{2^n-1} \mathbb{P}\left(\sup\left\{\left|B_s - B_{k/2^n}\right| : \frac{k}{2^n} \leq s \leq \frac{k+1}{2^n}, s \in \mathcal{D}\right\} > m\right) \\ &= 2^n \mathbb{P}\left(\sup\left\{|B_s| : 0 \leq s \leq 2^{-n}, s \in \mathcal{D}\right\} > m\right) \\ &= 2^n \mathbb{P}\left(\sup\{|B_s| : 0 \leq s \leq 1, s \in \mathcal{D}\} > 2^{n/2} m\right)\end{aligned}$$

Let $\kappa = \min\left\{k : B_{k/2^n} \geq m\right\}$. Then

$$\mathbb{P}(B_1 \geq m) = \mathbb{P}(\kappa \leq 2^n)\, \mathbb{P}(B_1 \geq m | \kappa \leq 2^n).$$

If $\kappa \leq 2^n$, then since $B_1 - B_\kappa$ is a Gaussian with mean zero and $B_1 = B_\kappa + B_1 - B_\kappa \geq m + B_1 - B_\kappa$, and so B_1 is a Gaussian with mean greater than m, and so $\mathbb{P}(B_1 \geq m | \kappa \leq 2^n) \geq 1/2$. Therefore

$$\mathbb{P}\left(\max_{t \in \mathcal{D}_n} B_t \geq m\right) \leq \mathbb{P}(\kappa \leq 2^n) \leq 2\mathbb{P}(B_1 \geq m).$$

Since $\{\max_{t \in \mathcal{D}_n} B_t \geq m\} \subseteq \left\{\max_{t \in \mathcal{D}_{n+1}} B_t \geq m\right\}$, i.e. is an increasing chain of events, then (by basic measure theory)

$$\mathbb{P}\left(\sup_{t\in\mathcal{D}} B_t \geq m\right) \leq 2\mathbb{P}\left(B_1 \geq m\right).$$

By symmetry

$$\mathbb{P}\left(\sup_{t\in\mathcal{D}} |B_t| \geq m\right) = 2\mathbb{P}\left(\sup_{t\in\mathcal{D}} B_t \geq m\right) \leq 4\mathbb{P}\left(B_1 \geq m\right).$$

Finding an upper bound for $\mathbb{P}\left(B_1 \geq m\right)$,

$$\begin{aligned}\mathbb{P}\left(B_1 \geq m\right) &= \int_m^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &\leq \int_m^\infty \frac{1}{\sqrt{2\pi}} e^{-mx/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{m} e^{-m^2/2}.\end{aligned}$$

Therefore, for a sequence $m_n \to 0$

$$\begin{aligned}\mathbb{P}\left(K_n > m_n\right) &\leq 2^n \mathbb{P}\left(\sup_{t\in\mathcal{D}} |B_t| \geq 2^{n/2} m_n\right) \leq \frac{4}{\sqrt{2\pi}} 2^n \frac{2}{2^{n/2} m_n} \exp\left[-2^{n-1} m_n^2\right] \\ &= \frac{8}{\sqrt{2\pi}} \frac{2^{n/2}}{m_n} \exp\left[-2^{n-1} m_n^2\right].\end{aligned}$$

Let $m_n = 2^{-n/4}$. Then

$$\mathbb{P}\left(K_n > m_n\right) \leq \frac{8}{\sqrt{2\pi}} 2^{3n/4} \exp\left[-2^{1+n/2}\right]$$

By the ratio test, $\sum_{m=0}^{\infty} \mathbb{P}\left(K_n > m_n\right)$ converges so by the Borel-Cantelli lemma, $\mathbb{P}\left(K_n > m_n \text{ i.o.}\right) = 0$ ('i.o.' means 'infinitely often'), and so with probability 1, $K_n \to 0$ as $n \to \infty$, i.e. $t \mapsto B_t$ is uniformly continuous.

Therefore, $t \mapsto B_t : \mathcal{D} \to \mathbb{R}$ can be uniquely extended to $[0,1]$. Are the increments $B_t - B_s$ and B_s still independent, for $s \leq t$ in $[0,1]$?

Definition 6. The *characteristic function* of a random variable X is $\mathbb{E}e^{iuX}$. The *characteristic function* of a joint distribution (X, Y) is

$$\varphi\left(u, v\right) \equiv \mathbb{E}e^{iuX+ivY}.$$

Proposition 7. *Random variables X and Y are independent if and only if, for all u and v, $\mathbb{E}e^{iuX+ivY} = \mathbb{E}e^{iuX}\mathbb{E}e^{iuY}$.*

Proposition 8. *If $r \leq s < t$ and $r, s, t \in [0,1]$, then $B_t - B_s$ is independent of B_r.*

Proof. By Proposition 7, it is enough to show that, for all u and v,

$$\mathbb{E}e^{iu(B_t - B_s) + ivB_r} = \mathbb{E}e^{iu(B_t - B_s)}\mathbb{E}e^{iuB_r}.$$

Then, for r, s, t,

$$B_t(\omega) = \lim_{t_n \to t, t_n \in \mathcal{D}} B_{t_n}(\omega)$$

i.e. $B_t(\omega)$ is the pointwise limit of $B_{t_n}(\omega)$ for almost every ω. Then by continuity

$$B_{t_n} - B_{s_n} \to B_t - B_s \text{ a.e. } \omega$$

$$\exp\left[iu\left(B_{t_n} - B_{s_n}\right) + ivB_{r_n}\right] \to \exp\left[iu\left(B_t - B_s\right) + ivB_r\right] \text{ a.e. } \omega.$$

Moreover, $\left|\exp\left[iu\left(B_{t_n} - B_{s_n}\right) + ivB_{r_n}\right]\right| = 1$ and so is bounded. Since the whole space has finite measure (i.e. $\mathbb{P}\Omega = 1$), then by the dominated convergence theorem

$$\mathbb{E}\exp\left[iu\left(B_{t_n} - B_{s_n}\right) + ivB_{r_n}\right] \to \mathbb{E}\exp\left[iu\left(B_t - B_s\right) + ivB_r\right].$$

For s_n choose a decreasing sequence in $\mathcal{D}$, and for r_n choose and increasing sequence in $\mathcal{D}$. Therefore, for all n, $r_n \leq s_n$, and since $s < t$, we can choose t_n so that $s_n < t_n$ for all n, i.e. $r_n \leq s_n < t_n$ for all n and $r_n, s_n, t_n \in \mathcal{D}$. Therefore, since $B_{t_n} - B_{s_n}$ and B_{r_n} must be independent, by Theorem 6

$$\mathbb{E}\exp\left[iu\left(B_{t_n} - B_{s_n}\right) + ivB_{r_n}\right] = \mathbb{E}\exp\left[iu\left(B_{t_n} - B_{s_n}\right)\right]\mathbb{E}\exp\left[ivB_{r_n}\right].$$

Therefore,

$$\mathbb{E}\exp\left[iu\left(B_t - B_s\right) + ivB_r\right] = \mathbb{E}\exp\left[iu\left(B_t - B_s\right)\right]\mathbb{E}\exp\left[ivB_r\right].$$

□

To define B_t on all of $\mathbb{R}_+$, let $B_t^1, B_t^2, \ldots$ be an independent family of Brownian motions on $[0,1]$. For $t \geq 0$, let T be the smallest integer $T \leq t$, and let

$$B_t = B_1^1 + B_1^2 + \cdots + B_1^T + B_{t-T}^{T+1}.$$

If (B_t) is a Brownian motion then $(B_t + x)_{t \geq 0}$ is a *Brownian motion starting at x*. It's probability measure is denoted $\mathbb{P}^x$ and its expectation denoted $\mathbb{E}^x$.

2.3 Markov Properties

Definition 9. A *filtration* is a family of σ-algebras indexed by time, $\{\mathcal{M}_t\}_{t\geq 0}$, such that if $s \leq t$, $\mathcal{M}_s \subseteq \mathcal{M}_t$.

For example, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration.

Definition 10. Given a filtration $\mathcal{M}_t$, a random variable T is an $\mathcal{M}$-*stopping time* if $\{T \leq t\} \in \mathcal{M}_t$ for all t. Intuitively: we can tell if T has happened yet given $\mathcal{M}_t$.

If $s \geq 0$ and B_t is a Brownian motion then $B_{t+s} - B_s$ is a B.M. independent of $\mathcal{F}_s$. This is the *weak Markov property.*

However, Brownian motion also satisfies the *strong Markov property.* If S is a $\mathcal{F}$-stopping time, then $B_{t+S} - B_S$ is a B.M. independent of $\mathcal{F}_S$.

Proof. See Chapter I.3 of Bass [1]. □

3 Itô Integrals

Suppose that ω is fixed and we want to calculate

$$\int_0^t f(s)\, dB_s.$$

If $t \mapsto B_t$ was differentiable (except for possibly finitely many points) then

$$\int_0^t f(s)\, dB_s = \int_0^t f(s) \frac{dB_s}{ds} ds.$$

But, with probability 1, $t \mapsto B_t$ is nowhere differentiable.

However, if $f(s) = c$ was constant, or depended only on ω, then the only sensible way to define $\int_S^T f(\omega)\, dB_s$ is

$$\int_S^T f(\omega)\, dB_s = f(\omega) \int_S^T dB_s = f(\omega) [B_T - B_S].$$

Recall that $\mathcal{F}_s \equiv \sigma\{B_r : r \leq s\}$.

An *elementary function* $\mathbb{R}_+ \times \Omega \to \mathbb{R}$ is one of the form

$$\phi(t, \omega) = \sum_{i\geq 1} e_{t_i}(\omega)\, \mathbf{1}_{[t_i, t_{i+1})}$$

where each $e_{t_i}(\omega)$ is $\mathcal{F}_{t_i}$-measurable, $\{t_i\}_{i\geq 0}$ is a partition of $\mathbb{R}_+$, and $\phi(t,\omega)$ is in L^2. For elementary functions, the *Itô integral* is defined as

$$\int_S^T \phi(t,\omega)\,dB_t(\omega) \equiv \sum_{i=0}^{M-1} e_{t_i}(\omega)\left[B_{t_{i+1}} - B_{t_i}\right](\omega).$$

where $t_0 = S$ and $t_M = T$, by subdividing the partition if necessary. This way, $e_{t_i}(\omega)$ depends on only what has happened up until t_i, and does not depend on $B_{t_{i+1}} - B_{t_i}$.

Definition 11. Given a filtration $\mathcal{M}_t$, a stochastic process $X_t(\omega)$ is $\mathcal{M}_t$*-adapted* if, for each $t \geq 0$, $X_t(\omega)$ is $\mathcal{M}_t$-measurable, i.e. if $U \subset \boldsymbol{R^n}$ is Borel measurable then $X_t^{-1}(U) \in \mathcal{M}_t$.

Definition 12. A stochastic process (X_t) is a $\mathcal{M}_t$*-martingale* if, for all t, X_t is integrable, $\mathcal{M}_t$-measurable, and if $s \leq t$,

$$\mathbb{E}\left[X_t|\mathcal{M}_s\right] = X_s \text{ a.s.}$$

Definition 13. A function $f(t,\omega)$ is *Itô integrable* on $[0,T]$ if it is

- measurable,
- $\mathcal{F}_t$-adapted, or, more generally, if B_t is a martingale with respect to $\mathcal{M}_t$ and $f(t,\omega)$ is $\mathcal{M}_t$-adapted,
- and $\mathbb{E}\left[\int_0^T f(t,\omega)^2\,dt\right] < \infty$

Proposition 14. *If $\mathcal{G} \subset \mathcal{F}$ are σ-algebras and X is a r.v., then*

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{F}\right]|\mathcal{G}\right] = \mathbb{E}\left[X|\mathcal{G}\right] \text{ (Tower Rule).}$$

Also, if Y is a r.v. that is $\mathcal{F}$ measurable,

$$\mathbb{E}\left[Y \cdot X|\mathcal{F}\right] = Y \cdot \mathbb{E}\left[X|\mathcal{F}\right].$$

Proposition 15. *Let B_t be a Brownian motion with respect to $\mathcal{F}_t$. The stochastic process $X_t \equiv \int_0^t \phi(s)\,dB_s$ is an $\mathcal{F}_t$-martingale.*

Proof. By the definition of the Itô integral on elementary functions, if $t_n = t$

$$\int_0^t \phi(s)\,dB_s \equiv \sum_{i=0}^{n-1} e_{t_i}(\omega)\left[B_{t_{i+1}} - B_{t_i}\right](\omega).$$

Suppose that $t_j = S$ (by subdividing if necessary). Then (X_t) is a martingale if $\mathbb{E}[X_t|\mathcal{F}_S] = X_S$. If $i+1 \leq j$, then $B_{t_{i+1}} - B_{t_i}$ and e_{t_i} are $\mathcal{F}_S$-measurable, so

$$\mathbb{E}\left[e_{t_i}\left[B_{t_{i+1}} - B_{t_i}\right]|\mathcal{F}_{t_j}\right] = e_{t_i}\left[B_{t_{i+1}} - B_{t_i}\right].$$

Otherwise if $i \geq j$, then $\mathcal{F}_{t_j} \subset \mathcal{F}_{t_i}$, so by the Tower Rule,

$$\begin{aligned}\mathbb{E}\left[e_{t_i}\left[B_{t_{i+1}} - B_{t_i}\right]|\mathcal{F}_{t_j}\right] &= \mathbb{E}\left[\mathbb{E}\left[e_{t_i}\left[B_{t_{i+1}} - B_{t_i}\right]|\mathcal{F}_{t_i}\right]|\mathcal{F}_{t_j}\right] \\ &= \mathbb{E}\left[0|\mathcal{F}_{t_j}\right] = 0.\end{aligned}$$

And so

$$\begin{aligned}\mathbb{E}\left[X_T|\mathcal{F}_S\right] &= \sum_{i=0}^{j-1}\mathbb{E}\left[e_{t_i}(\omega)\left[B_{t_{i+1}} - B_{t_i}\right](\omega)\right] \\ &= \sum_{i=0}^{j-1} e_{t_i}(\omega)\left[B_{t_{i+1}} - B_{t_i}\right](\omega) \\ &= X_S\end{aligned}$$

Is X_T $\mathcal{F}_T$-adapted? Yes, since if $t_{i+1} \leq t_n = T$, then both e_{t_i} and $B_{t_{i+1}} - B_t$ are both $\mathcal{F}_{t_{i+1}}$-adapted. Therefore X_T is a martingale. □

Theorem 16. *(Itô isometry). For a bounded elementary function $\phi(t,\omega)$,*

$$\mathbb{E}\left[\left(\int_S^T \phi(t,\omega)\,dB_t\right)^2\right] = \mathbb{E}\left[\int_S^T \phi(t,\omega)^2\,dt\right].$$

Proof. Since increments of Brownian motion are independent, and by the Tower Rule,

$$\begin{aligned}\mathbb{E}\left[\left(\int_S^T \phi(t,\omega)\,dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=0} e_{t_i}^2\left[B_{t_{i+1}} - B_{t_i}\right]\right)^2\right] \\ &= \sum_{i=0}\mathbb{E}\left[e_{t_i}^2\left[B_{t_{i+1}} - B_{t_i}\right]^2\right] \\ &= \sum_{i=0}\mathbb{E}\left[e_{t_i}^2\left(t_{i+1} - t_i\right)\right] \\ &= \mathbb{E}\left[\int_S^T \phi(t,\omega)^2\,dt\right].\end{aligned}$$

□

Theorem 17. *The elementary functions are dense in the class of Itô integrable functions with respect to the L^2 norm.*

Proof. See Ch 3.1 of Øksendal [3]. □

Definition 18. Suppose that $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is Itô integrable and ϕ_n is a sequence of elementary functions converging to f in $L^2(\mathbb{R}_+ \times \Omega)$, i.e. $\mathbb{E}\left[\int_S^T (\phi_n - f)^2\, dt\right] \to 0$.

Since (ϕ_n) is a Cauchy sequence then by the Itô isometry, and as $n, m \to \infty$

$$\mathbb{E}\left[\left(\int_S^T (\phi_n - \phi_m)\, dB_t\right)^2\right] = \mathbb{E}\left[\int_S^T (\phi_n - \phi_m)^2\, dt\right] \to 0$$

i.e. the sequence $\left(\int_S^T \phi_n dB_t\right)$ forms a Cauchy sequence in $L^2(\Omega)$, which is complete and therefore has a limit.

Then the *Itô integral* for f is defined as the L^2-limit

$$\int_S^T f(t)\, dB_t \equiv \lim_{n\to\infty} \int_S^T \phi_n(t)\, dB_t.$$

This makes the Itô isometry and 15 true for all Itô integrable functions.

Definition 19. A stochastic process X_t is an *Itô process* if

$$X_t - X_0 = \int_0^t U_s\, dB_s + \int_0^t V_s\, ds$$

which for short is written as

$$dX_t = U_t\, dB_t + V_t\, dt.$$

Theorem 20. *(Itô formula) If X_t is an Itô process, i.e. $dX_t = U_t dB_t + V_t dt$, $g(x)$ is C^2, and $Y_t \equiv g(X_t)$, then Y_t is an Itô process and*

$$dY_t = g'(X_t)\, dX_t + \frac{1}{2} g''(X_t)\, d\langle X_t, X_t\rangle$$

where $d\langle X_t, X_t\rangle \equiv d\langle X_t\rangle = V_t^2 dt$.

Proof. See p. 46-48 of Øksendal [3]. □

If $B_t(\omega) = \left(B_t^1, \ldots, B_t^m\right)$ is an m-dimensional Brownian motion, $U_t(\omega)$ an n-dimensional column vector, $V_t(\omega)$ an $n \times m$-matrix, and $X_t = \left(X_t^1, \ldots, X_t^n\right)$ satisfies

$$dX(t) = U_t\, dt + V_t \cdot dB_t$$

then X_t is an n-dimensional Itô process.

Theorem 21. *(Multidimensional Itô formula) If X_t is an n-dimensional Itô process, i.e. $dX(t) = U_t\, dt + V_t \cdot dB_t$, $g : \mathbb{R}^n \to \mathbb{R}^p$ is C^2 and $Y_t \equiv g(X_t)$, then Y_t is an Itô process with*

$$dY_k = \sum_i \frac{\partial g_k}{\partial x_i}(X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(X)\, dX_t^i \cdot dX_t^j$$

where $dB_t^i \cdot dB_t^j = \delta_{ij} dt$ and $dB_t^i dt = 0$.

4 Harmonic Functions

A subset $D \subset \mathbb{R}^n$ is a *domain* if it is open and path connected. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *harmonic* on D if f is C^2 and

$$\Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i^n \frac{\partial^2 f}{\partial x_i^2} = 0$$

everywhere on D.

A function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the *mean value property* on a domain D if, for all $x \in D$ and $0 < \epsilon < \operatorname{dist}(x, \partial D) \equiv \inf\{|x-y| : y \in \partial D\}$

$$f(x) = \int_{\partial B} f(x+\epsilon y) dA(y) \equiv MV(f, x, \epsilon)$$

where $dA(y)$ is the normalized surface measure for the unit ball B, i.e. $\int_{\partial B} dA(y) = 1$. That is, $f(x)$ is the average of the values of f on any sphere around x (contained wholly in D).

Proposition 22. *If a continuous $f : D\left(\subset \mathbb{R}^d\right) \to \mathbb{R}$ satisfies the mean value property then f is $C^\infty(D)$.*

Proof. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be spherically symmetric, C^∞, supported on $B = \{x : |x| < 1\}$ and $\int_{\mathbb{R}^d} \phi(x)\, d^d x = 1$. (Such a function exists). Then $\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right)$ is supported on ϵB.

Since f satisfies the mean value property and ϕ is spherically symmetric, for small enough ϵ

$$f(x) = \int_{\mathbb{R}^d} f(x+y)\, \phi_\epsilon(y)\, d^d y = \int_{\mathbb{R}^d} f(y)\, \phi_\epsilon(y-x)\, d^d y.$$

Therefore

$$\frac{d^n f(x)}{dx^n} = \int_{\mathbb{R}^d} f(y) \frac{d^n}{dx^n} \phi_\epsilon(y-x)\, d^d y$$

and so f is C^∞. □

Theorem 23. *A function $f : D\left(\subset \mathbb{R}^d\right) \to \mathbb{R}$ is harmonic on a domain D if and only if it satisfies the mean value property on D.*

Proof. Let $B = \{x : |x| < 1\}$. First, suppose that a continuous f satisfies the mean value property. Then, for all small enough ϵ, $MV(f, x, \epsilon) = f(x)$ and, for all small enough δ,

$$0 = \frac{MV(f, x, \epsilon+\delta) - MV(f, x\epsilon)}{\delta} = \int_{\partial B} \frac{f(x+(\epsilon+\delta)y) - f(x+\epsilon y)}{\delta} dA(y).$$

If f is C^2 then ∇f is continuous on D and so

$$\frac{f(x+(\epsilon+\delta)y) - f(x+\epsilon y)}{\delta}$$

is bounded. Then by the dominated convergence theorem

$$\begin{aligned} 0 = \frac{d}{d\epsilon} MV(f, x, \epsilon) &= \lim_{\delta \to 0} \int_{\partial B} \frac{f(x + (\epsilon + \delta) y) - f(x + \epsilon y)}{\delta} \, dA(y) \\ &= \int_{\partial B} \frac{d}{d\epsilon} f(x + \epsilon y) \, dA(y) \\ &= \int_{\partial B} y \cdot \nabla f(x + \epsilon y) \, dA(y). \end{aligned}$$

By the divergence theorem

$$\int_{\partial B} y \cdot \nabla f(x + \epsilon y) \, dA(y) = \int_B \Delta f(x + \epsilon y) \, dV = 0$$

since $y \in \partial B$, for all small ϵ.

If, for some $x \in D, \Delta f(x) = c > 0$, then since f is C^2 and so Δf is continuous, then $\Delta f(x) > 0$ on some small ball around x, say of radius δ, and so $\int_B \Delta f(x + \delta y) \, dV > 0$. Therefore $\Delta f(x) = 0$ for all $x \in D$ so f is harmonic.

Now, suppose that f is harmonic. Then

$$\frac{d}{d\epsilon} MV(f, x, \epsilon) = \int_B \Delta f(x + \epsilon y) \, dV = 0$$

for all small enough ϵ and all $x \in D$. Then since $MV(f, x, 0) = f(x)$, $MV(f, x, \epsilon) = f(x)$ for all small enough ϵ for all $x \in D$. □

Theorem 24. *(Maximum principle). Suppose D is a bounded domain and f is harmonic on D and continuous on $\overline{D}$. Then*

$$\max_{\partial D} f = \max_{\overline{D}} f.$$

Proof. Since $\overline{D}$ and ∂D are closed and bounded, both f achieves a maximum on both. Suppose that $x \in D$ is a maximal point for f.

Then $|p - x|$ for $p \in \partial D$ also achieves a minimum, say d at $p \in \partial D$. Let $d_n \nearrow d$. Then the balls $B(x, d_n) \subset D$ and let

$$y_n = x + \frac{d_n}{d}(p - x)$$

and so

$$y_n - p = \left(1 - \frac{d_n}{d}\right)(x - p)$$

so $y_n \to p$ as $n \to \infty$ and $|y_n - x| = d_n$ so y lies on the sphere of $B(x, d_n)$.

Since f is harmonic it satisfies the mean value property, i.e.

$$\max_{\overline{D}} f = f(x) = \int_{\partial B} f(x + d_n z) dA(z).$$

But this implies that f must equal $f(x)$ on the sphere of $B(x, d_n)$ and so $f(y_n) = \max_{\overline{D}} f$ also. Then since f is continuous $f(y_n) \to f(p) = \max_{\overline{D}} f$, so

$$\max_{\partial D} f = \max_{\overline{D}} f.$$

□

5 Dirichlet Problem

Suppose that D is a domain and $F : \partial D \to \mathbb{R}$. The Dirichlet problem is: does there exist a $f : D \to \mathbb{R}$ such that $\Delta f = 0$ on D, and f is continuous up to the boundary and approaches F, i.e. if $x_n \to x \in \partial D$ then $f(x_n) \to F(x)$?

We would like to solve this problem with Brownian motion. In particular, if B_t is a Brownian motion and $\tau_D = \inf\{t : B_t \notin D\}$, i.e. the time B_t leaves D, then $f(x) = \mathbb{E}^x F(B_{\tau_D})$ is a solution, but only for some boundary conditions.

Example 25. If $D = \mathbb{D} \setminus \{0\} \subset \mathbb{R}^2$, then $f(x) = \log|x|$ is harmonic on D and $f(x) = 0$ on $\partial\mathbb{D}$ and $f(x) \to -\infty$ as $x \to 0$. But Brownian motion hits single points with probability 0, so $\mathbb{E}^x F(B_{\tau_D}) = 0$. Therefore, changing F at single points will not change $\mathbb{E}^x F(B_{\tau_D})$.

On a domain D let B_t be a Brownian motion and define

$$f(x) = \mathbb{E}^x F(B_{\tau_D})$$

on $\overline{D}$, which is the expected value of F when B_t leaves the boundary, starting at x. If $B_\epsilon(x)$, the ball of radius ϵ around x, is inside D, let $S = \inf\{t : B_t \notin B_\epsilon(x)\}$, i.e. the time when B_t leaves the ball. By the a.s. continuity of B_t, $S < \tau_D$ a.s. too.

Let $\hat{B}_t = B_{t+S}$, and so $\hat{B}_t - B_S$ is a B.M. by the strong Markov property, and $\hat{\tau}_D = \inf\left\{t : \hat{B}_t \notin D\right\}$. Since $S < \tau_D$, the time shifted Brownian motion leaves D in the same place as B_t, i.e. $B_{\tau_D} = \hat{B}_{\hat{\tau}_D}$. By the Tower Rule

$$\begin{aligned} f(x) = \mathbb{E}^x F(B_{\tau_D}) &= \mathbb{E}^x F\left(\hat{B}_{\hat{\tau}_D} - B_S + B_S\right) \\ &= \mathbb{E}^x \mathbb{E}^x \left[F\left(\hat{B}_{\hat{\tau}_D} - B_S + B_S\right) \mid \mathcal{F}_S\right] \\ &= \mathbb{E}^x \mathbb{E}^{B_S} F\left(\hat{B}_{\hat{\tau}_D}\right) \\ &= \mathbb{E}^x f(B_S) \end{aligned}$$

since $f(B_S) = \mathbb{E}^{B_S} F\left(\hat{B}_{\hat{\tau}_D}\right)$. Moreover Brownian motion is spherically symmetric and so $\mathbb{E}^x f(B_S) = MV(f, x, \epsilon)$. Therefore f satisfies the mean value property in D and so is harmonic.

Proving continuity up to the boundary is harder, and here we'll consider just $\mathbb{R}^2$ (and $\mathbb{C}$).

Definition 26. A point $z \in \partial D$ is a *connected boundary point* if it is contained in a connected subset of ∂D that is not just a point.

Lemma 27. *If ∂D contains at least two points which are connected, then $\tau_D < \infty$ almost surely.*

Though we will not give a formal proof of this, it follows from the fact that Brownian motion has a positive probability of making a loop which separates the two connected boundary points. This loop must leave D. A corollary of this is the following:

Proposition 28. *In $\mathbb{R}^2$, if $z \in \partial D$ is a connected boundary point and $F : \partial D \to \mathbb{R}$ is continuous at z, then f is continuous up to z and approaches $F(z)$.*

Then if D is such that every $z \in \partial D$ is a connected boundary point, then $f(x) = \mathbb{E}^x F(B_{\tau_D})$ is harmonic on D, and if $x_n \to x \in \partial D$, then $f(x_n) = \mathbb{E}^{x_n} F(B_{\tau_D}) \to F(x)$ and so f solves the Dirichlet problem.

6 Lévy's Theorem

Definition 29. A stochastic process X_t is a *local martingale* for $t \leq \tau$, where τ is a stopping time $\mathcal{F}_t$-stopping time, if

- there exists a sequence of stopping times τ_n such that $\tau_n \leq \tau_{n+1}$ a.s. and $\tau_n \to \tau$ a.s.
- and for all n, the stochastic process $(X_{t\wedge\tau_n})_{t\geq 0}$ is a martingale.

Theorem 30. *If $f : \mathbb{R}^n \to \mathbb{R}$ is harmonic on D and $B_t = \left(B_t^1, \ldots, B_t^n\right)$ is a $\mathcal{M}_t$-Brownian motion then $f(B_t)$ is a local martingale for $t \leq \tau_D = \inf\{B_t \in \partial D\}$.*

Proof. Define the stopping times $\tau_n = \inf\{t : \text{dist}(B_t, \partial D) \leq 1/n\}$. Then $B^i_{t\wedge\tau_n} = \int_0^t \mathbf{1}_{[0,\tau_n]}(s) dB^i_s = \int_0^{t\wedge\tau_n} dB^i_s$ is an Itô process, i.e. $dB^i_{t\wedge\tau_n} = \mathbf{1}_{[0,\tau_n]} dB^i_t$. For all t, $B_{t\wedge\tau_n} \in D$.

Also, $dB^i_{t\wedge\tau_n} \cdot dB^j_{t\wedge\tau_n} = \mathbf{1}_{[0,\tau_n]} dB^i_t \cdot \mathbf{1}_{[0,\tau_n]} dB^j_t = \mathbf{1}_{[0,\tau_n]} \delta_{ij} t$.

Then since f is harmonic on D, $\Delta f = 0$, and so by the multidimensional Itô formula

$$
\begin{aligned}
df\left(B_{t\wedge\tau_n}\right) &= \sum_i \frac{\partial f}{\partial x_i}\left(B_{t\wedge\tau_n}\right) dB^i_{t\wedge\tau_n} + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}\left(B_{t\wedge\tau_n}\right) dB^i_{t\wedge\tau_n}\cdot dB^j_{t\wedge\tau_n} \\
&= \sum_i \frac{\partial f}{\partial x_i}\left(B_{t\wedge\tau_n}\right)\mathbf{1}_{[0,\tau_n]} dB^i_t + \frac{1}{2}\sum_i \frac{\partial^2 f}{\partial x_i^2}\left(B_{t\wedge\tau_n}\right)\mathbf{1}_{[0,\tau_n]} t \\
&= \mathbf{1}_{[0,\tau_n]}\nabla f\left(B_{t\wedge\tau_n}\right)\cdot dB_t.
\end{aligned}
$$

Then by Proposition 15, that Itô integrals are martingales, $f\left(B_{t\wedge\tau_n}\right)$ is a $\mathcal{M}_t$-martingale.

Since B.M. is continuous a.s. then $\tau_n \leq \tau_{n+1}$ a.s. and, for a given ω, $B_t(\omega)$ cannot go arbitrarily close to the boundary without also hitting it, so $\tau_n \to \tau$ almost surely.

Therefore $f(B_t)$ is a local martingale. □

6.1 Time changes

Suppose X_t is an Itô process given by $dX_t = U_t dB_t + V_t dt$. Let $\tilde{X}_u = X_{\sigma^{-1}(u)}$ where u is the new timescale, given by $u = \sigma(t)$.

Proposition 31. *Assume that $\sigma(0) = 0$ and $\sigma'(t) > 0$, and so the ordering of times is unchanged. The time changed $\tilde{X}_u$ is an Itô process and*

$$
d\tilde{X}_u = U_{\sigma^{-1}(u)}\sqrt{\left(\sigma^{-1}\right)'(u)}dW_u + V_{\sigma^{-1}(u)}\left(\sigma^{-1}\right)'(u)\,du
$$

where $W_u = \int_0^{\sigma^{-1}(u)}\sqrt{\sigma'(s)}dB_s$ is a Brownian motion.

Proof. That $V_t dt$ transforms to $V_{\sigma^{-1}(u)}\left(\sigma^{-1}\right)'(u)\,du$ is just due to the chain rule. Clearly W_u is an Itô process with no drift term, and so is a local martingale by Proposition 15.

By the Itô isometry,

$$
\begin{aligned}
\mathbb{E}W_u^2 &= \mathbb{E}\left[\left(\int_0^{\sigma^{-1}(u)}\sqrt{\sigma'(s)}dB_s\right)^2\right] \\
&= \mathbb{E}\left[\int_0^{\sigma^{-1}(u)}\sigma'(s)\,ds\right] \\
&= \mathbb{E}\left[\int_0^{\sigma^{-1}(u)}\sigma'(s)\,ds\right] \\
&= \mathbb{E}u = u
\end{aligned}
$$

Also $W_0 = 0$. By Theorem 32, W_u is a a Brownian motion.

Then since $u = \sigma(t)$,

$$\begin{aligned}\int_0^{\sigma^{-1}(u)} U_s\,dB_s &= \int_0^{\sigma^{-1}(u)} U_s \frac{\sqrt{\sigma'(s)}}{\sqrt{\sigma'(s)}}\,,dB_s \\ &= \int_0^{\sigma^{-1}(u)} \frac{U_{\sigma^{-1}(v)}}{\sqrt{\sigma'(\sigma^{-1}(v))}}\,dW_v \\ &= \int_0^{\sigma^{-1}(u)} U_{\sigma^{-1}(v)}\sqrt{(\sigma^{-1})'(v)}\,dW_v\end{aligned}$$

and so

$$d\tilde{X}_u = dX_{\sigma^{-1}(u)} = U_{\sigma^{-1}(u)}\sqrt{(\sigma^{-1})'(u)}dW_u + V_{\sigma^{-1}(u)}\left(\sigma^{-1}\right)'(u)\,du.$$

□

Theorem 32. *Let $X_t = \left(X_t^1, \ldots, X_t^d\right)$ be a d-dimensional stochastic process. If X_t^i is a continuous local $\mathcal{M}_t$-martingale, $X_0 = 0$ and $\left\langle X_t^i, X_t^j\right\rangle = \delta_{ij}t$, then X_t is a d-dimensional Brownian motion.*

Proof. Omitted. See Theorem 5.9 and Corollary 5.10 of Bass [1]. □

Definition 33. A function $f : D \to D'$ where $D, D' \subset \mathbb{C}$ are domains is *conformal* if it is holomorphic and one-to-one.

If f is conformal, then $f'(z) \neq 0$ for all $z \in D$.

Theorem 34. Lévy's Theorem *(Conformal Invariance of Brownian Motion). Let $D \subset \mathbb{C}$ be a domain, $f : D \to f(D)$ be a non-constant onto conformal map, and $B_t = B_t^1 + iB_t^2$ a complex Brownian motion. Let*

$$\sigma(t) \equiv \int_0^t \left|f'(B_s)\right|^2 ds.$$

Then $X_u = f\left(B_{\sigma^{-1}(u)}\right)$ is a complex Brownian motion and

$$dX_u = \frac{f'\left(B_{\sigma^{-1}(u)}\right)}{\left|f'\left(B_{\sigma^{-1}(u)}\right)\right|}dW_u$$

where $W_u = \int_0^{\sigma^{-1}(u)} \sqrt{\sigma'(s)}dB_s = \int_0^{\sigma^{-1}(u)} |f'(B_s)|\,dB_s$.

Proof. Let $f(z) = u(z) + iv(z)$ where $u, v : D \to \mathbb{R}$.

If f is holomorphic then u and v are harmonic, so by It's formula

$$\begin{aligned} du\left(B_t\right) &= \nabla u\left(B_t\right)\cdot dB_t \\ &= u_x dB_t^1 + iu_y dB_t^2 \\ &= u_x dB_t^1 - iv_x dB_t^2 \\ dv\left(B_t\right) &= \nabla v\left(B_t\right)\cdot dB_t \\ &= v_x dB_t^1 + iv_y dB_t^2 \\ &= v_x dB_t^1 + iu_x dB_t^2 \end{aligned}$$

Since f satisfies the Cauchy-Riemann equations, $u_x = v_y$ and $u_y = -v_x$. Also, $f' = u_x + iv_x$. Therefore

$$\begin{aligned} df\left(B_t\right) &= u_x dB_t^1 - iv_x dB_t^2 + i\left[v_x dB_t^1 + iu_x dB_t^2\right] \\ &= \left[u_x + iv_x\right] dB_t^1 + i\left[u_x + iv_x\right] dB_t^2 \\ &= f'\left(B_t\right)\left[dB_t^1 + i dB_t^2\right] \\ &= f'\left(B_t\right) dB_t \end{aligned}$$

By Proposition 31 W_u is a complex Brownian motion and

$$dX_u = \frac{f'\left(B_{\sigma^{-1}(u)}\right)}{\left|f'\left(B_{\sigma^{-1}(u)}\right)\right|} dW_u$$

is an It process. Note that $f'\left(B_{\sigma^{-1}(u)}\right)/\left|f'\left(B_{\sigma^{-1}(u)}\right)\right|$ is a phase, say, $\exp i\theta\left(u,\omega\right)$, and $X_0 = 0$. If $W_u = W_u^1 + iW_u^2$ and $X_u = X_u^1 + iX_u^2$, then

$$\begin{aligned} X_u^1 &= \int_0^u \cos\theta\left(v\right) dW_v^1 - \int_0^u \sin\theta\left(v\right) dW_v^2 \\ X_u^2 &= \int_0^u \sin\theta\left(v\right) dW_v^1 + \int_0^u \cos\theta\left(v\right) dW_v^2. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}\left[\left(X_u^1\right)^2\right] &= \mathbb{E}\left[\left(\int_0^u \cos\theta\left(v,\omega\right) dW_v^1 - \int_0^u \sin\theta\left(v,\omega\right) dW_v^2\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_0^u \cos\theta dW_v^1\right)^2\right] \\ &\quad - 2\mathbb{E}\left[\left(\int_0^u \cos\theta dW_v^1\right)\left(\int_0^u \sin\theta dW_v^2\right)\right] \\ &\quad + \mathbb{E}\left[\left(\left(\int_0^u \sin\theta dW_v^2\right)\right)^2\right] \end{aligned}$$

Since B_t^1 and B_t^2 are independent, so are W_u^1 and W_u^2, and $\int_0^u \cos\theta dW_v^1$ and $\int_0^u \sin\theta dW_v^2$. Therefore, by the It isometry

$$\begin{aligned}\mathbb{E}\left[\left(X_u^1\right)^2\right] =& \mathbb{E}\left[\left(\int_0^u \cos\theta dW_v^1\right)^2\right] + \mathbb{E}\left[\left(\left(\int_0^u \sin\theta dW_v^2\right)\right)^2\right] \\ =& \mathbb{E}\left[\int_0^u \cos^2\theta dv\right] + \mathbb{E}\left[\int_0^u \sin^2\theta dv\right] \\ =& \mathbb{E}\left[\int_0^u 1dv\right] = u.\end{aligned}$$

Similarly $\mathbb{E}\left[\left(X_u^1\right)^2\right] = u$. Finally, the covariance is zero:

$$\begin{aligned}\mathbb{E}\left[X_u^1 X_u^2\right] &= \mathbb{E}\left[\left(\int_0^u \cos\theta\left(v\right)dW_v^1 - \int_0^u \sin\theta\left(v\right)dW_v^2\right)\left(\int_0^u \sin\theta\left(v\right)dW_v^1 + \int_0^u \cos\theta\left(v\right)dW_v^2\right)\right] \\ &= \mathbb{E}\left[\int_0^u \cos\theta\left(v\right)dW_v^1 \int_0^u \sin\theta\left(v\right)dW_v^1\right] + \mathbb{E}\left[\int_0^u \cos\theta\left(v\right)dW_v^1 \int_0^u \cos\theta\left(v\right)dW_v^2\right] \\ &\quad - \mathbb{E}\left[\int_0^u \sin\theta\left(v\right)dW_v^2 \int_0^u \sin\theta\left(v\right)dW_v^1\right] - \mathbb{E}\left[\int_0^u \sin\theta\left(v\right)dW_v^2 \int_0^u \cos\theta\left(v\right)dW_v^2\right] \\ &= 0\end{aligned}$$

and so by Theorem 32, X_u is a standard complex Brownian motion. □

7 Harmonic Measure

For $z \in D \subseteq \mathbb{R}^d$ and $E \subset \partial D$, let $\tau_D = \inf\{t > 0 : B_t \notin D\}$, i.e. the time B_t leaves D. The *harmonic measure* is

$$H_D(z, E) = \mathbb{P}^z\left(B_{\tau_D} \in E\right) = \mathbb{E}^z \mathbf{1}_E\left(B_{\tau_D}\right).$$

Since $\mathbf{1}_E$ is bounded on ∂D then $H_D\left(z, E\right)$ is harmonic in z.

It is also the distribution of the random variable B_{τ_D} and so is a probability distribution.

Example 35. Consider the open annulus $D = \{r < |z| < R\}$ in $\mathbb{R}^2$. What is the probability of a Brownian motion leaving through the inner radius, i.e. what is $H_D\left(z, r\mathbb{T}\right)$ (where $\mathbb{T}$ is the unit circle) for some $z \in D$?

Let

$$u(z) = \frac{\log R - \log|z|}{\log R - \log r}.$$

Note that $\Delta \log|z| = 0$ in $\mathbb{R}^2 \setminus \{0\}$, so $\Delta u(z) = 0$ on D. If $|z| = r$ then $u(z) = 1$ and if $|z| = R$ then $u(z) = 0$.

Then, by uniqueness, $H_D\left(z, r\mathbb{T}\right) = u(z)$.

8 Green's Functions

Given a domain $D \subset \mathbb{R}^2$ (or $\mathbb{C}$), a Green's function on D, $G_D(x, y)$ is characterized by

1. $G_D(x, y)$ is C^2 and $\Delta G_D(x, y) = 0$ if $x \neq y$,
2. $G_D(x, y) = 0$ if x or y are in ∂D,
3. $\Delta G_D(x, y)$ acts like the delta function. More precisely, if $g \in C^\infty$ and compactly supported inside D, then $\int_D \Delta g(x) G_D(x, y) d^2x = -2\pi\delta g(y)$.

These imply $G_D(x, y) = G_D(y, x)$. Then if ρ is C^∞ and compactly supported inside D (and possibly with weaker assumptions), then

$$f(x) = \int_D \rho(y) G_D(y, x) d^2y$$

solves the general Dirichlet problem $\Delta f = \rho$ on D and $f|_{\partial D} = 0$.

Here we will define the Green's function on D probabilistically. The existence proof uses an idea due to Lawler.

Theorem 36. *Let $D \subset \mathbb{C}$ be a domain and suppose $z, w \in D$. Then the limit*

$$\begin{aligned} G_D(z, w) &= -\lim_{\epsilon \to 0} \log \epsilon \cdot H_{D \setminus \overline{B_\epsilon(w)}}(z, \partial B_\epsilon(w)) \\ &= -\lim_{\epsilon \to 0} \log \epsilon \cdot \mathbb{P}^z \left\{ \tau_{\partial B_\epsilon(w)} < \tau_{\partial D} \right\} \end{aligned}$$

exists and is called the Green's function *on D.*

By translation invariance, it is enough to show that $G_D(z, 0)$ exists.

Also, let $D_u = D \setminus \overline{e^{-u}\mathbb{D}}$ and $\mathbb{D}_u = \mathbb{D} \setminus \overline{e^{-u}\mathbb{D}}$.

Lemma 37. *If D is a domain such that ∂D contains at least two connected points then*

$$H_{D_u}\left(e^{i\theta}, e^{-u}\mathbb{T}\right) \leq \frac{K}{u}$$

for all θ, and where K depends only on D.

Proof. By scaling invariance assume that $e\mathbb{D} \subset D$. For a Brownian motion starting at $e^{i\theta}$ and ending at $e^{-u}\mathbb{T}$, like in Figure 8.1, either it stays entirely within $e\mathbb{D}$ or leaves $e\mathbb{D}$ and then hits $e^{-u}\mathbb{T}$, after passing through the unit circle.

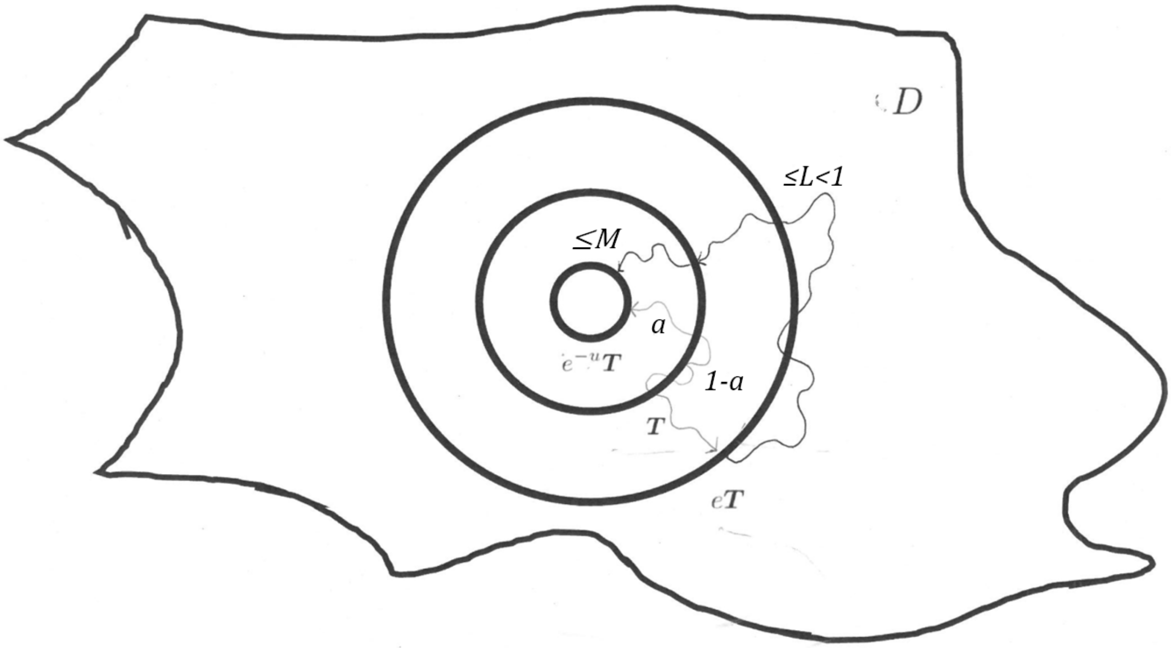

Figure 8.1:

Let

$$a = H_{e\mathbb{D}\setminus\overline{e^{-u}\mathbb{D}}}\left(e^{i\theta}, e^{-u}\mathbb{T}\right)$$

i.e. the probability that a Brownian motion starting at $e^{i\theta}$ hits $e^{-u}\mathbb{T}$ before $e\mathbb{T}$,

$$M = \max_{\theta} H_{D_u}\left(e^{i\theta}, e^{-u}\mathbb{T}\right)$$

i.e. the maximum probability that a Brownian motion starting on $\mathbb{T}$ exits through $e^{-u}\mathbb{T}$,

$$L = \max_{\theta} H_{D_0}\left(e^{1+i\theta}, \mathbb{T}\right)$$

i.e. the maximum probability that a Brownian motion starting on $e\mathbb{T}$ will exit through $\mathbb{T}$.

Then

$$H_{D_u}\left(e^{i\theta}, e^{-u}\mathbb{T}\right) \leq a + (1-a)\,ML.$$

Since this is true for all θ and since H_{D_u} is continuous, then $M \leq a + (1-a)\,ML$. Therefore $M \leq \frac{a}{1-(1-a)L}$. From Example 35, $a = \frac{1}{u+1}$. Also, since ∂D contains a connected component, then for any Brownian motion starting on $e\mathbb{T}$, there is some positive probability that the B.M. will hit this component before it hits $\mathbb{T}$, so $L < 1$. Then

$$M \leq \frac{1}{1+u\,(1-L)} \leq \frac{(1-L)^{-1}}{u}.$$

□

We've used the strong Markov properties implicitly above in restarting the Brownian motion at the circles $\mathbb{T}$ and $e\mathbb{T}$.

Proof. Let $\sigma_u = \tau_{e^{-u}\mathbb{T}} = \inf\{t : |B_t| = e^{-u}\}$. Then

$$\begin{aligned}\mathbb{P}^{e^{i\theta}}\left\{B_{\tau_{D_u}} \in e^{-u}\mathbb{T}\right\} &= \mathbb{P}^{e^{i\theta}}\{\sigma_u < \tau_D\} \\ &= \mathbb{P}^{e^{i\theta}}\{\sigma_u < \sigma_{-1}\} + \mathbb{P}^{e^{i\theta}}\{\sigma_{-1} < \sigma_u \& \sigma_u < \tau_D\} \\ &= \frac{1}{u+1} + \mathbb{P}^{e^{i\theta}}\{\sigma_{-1} < \sigma_u\}\mathbb{P}^{e^{i\theta}}\{\sigma_u < \tau_D | \sigma_{-1} < \sigma_u\}.\end{aligned}$$

By the strong Markov property, if $\tilde{B_S} = B_{\sigma_{-1}+S}$ then $\tilde{\sigma_u} = \inf\left\{t : \left|\tilde{B_s}\right| = e^{-u}\right\} = \sigma_u - \sigma_{-1}$. Then

$$\begin{aligned}\mathbb{P}^{e^{i\theta}}\{\sigma_u < \tau_D\} &= \frac{1}{u+1} + \frac{u}{u+1}\mathbb{P}^{B_{\sigma_{-1}}}\{\tilde{\sigma_0} < \tilde{\sigma_u} < \tilde{\tau_D}\} \\ &= \frac{1}{u+1} + \frac{u}{u+1}\mathbb{P}^{B_{\sigma_{-1}}}\{\tilde{\sigma_0} < \tilde{\tau_D}\}\mathbb{P}^{B_{\sigma_{-1}}}\{\tilde{\sigma_u} < \tilde{\tau_D} | \tilde{\sigma_0} < \tilde{\tau_D}\} \\ &\le \frac{1}{u+1} + \frac{u}{u+1}L\mathbb{P}^{\hat{B}_{\tilde{\sigma_0}}}\{\hat{\sigma_u} < \hat{\tau_D}\}\end{aligned}$$

And so $M \le a + (1-a)LM$. Then the proof proceeds as before to show

$$M \le \frac{1}{1+u(1-L)} \le \frac{(1-L)^{-1}}{u}.$$

□

Proof. (Theorem 36)

The proof of Theorem 36 is very similar, except now one bound is tighter.

Again, for a Brownian motion starting at $e^{-u+i\theta}$ and ending at $e^{-v}\mathbb{T}$ and if $u < v$, like in Figure 8.2, either it stays within $\mathbb{D}$ or leaves $\mathbb{D}$ and then hits $e^{-v}\mathbb{T}$, after passing through $e^{-u}\mathbb{T}$.

Let

$$a = H_{\mathbb{D}_v}\left(e^{-u+i\theta}, e^{-v}\mathbb{T}\right)$$

i.e. the probability that a Brownian motion starting at $e^{-u+i\theta}$ hits $e^{-v}\mathbb{T}$ before $\mathbb{T}$,

$$M = \max_\theta H_{D_v}\left(e^{-u+i\theta}, e^{-v}\mathbb{T}\right)$$

i.e. the maximum probability that a Brownian motion starting on $e^{-u}\mathbb{T}$ exits through $e^{-v}\mathbb{T}$,

$$L = \max_\theta H_{D_u}\left(e^{i\theta}, e^{-u}\mathbb{T}\right) \le \frac{K}{u}$$

i.e. the maximum probability that a Brownian motion starting on $\mathbb{T}$ will exit through $e^{-u}\mathbb{T}$.

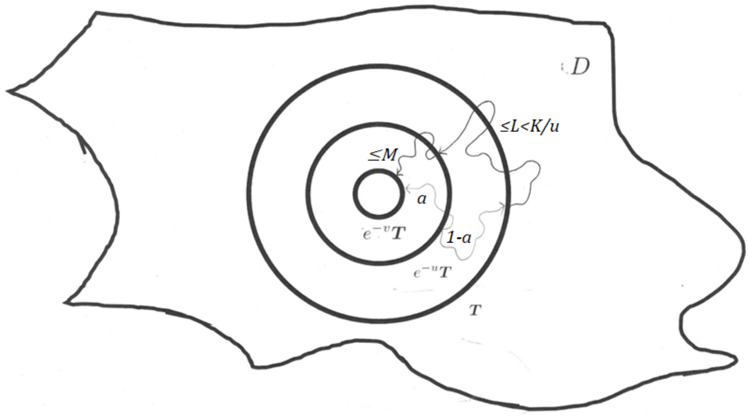

Figure 8.2:

By the same reasoning,

$$H_{D_v}\left(e^{-u+i\theta}, e^{-v}\mathbb{T}\right) \leq a + (1-a)\, ML.$$

Again, since this is true for all θ, and since $a = u/v$, then $M \leq a + (1-a)\, ML$ and so

$$M \leq \frac{a}{1-(1-a)\, L} \leq \frac{u/v}{1-K/u}.$$

Of course, to find $H_{D_v}(z, e^{-v}\mathbb{T})$ we start at a $z \in D$. If a Brownian motion starting at z hits $e^{-v}\mathbb{T}$ it must pass through $e^{-u}\mathbb{T}$ and so

$$H_{D_v}\left(z, e^{-v}\mathbb{T}\right) \leq M \cdot H_{D_u}\left(z, e^{-u}\mathbb{T}\right).$$

The probability of going from $e^{-u}\mathbb{T}$ to $e^{-v}\mathbb{T}$ is in D greater than the probability inside $\mathbb{D}\,(\subset D)$ so

$$H_{D_v}\left(z, e^{-v}\mathbb{T}\right) \geq \frac{u}{v} \cdot H_{D_u}\left(z, e^{-u}\mathbb{T}\right).$$

Therefore

$$uH_{D_u}\left(z, e^{-u}\mathbb{T}\right) \leq vH_{D_v}\left(z, e^{-v}\mathbb{T}\right) \leq \frac{u}{1-K/u} \cdot H_{D_u}\left(z, e^{-u}\mathbb{T}\right).$$

Note that the middle term depends on v and not u, and the sandwiching terms depend on u and not v. Moreover, the sandwiching terms become close together as $u \to \infty$, so $vH_{D_v}(z, e^{-v}\mathbb{T})$ is a Cauchy sequence and so has a limit as $v \to \infty$.

In particular,

$$\left|u - \frac{u}{1-K/u}\right| = \left|\frac{K}{1-K/u}\right| \leq 2K$$

for all large enough u. Also as $u \to \infty$, $H_{D_u}(z, e^{-u}\mathbb{T}) \to 0$ by Lemma 37. □

Proposition 38. *Let U be some neighborhood of 0 in D. Then $G_D(z,0)+\log|z|$ is bounded on U, i.e. 0 is a removable singularity. Therefore $G_D(z,0)+\log|z|$ can be extended to be continuous on all of D.*

Proof. Without loss of generality assume that $\mathbb{D}\subset D$. Then for $|z|<1$

$$-\frac{\log|z|}{v}\leq H_{D_v}\left(z,e^{-v}\mathbb{T}\right)\leq-\frac{\log|z|}{v}+\frac{K}{v}$$

i.e. the probability of hitting $e^{-v}\mathbb{T}$ from z is greater than if the Brownian motion is contained within the disk, and is equal to going in inside the disk or, starting at $\mathbb{T}$ going to $e^{-v}\mathbb{T}$, which is bounded by K/v. Then

$$0\leq vH_{D_v}\left(z,e^{-v}\mathbb{T}\right)+\log|z|\leq K$$

and so by taking limits, for all $0<|z|<1$

$$0\leq G_D(z,0)+\log|z|\leq K.$$

□

Proposition 39. *For all $z\in D\setminus\{0\}$, $G_D(z,0)$ is harmonic.*

Proof. Let $f(z)=G_D(z,0)$ and

$$f_n(z)=nH_{D_n}\left(z,e^{-n}\mathbb{T}\right)$$

and so for all $z\in D\setminus\{0\}$, $f_n(z)\to f(z)$. From the proof of Proposition 36 for all $n>u$ and for some large enough u,

$$f_n(z)=nH_{D_n}\left(z,e^{-n}\mathbb{T}\right)\leq H_{D_u}\left(z,e^{-u}\partial\mathbb{D}\right)\frac{u}{1-K/u}\leq\frac{u}{1-K/u}.$$

The harmonic measure is harmonic in z and so satisfies the mean value property. So for all sufficiently small δ

$$f_n(z)=\int_0^{2\pi}nH_{D_n}\left(z+e^{\delta+i\theta},e^{-n}\partial\mathbb{D}\right)\frac{d\theta}{2\pi}.$$

By the dominated convergence theorem

$$\begin{aligned}f(z)=\lim_{n\to\infty}f_n(z)&=\lim_{n\to\infty}\int_0^{2\pi}nH_{D_n}\left(z+e^{\delta+i\theta},e^{-n}\partial\mathbb{D}\right)\frac{d\theta}{2\pi}\\&=\int_0^{2\pi}\lim_{n\to\infty}nH_{D_n}\left(z+e^{\delta+i\theta},e^{-n}\partial\mathbb{D}\right)\frac{d\theta}{2\pi}\\&=\int_0^{2\pi}G_D\left(z+e^{\delta+i\theta},0\right)\frac{d\theta}{2\pi}\end{aligned}$$

and so f satisfies the mean value theorem for all $z\in D$, so f is harmonic. □

Lemma 40. *Suppose that u is harmonic on a bounded domain D, continuous on $\overline{D}$ and $u = f$ on ∂D. Then u is unique.*

Proof. Suppose that v is also harmonic on D, continuous on $\overline{D}$ and $v = f$ on ∂D. Then by the maximum principle,

$$\max_{\overline{D}} (u - v) = \max_{\partial D} (u - v) = \max_{\partial D} (f - f) = 0$$

so $u = v$ on $\overline{D}$. □

Proposition 41. *If g is harmonic and bounded on $\epsilon\mathbb{D} \setminus \{0\}$ for some $\epsilon > 0$ then $g(0) = \lim_{z->0} g(z)$ exists and the extended function $g : \epsilon\mathbb{D} \to \mathbb{R}$ is harmonic.*

Proof. Without loss of generality, by scaling, suppose that $\mathbb{D}$ is compactly contained in D and let $D' = \mathbb{D}\backslash\overline{\epsilon\mathbb{D}}$ and $g(z) = G_D(z, 0) + \log|z|$. Again let B_t be a B.M. and $\tau_{D'} = \inf\{t : B_t \notin D'\}$.

Then by Lemma 40, and since $\mathbb{E}^z g\left(B_{\tau_{D'}}\right)$ is harmonic on D' and takes the same boundary values,

$$g(z) = \mathbb{E}^z g\left(B_{\tau_{D'}}\right)$$

for all $z \in D'$. Denote the characteristic function by $\mathbf{1}$. Then, for $z \in \mathbb{D}\setminus\{0\}$

$$\begin{aligned}\mathbb{E}^z g\left(B_{\tau_{D'}}\right) &= \mathbb{E}^z\left[\left(\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}} + \mathbf{1}_{B_{\tau_{D'}}\in\epsilon\mathbb{T}}\right) g\left(B_{\tau_{D'}}\right)\right] \\ &= \mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right] + \mathbb{E}^z\left[\mathbf{1}_{B_{\tau_{D'}}\in\epsilon\mathbb{T}} g\left(B_{\tau_{D'}}\right)\right].\end{aligned}$$

Since g is bounded on $\mathbb{D}$ by Proposition 38, by K say

$$\begin{aligned}\mathbb{E}^z g\left(B_{\tau_{D'}}\right) &\leq \mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right] + \mathbb{E}^z\left[\mathbf{1}_{B_{\tau_{D'}}\in\epsilon\mathbb{T}} K\right] \\ &= \mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right] + K\mathbb{P}^z\left(B_{\tau_{D'}} \in \epsilon\mathbb{T}\right)\end{aligned}$$

and similarly

$$\mathbb{E}^z g\left(B_{\tau_{D'}}\right) \geq \mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right] - K\mathbb{P}^z\left(B_{\tau_{D'}} \in \epsilon\mathbb{T}\right).$$

And so, for all $\epsilon > 0$

$$\left|g(z) - \mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right]\right| \leq K\mathbb{P}^z\left(B_{\tau_{D'}} \in \epsilon\mathbb{T}\right).$$

Almost surely B_t leaves $\mathbb{D}$ without passing through 0, so for a fixed $\omega \in \Omega$, B_t from $t = 0$ to $t = \tau_D$ is a closed path, and so for all $\epsilon < \delta$ for some δ, $\tau_{D'} = \tau_D$ and so as $\epsilon \to 0$, almost surely

$$\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}} g\left(B_{\tau_{D'}}\right) \to g\left(B_{\tau_D}\right).$$

Then since both are bounded by K and by the dominated convergence theorem

$$\mathbb{E}^z\left[g\left(B_{\tau_{D'}}\right)\mathbf{1}_{B_{\tau_{D'}}\in\mathbb{T}}\right]\to\mathbb{E}^z g\left(B_{\tau_D}\right)$$

as $\epsilon\to 0$. Similarly as $\epsilon\to 0$, $\mathbb{P}^z\left(B_{\tau_{D'}}\in\epsilon\mathbb{T}\right)\to 0$. By taking limits as $\epsilon\to 0$ then

$$\left|g\left(z\right)-\mathbb{E}^z g\left(B_{\tau_D}\right)\right|\leq 0$$

i.e. $g\left(z\right)=\mathbb{E}^z g\left(B_{\tau_D}\right)$ for all $z\in\mathbb{D}\backslash\left\{0\right\}$. Then the harmonic extension of g to $\mathbb{D}$ is just $g\left(z\right)=\mathbb{E}^z g\left(B_{\tau_D}\right)$.

□

Lemma 42. *Let φ be a C^2 function compactly supported in D. Then, integrating by parts gives,*

$$\int_D\Delta\varphi\left(z\right)\log\left|z\right|d^2z=2\pi\varphi\left(0\right).$$

This is a standard analytic fact proved using integration by parts. Therefore

$$\begin{aligned}0&=\int_D\varphi\left(z\right)\Delta\left[G_D\left(z,0\right)+\log\left|z\right|\right]d^2z\\&=\int_D\varphi\left(z\right)\Delta G_D\left(z,0\right)d^2z+\int_D\varphi\left(z\right)\Delta\log\left|z\right|d^2z\\\therefore&\int_D\varphi\left(z\right)\Delta G_D\left(z,0\right)d^2z=-2\pi\varphi\left(0\right)\end{aligned}$$

i.e. $\Delta G_D\left(z,0\right)$"$=$"$-2\pi\delta\left(z\right)$ in the sense of distributions.

Lemma 43. *If $z\in\partial D$ is a connected boundary point and $z_i\to z$ with $z_i\in D$, then $G_D\left(z_i,0\right)\to 0$.*

Proof. From the proof of Theorem 36, we had the following inequality, for all $n>u$ and a large enough u:

$$f_n\left(z_i\right)=nH_{D_n}\left(z_i,e^{-n}\mathbb{T}\right)\leq H_{D_u}\left(z_i,e^{-u}\mathbb{T}\right)\frac{u}{1-K/u}.$$

We can assume that z_i and z are away from 0, so by taking limits

$$G_D\left(z_i,0\right)\leq H_{D_u}\left(z_i,e^{-u}\mathbb{T}\right)\frac{u}{1-K/u}.$$

However as $z_i\to z\in\partial D$ then $H_{D_u}\left(z_i,e^{-u}\mathbb{T}\right)\to 0$ so

$$G_D\left(z_i,0\right)\to 0.$$

□

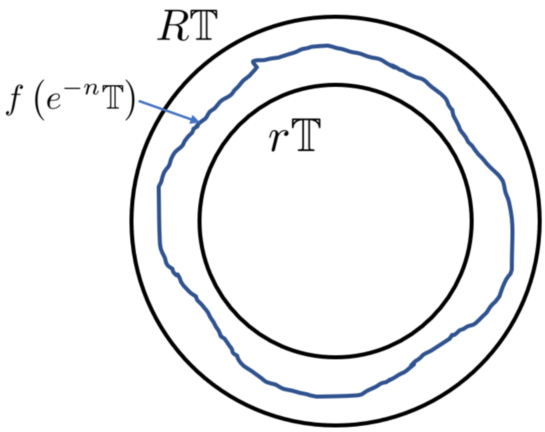

Figure 8.3: The image $f\left(e^{-n}\mathbb{T}\right)$ is sandwiched between $r\mathbb{T}$ and $R\mathbb{T}$, which converge to $e^{-n}\left|f'(z)\right|\mathbb{T}$.

Therefore the probabilistic definition of the Green's function satisfies the usual properties, and so is the unique Green's function on D.

Lemma 44. *Green's functions are unique if D is a bounded domain.*

Proof. Suppose $G_D(z,0)$ and $G'_D(z,0)$ are both Green's functions on D. Then $G_D(z,0)+\log|z|$ and $G'_D(z,0)+\log|z|$ are both harmonic on D and have the same boundary values so by Lemma 40 they must be equal and so

$$G_D(z,0) = G'_D(z,0).$$

□

It turns out that $G_D(x,y) = G_D(y,x)$, which can be proved from the analytic characterization of the Green's function. More surprising is that the Green's function is invariant under conformal transformations:

Theorem 45. *Suppose that D is a domain and f is conformal and onto. Then*

$$G_D(z,w) = G_{f(D)}(f(z), f(w)).$$

Proof. That $G_D(z,w) = G_{D+x}(z+x, w+z)$ follows easily from the definition of G_D and the translation invariance of Brownian motion.

Then we can assume that $f(0)=0$ and it is enough to show that

$$G_D(z,0) = G_{f(D)}(f(z),0).$$

By the conformal invariance of the harmonic measure

$$\begin{aligned} G_D(z,0) &= \lim_{n\to\infty} nH_{D_n}\left(z, e^{-n}\mathbb{T}\right) \\ &= \lim_{n\to\infty} nH_{f(D)\backslash f\left(\overline{e^{-n}\mathbb{D}}\right)}\left(f(z), f\left(e^{-n}\mathbb{T}\right)\right) \end{aligned}$$

where $f(D)\backslash f\left(\overline{e^{-n}\mathbb{D}}\right) = f(D_n)$ since f is bijective. Also

$$f(z+\delta) - f(z) = \delta f'(z) + \delta^2 h(\delta)$$

where

$$h(\delta) = \frac{1}{2!}f^{(2)}(z) + \frac{1}{3!}\delta f^{(3)}(z) + \frac{1}{4!}\delta^2 f^{(4)}(z) + \cdots.$$

Let $\delta = e^{i\theta - n}$ and $m = \min_\theta \left|h\left(e^{i\theta-n}\right)\right|$. By the triangle inequality and reverse triangle inequality

$$\left|e^{-n}\left|f'(z)\right| - e^{-2n}m\right| \le \left|f(z+\delta) - f(z)\right| \le e^{-n}\left|f'(z)\right| + e^{-2n}m$$

and so for large enough n

$$r \equiv e^{-n}\left|f'(z)\right| - e^{-2n}m \le \left|f(z+\delta) - f(z)\right| \le e^{-n}\left|f'(z)\right| + e^{-2n}m \equiv R.$$

Note that δ, m, r and R all depend on n. As $n\to\infty$ then $m \to \frac{1}{2!}\left|f^{(2)}(z)\right|$. Then by Figure 8.3

$$nH_{f(D)\backslash\overline{r\mathbb{D}}}\left(f(z), r\mathbb{T}\right) \le nH_{f(D_n)}\left(f(z), f\left(e^{-n}\mathbb{T}\right)\right) \le nH_{f(D)\backslash\overline{R\mathbb{D}}}\left(f(z), R\mathbb{T}\right).$$

Also since $r \le |f'(z)| \le R$,

$$nH_{f(D)\backslash\overline{r\mathbb{D}}}\left(f(z), r\mathbb{T}\right) \le nH_{f(D)\backslash\overline{e^{-n}|f'(z)|\mathbb{D}}}\left(f(z), e^{-n}\left|f'(z)\right|\mathbb{T}\right) \le nH_{f(D)\backslash\overline{R\mathbb{D}}}\left(f(z), R\mathbb{T}\right).$$

Also

$$n\left|H_{f(D)\backslash\overline{R\mathbb{D}}}\left(f(z), R\mathbb{T}\right) - H_{f(D)\backslash\overline{r\mathbb{D}}}\left(f(z), r\mathbb{T}\right)\right| \to 0$$

as $n\to\infty$. Then

$$nH_{f(D_n)}\left(f(z), f\left(e^{-n}\mathbb{T}\right)\right) \to nH_{f(D)\backslash\overline{e^{-n}|f'(z)|\mathbb{D}}}\left(f(z), e^{-n}\left|f'(z)\right|\mathbb{T}\right)$$

as $n\to\infty$ and so

$$\begin{aligned} G_D(z,0) &= -\lim_{n\to\infty} \log e^{-n} \cdot H_{f(D)\backslash\overline{e^{-n}|f'(z)|\mathbb{D}}}\left(f(z), e^{-n}\left|f'(z)\right|\mathbb{T}\right) \\ &= -\lim_{n\to\infty}\left[\log\left|f'(z)\right|e^{-n} - \log\left|f'(z)\right|\right] \cdot H_{f(D)\backslash\overline{e^{-n}|f'(z)|\mathbb{D}}}\left(f(z), e^{-n}\left|f'(z)\right|\mathbb{T}\right) \\ &= -\lim_{n\to\infty} \log\left|f'(z)\right|e^{-n} \cdot H_{f(D)\backslash\overline{e^{-n}|f'(z)|\mathbb{D}}}\left(f(z), e^{-n}\left|f'(z)\right|\mathbb{T}\right) \\ &= G_{f(D)}\left(f(z), 0\right). \end{aligned}$$

□

Example 46. Suppose that $D = \mathbb{D}$. What is $G_{\mathbb{D}}(z, 0)$? From Example 35,

$$nH_{\mathbb{D}_n}\left(z, e^{-n}\mathbb{T}\right) = n\frac{0 - \log|z|}{0 - \log e^{-n}} = -\log|z|$$

and so $G_{\mathbb{D}}(z, 0) = -\log|z|$.

From complex analysis, all the Möbius transformations (which are conformal) that map $\mathbb{D}$ onto itself are given by

$$\lambda(z) = \alpha\frac{z - a}{1 - \overline{a}z}$$

where $a \in \mathbb{D}$ and $|\alpha| = 1$. Let

$$\lambda(z) = \frac{z - w}{1 - \overline{w}z}.$$

Then by Theorem 45

$$\begin{aligned} G_{\mathbb{D}}(z, w) &= G_{\lambda\mathbb{D}}(\lambda(z), \lambda(w)) \\ &= G_{\mathbb{D}}\left(\frac{z - w}{1 - \overline{w}z}, 0\right) \\ &= -\log\left|\frac{z - w}{1 - \overline{w}z}\right|. \end{aligned}$$

One use of Green's functions is to prove the Riemann mapping theorem.

Theorem 47. *(Riemann Mapping Theorem). Suppose that $D \subset \mathbb{C}$ is a simply connected domain and $D \neq \mathbb{C}$. Then there is an onto conformal map $f : D \to \mathbb{D}$.*

In particular, the onto conformal map is given by

$$f(z) = \exp\left[-G_D(z, 0) + ih(z)\right]$$

where h is a function making f holomorphic. We can show that such a h exists. For a proof of the Riemann Mapping Theorem using Green's functions see pp. 317–320 of Bass [1].

9 Poisson Kernels

Again we consider only $\mathbb{C}$ and $\mathbb{R}^2$.

Definition 48. Let D be a domain, $z \in D$ and suppose that γ is a suitable subset of ∂D. Then

$$H_D(z, \gamma) = \int_\gamma P_D(z, w)\,|dw|$$

determines the *Poisson kernel* $P_D(z, w) : D \times \partial D \to \mathbb{R}^+$.

The $P_D(z, \cdot)$ is the Radon-Nikodym derivative of $H_D(z, \cdot)$ with respect to the arc length measure $|\cdot|$.

If γ is parameterized by Γ, i.e. $\gamma = \Gamma([0,1])$ then the arc length of γ is

$$|\gamma| = \int_0^1 \left|\Gamma'(t)\right| dt$$

and so $|dw| = |\Gamma'(t)|\, dt$.

But what does it mean for an arc in ∂D to be suitable? At the very least $\Gamma'(t)$ needs to be defined on $[0,1]$ except perhaps on a measure 0 subset.

What we'll actually define the Poisson kernel on is the set of *analytic boundary arcs.*

Definition 49. Let D be a domain with boundary ∂D. Then $w \in \partial D$ is an *analytic boundary point (a.b.p.)* if there exists a domain D' and an onto conformal map $f : \mathbb{D} \to D'$ such that $f(0) = w$ and $f(\mathbb{D} \cap \mathbb{H}) = D \cap D'$. Then f maps bijectively $(-1,1)$ to $\partial D \cap D'$.

An subset $\gamma \subseteq \partial D$ is an *analytic boundary arc (a.b.a.)* if every $w \in \gamma$ is an analytic boundary point.

Lemma 50. *Suppose that f is onto and conformal on D and continuous on $\overline{D}$. Then $f(\partial D) \subseteq \partial f(D)$.*

Proof. Let $w \in \partial D$. Then $f(w)$ is a boundary point of $f(D)$ if $f(w) \notin f(D)$ and $f(w)$ is arbitrarily close to $f(D)$.

Suppose that $f(w) \in f(D)$. Then since f is a bijection, $f^{-1}f(w) = z$ must be in D, which is false. Since $w \in \partial D$, there exists a sequence $w_n \to w$ with $w_n \in D$. Then since f is continuous on $\overline{D}$, $f(w_n) \to f(w)$ and $f(w_n) \in f(D)$. $\square$

Fact 51. *Suppose that a conformal mapping $g : \mathbb{H} \cap \mathbb{D} \to D$ is continuous up to $\mathbb{R} \cap \mathbb{D}$ and $|g'(z)|$ is bounded on $\mathbb{H} \cap \epsilon\mathbb{D}$, then g extends to an analytic function $\tilde{g} : \epsilon'\mathbb{D} \to B_\delta(g(0))$ for some ϵ' and δ. Hence, $g(0)$ is an analytic boundary point of ∂D.*

Proposition 52. *Suppose that $g : D \to \hat{D}$ is conformal and onto, and continuous on $\overline{D}$, $w \in \partial D$ is an analytic boundary point, and $|g'|$ is bounded around w. Then $g(w)$ is an analytic boundary point of $\partial g(D)$.*

Proof. Since $w \in \partial D$ is an a.b.p. there is a conformal map $f : \mathbb{D} \to D'$ and a domain D'. Then since $f(\mathbb{D} \cap \mathbb{H}) = D \cap D'$ it follows that $g \circ f$ is defined on $\mathbb{D} \cap \mathbb{H}$ and is conformal.

Also $f\left(\mathbb{D} \cap \overline{\mathbb{H}}\right) = \overline{D} \cap D'$ and so $g \circ f$ is continuous on $\mathbb{D} \cap \overline{\mathbb{H}}$. Moreover $(g \circ f)'(z) = f'(z)\, g'(f'(z))$ and so $\left|(g \circ f(z))'\right|$ is bounded on $\mathbb{D} \cap \epsilon\mathbb{H}$ for some ϵ. Then by Fact 51, $g \circ f(0) = g(w)$ is an analytic boundary point of $\partial g(D)$. $\square$

Theorem 53. *Let D be a domain and $f : D \to f(D)$ an onto conformal map and holomorphic on $\overline{D}$. Then*

$$P_D(z,w) = P_{f(D)}(f(z), f(w)) \cdot |f'(w)|$$

for all analytic boundary points w if both Poisson kernels exist.

Let D be a domain and f a conformal map on $\overline{D}$ (or more precisely, a conformal map on a domain containing $\overline{D}$). Then

$$P_D(z,w) = P_{f(D)}(f(z), f(w)) \cdot |f'(w)|$$

for all a.b.p.'s w if both Poisson kernels exist.

Proof. Let γ be an analytic boundary arc in ∂D containing w. Then $f(w)$ is an a.b.p. of $\partial f(D)$. Similarly for all a.b.a.'s $E \subseteq \gamma$, $f(E)$ is an a.b.a. in $\partial f(D)$.

By the conformal invariance of the harmonic measure, for all $z \in D$

$$\int_E P_D(z,w)|dw| = H_D(z,E) = H_{f(D)}(f(z), f(E)) = \int_{f(E)} P_{f(D)}(f(z), v)|dv|.$$

Let $u = f^{-1}(v)$. Then $|dv| = |f'(u)||du|$ and by change of variables

$$\int_{f(E)} P_{f(D)}(f(z), v)|dv| = \int_E P_{f(D)}(f(z), f(u))|f'(u)||du|.$$

Since this is true for all a.b.a.'s E then

$$P_D(z,w) = P_{f(D)}(f(z), f(w)) \cdot |f'(w)|.$$

□

Example 54. What is $P_{\mathbb{D}}(0,w)$ for $w \in \partial\mathbb{D}$? Since for any analytic boundary arc $E \subset \partial\mathbb{D}$, by the rotational symmetry of Brownian motion

$$H_{\mathbb{D}}(0,E) = \mathbb{P}^0(B_{\tau_{\mathbb{D}}} \in E) = \frac{|E|}{2\pi}$$

and so

$$\int_E \frac{1}{2\pi}|dw| = \frac{|E|}{2\pi} = H_{\mathbb{D}}(0,E).$$

Therefore $P_{\mathbb{D}}(0,w)$ exists and is $1/2\pi$.

Then using Theorem 53 we can find $P_{\mathbb{D}}(z,w)$. Recall that

$$\lambda(z) = \frac{z-w}{1-\overline{w}z}$$

is a conformal and maps $\mathbb{D}$ onto itself. Then

$$\begin{aligned} P_{\mathbb{D}}(z,w) &= P_{\mathbb{D}}(\lambda(z),\lambda(w)) \cdot \left|\lambda'(w)\right| \\ &= P_{\mathbb{D}}\left(\frac{z-w}{1-\overline{w}z},0\right) \cdot \left|\frac{1}{1-|w|^2}\right| \\ &= \frac{1}{2\pi}\frac{1}{1-|w|^2}. \end{aligned}$$

Proposition 55. *Let D be a simply connected domain, $z \in D$ and $\gamma \subseteq \partial D$ an analytic boundary arc. Then for all a.b.a.'s $E \subseteq \gamma$ the Poisson kernel exists.*

Proof. Let $w \in \gamma$. By the Riemann mapping theorem there exists an onto conformal map $f : D \to \mathbb{D}$ such that $z \mapsto 0$ and $w \mapsto 0$.

By the conformal invariance of the harmonic measure, for all a.b.a.'s $E \subseteq \gamma$,

$$H_D(z,E) = H_{\mathbb{D}}(0,f(E)) = \frac{|f(E)|}{2\pi}.$$

Also, by change of variables $v = f(w)$,

$$\int_E \frac{1}{2\pi}\left|f'(w)\right||dw| = \int_{f(E)} \frac{1}{2\pi}|dv| = \frac{|f(E)|}{2\pi}$$

and so for all a.b.a.'s E

$$H_D(z,E) = \int_E \frac{1}{2\pi}\left|f'(w)\right||dw|.$$

□

Therefore $\frac{1}{2\pi}|f'(w)|$ is the Radon-Nikodym derivative with respect to $|dw|$on γ, and so the Poisson kernel exists for D and

$$P_D(z,w) = \frac{1}{2\pi}\left|f'(w)\right|.$$

Proposition 56. *Let D be a domain (not necessarily simply connected), $z \in D$ and $\gamma \subseteq \partial D$ an analytic boundary arc. Then for all subarcs $E \subset \gamma$ the Poisson kernel exists.*

Proof. Without loss of generality suppose that D is bounded. (If D is not bounded, choose an interior point of D^c, w say, and let λ be a Möbius transformation sending w to ∞. Then $\lambda(D)$ will be bounded. If there are no interior points of D^c, it is still the case that we can make the image of D to be bounded under some conformal map.)

Let $D' =$ "D filled in", that is, if γ is a closed Jordan curve in D, then the interior of γ is in D'. Then D' is a bounded simply connected domain and so by Proposition 55, the Poisson kernel exists for all analytic boundary arcs in $\partial D'$.

Let $\tau_S = \inf\{t : B_t \in D' \backslash D\}$, i.e. the time when B_t first hits one of the filled in parts of D^c, and $S = D' \backslash D$, i.e. the filled in parts. Suppose that E is an analytic boundary arc of $\partial D'$ (and so of ∂D too). By Proposition 55,

$$\mathbb{P}^z \left\{B_{\tau_{D'}} \in E\right\} = \int_E P_{D'}(z, w) |dw|$$

for some $P_{D'}(z, w)$. For a path in D', either B_t hits one of the filled in parts before leaving E, and so the path does not leave E in D, or doesn't hit one of the filled in parts, and so leaves D through E. I.e. either $\tau_S < \tau_{D'}$ or $\tau_{D'} < \tau_S$.

$$\mathbb{P}^z \left\{B_{\tau_{D'}} \in E\right\} = \mathbb{P}^z \left\{B_{\tau_{D'}} \in E, \tau_{D'} < \tau_S\right\} + \mathbb{P}^z \left\{B_{\tau_{D'}} \in E, \tau_S < \tau_{D'}\right\}$$

and so $\mathbb{P}^z \left\{B_{\tau_{D'}} \in E, \tau_{D'} < \tau_S\right\} = \mathbb{P}^z \left\{B_{\tau_D} \in E\right\}$. Also if $\zeta = B_{\tau_S}$,

$$\begin{aligned}
\mathbb{P}^z \left\{B_{\tau_{D'}} \in E, \tau_S < \tau_{D'}\right\} &= \mathbb{P}^z \left\{\tau_S < \tau_{D'}\right\} \mathbb{P}^z \left\{B_{\tau_{D'}} \in E | \tau_S < \tau_{D'}\right\} \\
&= \int_{\partial(D' \backslash D)} H_D(z, d\zeta) \mathbb{P}^\zeta \left\{B_{\tau_{D'}} \in E\right\} \\
&= \int_{\partial(D' \backslash D)} H_D(z, d\zeta) \mathbb{P}^\zeta \left\{B_{\tau_{D'}} \in E\right\} \\
&= \int_{\partial(D' \backslash D)} H_D(z, d\zeta) \int_E P_{D'}(\zeta, w) |dw| \\
&= \int_E \int_{\partial(D' \backslash D)} P_{D'}(\zeta, w) H_D(z, d\zeta) |dw| .
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{P}^z \left\{B_{\tau_D} \in E\right\} &= \mathbb{P}^z \left\{B_{\tau_{D'}} \in E\right\} - \mathbb{P}^z \left\{B_{\tau_{D'}} \in E, \tau_S < \tau_{D'}\right\} \\
&= \int_E \left[P_{D'}(z, w) - \int_{\partial(D' \backslash D)} P_{D'}(\zeta, w) H_D(z, d\zeta)\right] |dw|
\end{aligned}$$

and so the Poisson kernel exists for D. □

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